Incorporating MR, ER and Robustness in Mean Variance Portfolio Choice

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1 Contents

- ER and MR: a (partial) literature overview
- Mean Variance Portfolio Model, ER and MR
- Robust Statistics
- Robust Corrections for ER Using Robust Statistics
- Conclusions, Discussion

2 ER and MR

- ER:
  - Uncertainty about the true model parameter
  - Arises by the fact that econometricians have to work with finite samples of asset returns
- MR:
  - The class of relevant model distributions may not be fully consistent with the genuine stochastic structure of returns
  - Arises through model assumptions which cannot be always completely satisfied by the financial reality

2.1 Portfolio Selection and ER

- Bayesian approaches: For instance, Alexander and Resnick (1985), Barry (1974), Brown (1979), Chen and Brown (1983), Jorion (1985, 1986, 1991), Klein and Bawa (1976), Balduzzi and Liu (2000), Barberis (2000), Pastor (2000)

- "Frequentist" approaches: For instance, ter Horst, de Roon and Werker (2000)

2.1.1 Bayesian Approaches to ER

- Specification of an initial prior distribution on the opportunity set parameters (for instance for the mean return and the covariance matrix of returns)

- Maximization of expected utility under the predictive distribution of returns

- Bayesian estimators for the opportunity set parameters (for instance Jorion (1986))

- Performance of single prior Bayesian portfolio policies is sensitive to slight perturbations of the data generating and the initial prior distributions

2.1.2 Frequentist Approaches to ER

- Implicit prior specification through the finite sample distribution of the (given) opportunity set estimators

- Maximization of expected utility under the joint distribution of returns and opportunity set estimators yields a correction of (pseudo) risk aversion for ER

- Maximum-Likelihood estimators for the opportunity set parameters (for instance ter Horst et al. (2000))

- Performance of ML estimators and implied portfolio policies is sensitive to slight perturbations of the data generating distribution (cf. Huber (1981))

2.2 Portfolio Selection and MR

- Bayesian approaches:
  - For instance, Chamberlain (2000)

- Preferences for Robustness/Ambiguity Models:
  - For instance, Chen and Epstein (2000), Epstein and Miao (2001), Lei (2001a, 2001b), Maenhout (1999), Trojani and Vanini (2001a, 2001b), Uppal and Wang (2001)

2.3 Bayesian Approaches to MR

- MR is described by a (typically finite dimensional) set of multiple priors
- Numerical Max-Min expected utility optimization following the axiomatic foundation in Gilboa and Schmeidler (1986)
- Extensions to richer structures of multiple priors are difficult: local stability is not easily granted

2.4 Preferences for Robustness, Ambiguity Models

- Estimation of the reference model parameters and portfolio optimization are separated steps

- ER and MR can jointly lead to select non-local alternatives relatively to the underlying data distribution when non robust statistics are used

- In applications large utility losses can arise when using non-robust estimators for the reference model distribution

Wanted:

- Portfolio selection procedures that take into account ER and MR both
  - when determining optimal policies in financial models
  - when estimating the parameter inputs for a financial model
- Robust financial and econometric models putting modelbuilders and econometricians on the same foot!

3 Mean Variance Portfolio Model, ER and MR

- Optimization problem and solution

$$\max_{w}\left\{w'\mu_0 - \frac{\lambda}{2}w'\Sigma_0 w\right\} \quad , \quad w^*(\lambda) = \frac{1}{\lambda}\Sigma_0^{-1}(\mu_0 - \eta_0 1)$$

- $\eta_0 = \frac{B_0 - \lambda}{A_0}$ is expected return on zero beta portfolio, where

$$A_0 = 1'\Sigma_0^{-1}1 \quad , \quad B_0 = 1'\Sigma_0^{-1}\mu_0 \quad , \quad C_0 = \mu_0'\Sigma_0^{-1}\mu_0$$

- Utility

$$U_0(\lambda) = \frac{1}{\lambda}(\mu_0 - \eta_0 1)'\Sigma_0^{-1}\mu_0 - \frac{1}{2\lambda}(\mu_0 - \eta_0 1)'\Sigma_0^{-1}(\mu_0 - \eta_0 1)$$

3.1 ER in Mean Variance Portfolio Choice

- Estimated (naive) optimal policies

$$\widehat{w}_T^*(\lambda) = \frac{1}{\lambda}\widehat{\Sigma}_T^{-1}(\widehat{\mu}_T - \widehat{\eta}_T \mathbf{1}) \quad , \quad \widehat{\eta}_T = \frac{\widehat{B}_T - \alpha}{\widehat{A}_T}$$

- Realized utilities

$$\widehat{U}_T(\lambda) = \widehat{w}_T^*(\lambda)' \mu_0 - \frac{\lambda}{2}\widehat{w}_T^*(\lambda)' \Sigma_0 \widehat{w}_T^*(\lambda) \ ,$$

- Realized utility losses

$$\widehat{L}_T(\lambda) = U_0(\lambda) - \widehat{U}_T(\lambda)$$

3.2 ER and Corrections for (Pseudo) Risk Aversion

- Mean variance policy and pseudo risk aversion α:

$$\widehat{w}_T^*(\alpha) = \frac{1}{\alpha}\widehat{\Sigma}_T^{-1}(\widehat{\mu}_T - \widehat{\eta}_T 1) \quad , \quad \widehat{\eta}_T = \frac{\widehat{B}_T - \alpha}{\widehat{A}_T}$$

- Realized mean variance objective

$$\widehat{U}_T(\alpha) = \widehat{w}_T^*(\alpha)' \mu_0 - \frac{\lambda}{2}\widehat{w}_T^*(\alpha)' \Sigma_0 \widehat{w}_T^*(\alpha)$$

- Pseudo risk aversion correction

$$\alpha_T^* := \alpha_T^*(F_0) := \arg\sup_{\alpha} \widehat{\delta}_T(\alpha, F_0) \quad , \quad \widehat{\delta}_T(\alpha, F_0) := E_{F_0}\left(\widehat{U}_T(\alpha)\right)$$

3.2.1 Corrections for ER under Normality of Returns

Property 3.2.1 Under a normal model F_0 and a known covariance matrix Σ_0 it follows for the sample mean and sample covariance matrix estimators

1. Expected Utility:

$$\widehat{\delta}_T(\alpha, F_0) = \alpha^{-1}\left(C_0 - \frac{B_0^2}{A_0}\right) - \alpha^{-2}\frac{\lambda}{2}\left(\frac{K-1}{T} + C_0 - \frac{B_0^2}{A_0}\right) + \frac{1}{A_0}\left(B_0 - \frac{1}{2}\lambda\right)$$

2. Pseudo risk aversion correction:

$$\alpha_T^* = \lambda\left(1 + \frac{K-1}{T}\frac{A_0}{A_0 C_0 - B_0^2}\right)$$

3.2.2 Simulation Parameters

- Unhedged monthly USD MSCI stock index returns for the U.S., France, Germany, Japan and the U.K. from 1974.1 to 1998.12

- Opportunity Set

$$
\begin{aligned}
\mu_0 &= \begin{pmatrix} 0.014 & 0.013 & 0.011 & 0.015 & 0.012 \end{pmatrix}' \\
\sigma_0 &= \begin{pmatrix} 0.069 & 0.059 & 0.067 & 0.073 & 0.044 \end{pmatrix}' \\
\rho_0 &= \begin{pmatrix} 1 & 0.590 & 0.390 & 0.541 & 0.456 \\ & 1 & 0.338 & 0.424 & 0.347 \\ & & 1 & 0.342 & 0.221 \\ & & & 1 & 0.506 \\ & & & & 1 \end{pmatrix}
\end{aligned}
$$

3.2.3 Utility Losses Under Normality of Returns $F_0 = \mathcal{N}(\mu_0, \Sigma_0)$; $\lambda = 1, 10, 20$

- Mean losses and percentage losses (in parentheses):

$$U_0(\lambda) - E_{F_0}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right); \left[U_0(\lambda) - E_{F_0}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right)\right] / |U_0(\lambda)|$$

T	Naive	Corr.	Naive	Corr.	Naive	Corr.
50	0.052 (33%)	0.027 (17%)	0.012 (21%)	0.001 (17%)	0.017 (55%)	0.016 (52%)
125	0.019 (12%)	0.013 (8%)	0.005 (9%)	0.005 (8%)	0.007 (24%)	0.007 (24%)

- $U_0(\lambda) = 0.158,\ 0.058,\ -0.031$

3.3 MR and Corrections for (Pseudo) Risk Aversion

- ε-contaminated neighborhood

$$\mathcal{G} = \{G_\varepsilon \mid G_\varepsilon = (1-\varepsilon)\cdot F_0 + \varepsilon \cdot G \; ; \; \varepsilon \le \zeta\} \quad ,$$

where $F_0 \vee \mathcal{N}(\mu_0, \Sigma_0)$.

- Model 1: Simple outliers model, $\varepsilon = 0.01$

$$G = \delta_{\mu_0 + \Sigma_0^{\frac{1}{2}} x} \quad , \quad \Sigma_0^{\frac{1}{2}} \Sigma_0^{\frac{1}{2}} = \Sigma_0 \quad , \quad x = -5 \cdot 1_K$$

- Model 2: Simple contaminated normal model, $\varepsilon = 0.01$

$$G = \mathcal{N}(\mu_0, 25 \cdot \Sigma_0)$$

3.3.1 Utility Losses under Model 1; $\lambda = 1, 10, 20$

- Mean losses and percentage losses (in parentheses):

$$U_0(\lambda) - E_{G_\varepsilon}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right);\ \left[U_0(\lambda) - E_{G_\varepsilon}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right)\right] / \left|U_0(\lambda)\right|$$

T	Naive	Corr.	Naive	Corr.	Naive	Corr.
50	0.058 (37%)	0.034 (21%)	0.038 (65%)	0.036 (61%)	0.066 (212%)	0.064 (208%)
125	0.024 (15%)	0.020 (12%)	0.027 (46%)	0.027 (46%)	0.048 (155%)	0.049 (157%)

- $U_0(\lambda) = 0.158,\ 0.058,\ -0.031$

3.3.2 Utility Losses under Model 2; $\lambda = 1, 10, 20$

- Mean losses and percentage losses (in parentheses):

$$U_0(\lambda) - E_{G_\varepsilon}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right);\ \left[U_0(\lambda) - E_{G_\varepsilon}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right)\right] / \left|U_0(\lambda)\right|$$

T	Naive	Corr.	Naive	Corr.M.L	Naive	Corr.
50	0.052 (32%)	0.028 (18%)	0.016 (28%)	0.014 (24%)	0.026 (84%)	0.025 (80%)
125	0.020 (13%)	0.015 (9%)	0.010 (18%)	0.010 (17%)	0.018 (57%)	0.018 (57%)

- $U_0(\lambda) = 0.158,\ 0.058,\ -0.031$

3.4 1st Summary on ER an MR in Mean Variance Portfolio Choice

- Slight deviations from normality can account for a large part of the utility losses arising when estimating the opportunity set

- Pseudo risk aversion corrections based on a normality assumption can loose a part of their efficacy in the presence of MR

- The impact of MR depends on the location of optimal portfolios on the underlying mean variance frontier

- Robust estimators and pseudo risk aversion corrections suited to maintain a satisfactory performance in the presence of MR are needed

4 Robust Statistics

- Robust Statistics (RS)
  - deals with deviations from ideal models and their dangers for corresponding inference procedures
  - primary goal is the development of procedures which are still reliable and reasonably efficient under deviations from the model used.
- RS care about the stability properties of a statistical procedure:
  - Influence Function: analyzes local stability
  - Breakdown Point: analyzes global reliability

4.1 Bounded-Influence Statistics

Definition 4.1 The Influence Function (IF, Hampel (1974)) of a statistic T is defined by

$$IF(T;x) = \lim_{\varepsilon \longrightarrow 0} \frac{T((1-\varepsilon)F_0 + \varepsilon\delta_x) - T(F_0)}{\varepsilon} \quad ,$$

for all x where the limit exists.

- The IF describes the asymptotic infinitesimal normalized influence of an additional observation x on the value of a statistic

- Examples of "interesting" statistics: an estimator, its mean and variance, the power and the level of a test, a portfolio allocation, its indirect utility.......

4.2 The Bounded-Influence "Paradigm"

- For a bounded bias of a statistic on contamined neighborhoods it is necessary that its IF is bounded.

- Procedures are needed that construct statistics with bounded IF for models where "standard" procedures imply unbounded IF.

- Many models in econometrics/finance imply policies/statistics with unbounded IF

- Well known examples: OLS–, TSLS–, NLLS–methods, many ML-based and GMM statistics; optimal portfolios and indirect utilities in mean variance optimization problems

4.2.1 Example of Unbounded IF; Sample Mean and Sample Covariance Matrix Estimators

- Asymptotic orthogonality conditions

$$E\left[X-\widehat{\mu}\right]=0 \quad , \quad E\left[\text{vech}\left(\left(X-\widehat{\mu}\right)\left(X-\widehat{\mu}\right)-\widehat{\Sigma}\right)\right]=0$$

- Orthogonality functions

$$v\left(x,\mu\right)=x-\mu \quad , \quad s\left(x;\mu,\Sigma\right)=\text{vech}\left(v\left(x,\mu\right)v\left(x,\mu\right)'-\Sigma\right)$$

- Influence Function

$$IF\left(\widehat{\mu};x\right)=v\left(x,\mu_0\right) \quad , \quad IF\left(\text{vech}\left(\widehat{\Sigma}\right);x\right)=s\left(x;\mu_0,\Sigma_0\right) \quad .$$

4.2.2 Robust Location-Scale Estimators as GMM Estimators (Krishnakumar and Ronchetti (1997))

- Weight functions

$$w_1 = \min\left(1, \frac{c_1}{\left\|A_1 \widehat{\Sigma}^{-1} v\right\|}\right) \quad , \quad w_2 = \min\left(1, \frac{c_2}{\|A_2 (s - a_2)\|}\right) \quad ,$$

where $v = x - \widehat{\mu}$, $s = \text{vech}\left(vv' - \widehat{\Sigma}\right)$ and for suitable constants c_1, c_2, and matrices A_1, A_2, a_2.

- Asymptotic orthogonality conditions

$$E\left[w_1 v\left(X, \widehat{\mu}\right)\right] = 0 \quad , \quad E\left[w_2\left(s\left(X; \widehat{\mu}, \widehat{\Sigma}\right) - a_2\right)\right] = 0 \ .$$

4.2.3 Robust Location-Scale Estimators as Iterated WLS Estimators

WLS equations, $s_t = \text{vech}\Big((x_t - \mu_0)(x_t - \mu_0)' - \Sigma_0 \Big)$:

$$\widehat{\mu}_T = \left(\sum_{t=1}^{T} w_{1t} \right)^{-1} \sum_{t=1}^{T} w_{1t} x_t, \text{vech}\left(\widehat{\Sigma}_T \right) = \left(\sum_{t=1}^{T} w_{2t} \right)^{-1} \sum_{t=1}^{T} w_{2t} (s_t - a_2)$$

where $w_{1t} = w_1(x_t, \mu_0, \Sigma_0)$, $w_{2t} = w_2(x_t, \mu_0, \Sigma_0)$, and A_1, A_2, a_2 given by

$$\begin{aligned} a_2 &= \left(\sum_{t=1}^{T} w_{2t} \right)^{-1} \sum_{t=1}^{T} w_{2t} s_t \quad , \quad A_1' A_1 = \left(\frac{1}{T} \sum_{t=1}^{T} w_{1t}^2 v_t v_t' \right)^{-1} , \\ A_2' A_2 &= \left(\frac{1}{T} \sum_{t=1}^{T} w_{2t}^2 (s_t - a_2)(s_t - a_2)' \right)^{-1} , \end{aligned}$$

with $v_t = x_t - \mu_0$.

4.2.4 Remarks on Robust Location-Scale Estimators

- These estimators have IF with (selfstandardized) norm bounded by c_1 and c_2, respectively.

- c_1, c_2 can be interpreted as a regulator between robustness and efficiency of our estimators
  - For lower values of c_1, c_2, one gains robustness but looses efficiency, and vice versa for higher values of these constants
  - Letting c_1, $c_2 \to \infty$, gives the Maximum Likelihood sample mean and sample covariance estimators, the most efficient but non robust estimators.

5 Robust Corrections for ER Using Robust Opportunity Set Estimators

- In order to preserve portfolio performance in the presence of MR, a robust estimation of the opportunity set parameters is needed

- Robust opportunity set estimates ensure bounded asymptotic utility losses of a portfolio strategy in the presence of MR

- Simple pseudo risk aversion corrections that are robust to MR can be incorporated using robust opportunity set estimates

5.1 Utility Losses Under Normality of Returns

- $T = 50$, mean losses and percentage losses (in parentheses):

$$U_0(\lambda) - E_{F_0}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right);\ \left[U_0(\lambda) - E_{F_0}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right)\right] / \left|U_0(\lambda)\right|$$

$\mathcal{N}(\mu_0, \Sigma_0)$	Naive	Naive Rob.	Corr.	Corr. Rob.
$\lambda = 1$	0.052 (33%)	0.053 (33%)	0.027 (17%)	0.028 (17%)
$\lambda = 10$	0.012 (21%)	0.012 (21%)	0.010 (17%)	0.010 (17%)
$\lambda = 20$	0.017 (55%)	0.017 (56%)	0.016 (52%)	0.016 (53%)

- The marginal mean utility loss under pure normality is moderate, when adopting robust opportunity set estimators instead of non-robust ones

5.2 Utility Losses Under Model 1

- $T = 50$, mean losses and percentage losses (in parentheses):

$$U_0(\lambda) - E_{F_0}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right); \left[U_0(\lambda) - E_{F_0}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right)\right] / \left|U_0(\lambda)\right|$$

Model 1	Naive	Naive Rob.	Corr.	Corr. Rob.
$\lambda = 1$	0.058 (37%)	0.055 (35%)	0.034 (21%)	0.030 (19%)
$\lambda = 10$	0.038 (65%)	0.013 (23%)	0.036 (61%)	0.011 (19%)
$\lambda = 20$	0.066 (212%)	0.019 (61%)	0.064 (208%)	0.018 (58%)

- Under Model 1 the performance of corrections for ER is enhanced when using robust opportunity set estimators.

5.3 Utility Losses Under Model 2

- $T = 50$, mean losses and percentage losses (in parentheses):

$$U_0(\lambda) - E_{F_0}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right);\ \left[U_0(\lambda) - E_{F_0}\left(\widehat{U}_T\left(\widehat{\alpha}_T^*\right)\right)\right] / \left|U_0(\lambda)\right|$$

Model 2	Naive	Naive Rob.	Corr.	Corr. Rob.
$\lambda = 1$	0.052 (32%)	0.054 (34%)	0.028 (18%)	0.029 (18%)
$\lambda = 10$	0.016 (28%)	0.013 (21%)	0.014 (24%)	0.010 (18%)
$\lambda = 20$	0.052 (32%)	0.054 (34%)	0.028 (18%)	0.029 (18%)

- Under Model 2 the performance of corrections for ER is essentially enhanced when using robust opportunity set estimators.

5.4 2nd Summary on ER an MR in Mean Variancer Portfolio Choice

- Utility performance of robust portfolio stategies in Model 1, 2, is satisfactory and comparable to that under pure normality of asset returns

- Corrections for ER based on robust statistics work well in the presence of MR, with an efficacy comparable to that arising under pure normality of asset returns

- Taking ER or MR separately into account does not generally produce portfolios with a satisfactory performance under local deviations from normality (cf. the results for Model 1)

6 Incorporating MR and ER Explicitly Using Robust Pseudo Risk Aversion Estimators

- We consider pseudo risk aversion corrections that explicitly incorporate MR and ER and which are robust to local departures from normality

- We approximate analytically the dependence of expected utilities and pseudo risk aversion corrections on the type of model deviation present in the data

- We adopt an approximate max-min expected utility approach (cf. also Gilboa and Schmeidler (1989)) to incorporate ER in a way that is robust to MR

6.1 Asymptotic Local Deviations from Normality and Indirect Utilities

- We consider model deviations standardized with respect to sample size:
$$\mathcal{G}_T = \left\{ G_{\varepsilon,T} \mid G_{\varepsilon,T} = \left(1 - \frac{\varepsilon}{T^{1/2}}\right) \cdot \mathcal{N}\left(\mu_0, \boldsymbol{\Sigma}_0\right) + \frac{\varepsilon}{T^{1/2}} \cdot G \ ; \ \varepsilon \leq \zeta \right\} \quad .$$

- $\mathcal{G}_T$ represents a set of nonparametric contaminations of a normal distribution where the maximal probability of contamination is $\zeta \cdot T^{-\frac{1}{2}}$

- $\mathcal{G}_T \downarrow \{F_0\}$ as $T \to \infty$, reflecting the fact that in larger samples MR is less dangerous because of the larger amount of data we can use to statistically distinguish two competing models.

6.1.1 Pseudo Risk Aversion Corrections and MR

- Mean variance policy and pseudo risk aversion α:

$$\widehat{w}_T^*(\alpha) = \frac{1}{\alpha}\widehat{\Sigma}_T^{-1}\left(\widehat{\mu}_T - \widehat{\eta}_T \mathbf{1}\right) \quad , \quad \widehat{\eta}_T = \frac{\widehat{B}_T - \alpha}{\widehat{A}_T}$$

- Realized mean variance objective

$$\widehat{U}_T(\alpha) = \widehat{w}_T^*(\alpha)' \mu_0 - \frac{\lambda}{2}\widehat{w}_T^*(\alpha)' \Sigma_0 \widehat{w}_T^*(\alpha)$$

- Pseudo risk aversion correction

$$\alpha_T^*\left(G_{\varepsilon,T}\right) := \arg\sup_{\alpha} \widehat{\delta}_T\left(\alpha, G_{\varepsilon,T}\right) \ , \ \widehat{\delta}_T\left(\alpha, G_{\varepsilon,T}\right) := E_{G_{\varepsilon,T}}\left(\widehat{U}_T(\alpha)\right)$$

6.1.2 Uniform Asymptotic Normality

Property 6.1.2. Let $\widehat{\mu}$ be a robust estimator of μ_0 and denote by $\Phi_0 = E_{F_0}\left(IF\left(\widehat{\mu};X\right)IF'\left(\widehat{\mu};X\right)\right)$ its asymptotic covariance matrix at the reference model F_0. It then follows

$$T^{\frac{1}{2}}\left(\widehat{\mu}_T - \widehat{\mu}_{G_{\varepsilon,T}}\right) \rightarrow \mathcal{N}\left(0, \Phi_0\right) \quad ,$$

in distribution as $T \rightarrow \infty$, uniformly in $G_{\varepsilon,T} \in G_T$.

- Important implication: we can approximate the finite-sample distribution of $\widehat{\mu}_T$ under a model contamination $G_{\varepsilon,T} \in \mathcal{G}_T$ by a normal one with covariance matrix Φ_0 and a mean that depends on the contaminating direction G.

6.2 Expected Utility Asymptotics under MR

Proposition 6.2 If Σ_0 is known it follows up to terms of order $o\left(\frac{\varepsilon}{T^{\frac{1}{2}}}\right)$:

$$\widehat{\delta}_T\left(\alpha, G_{\varepsilon,T}\right) = \widehat{\delta}_T\left(\alpha, F_0\right) + \frac{\varepsilon\left(\alpha-\lambda\right)}{T^{\frac{1}{2}}\alpha^2}\left(\mu_0 - \frac{B_0}{A_0}1\right)' \Sigma_0^{-1} E_G\left[IF\left(\widehat{\mu}; X\right)\right] \quad ,$$

where (denoting by $tr\left(\cdot\right)$ the trace operator)

$$\begin{aligned}
\widehat{\delta}_T\left(\alpha, F_0\right) &= -\frac{\lambda}{2\alpha^2}\left[C_0 - \frac{B_0^2}{A_0} + \frac{tr\left(\Phi_0 \Sigma_0^{-1} \Pi_0\right)}{T}\right] + \frac{1}{\alpha}\left(C_0 - \frac{B_0^2}{A_0}\right) \\
&\quad + \frac{1}{A_0}\left(B_0 - \frac{\lambda}{2}\right) \quad , \\
\Pi_0 &= \left(id_K - \frac{1}{A_0} 1 1' \Sigma_0^{-1}\right) \quad .
\end{aligned}$$

6.2.1 Remarks on Expected Utility Asymptotics

- $tr\left(\Phi_0\Sigma_0^{-1}\Pi_0\right)$ reflects the higher ER implied by a robust estimation of expected returns; if $\Phi_0 = \Sigma_0$ (cf. also ter Horst et a. (2000)) then $tr\left(\Phi_0\Sigma_0^{-1}\Pi_0\right) = K - 1$

- A MR-induced expected utility loss arises if and only if

$$\frac{\varepsilon}{T^{\frac{1}{2}}}\left(\mu_0 - \frac{B_0}{A_0}1\right)' \Sigma_0^{-1} E_G\left[IF\left(\widehat{\mu}; X\right)\right] < 0 \quad . \tag{1}$$

- The LHS of (1) is a first order asymptotics for the MR-induced change in the expected return of a zero-investment portfolio $\Sigma_0^{-1}\left(\mu_0 - \frac{B_0}{A_0}1\right)$.

7 Approximate Max-Min Utility Approach

- Worst case expected utility

$$\widehat{\delta}_T^{wc}(\alpha) = \inf_{G_{\varepsilon,T}\in\mathcal{G}_T} \widehat{\delta}_T\left(\alpha, G_{\varepsilon,T}\right) \quad ,$$

- Implied pseudo risk aversion correction for ER

$$\alpha_T^{*wc} := \arg\sup_{\alpha} \widehat{\delta}_T^{wc}(\alpha) = \arg\sup_{\alpha} \inf_{G_{\varepsilon,T}\in\mathcal{G}_T} \left[\widehat{\delta}_T\left(\alpha, G_{\varepsilon,T}\right)\right] \quad .$$

7.1 Worst Case Utilities and Correction for MR

Proposition 7.1 If Σ_0 is known it follows up to terms of order $o\left(\frac{\zeta}{T^{\frac{1}{2}}}\right)$:

$$\widehat{\delta}_T^{wc}(\alpha) = \widehat{\delta}_T(\alpha, F_0) - \frac{\zeta}{T^{\frac{1}{2}}} \frac{c_1\left(\frac{1}{\alpha} - \frac{\lambda}{\alpha^2}\right)}{\left\|\Sigma_0^{-1}\left(\mu_0 - \frac{B_0}{A_0}1\right)\right\|} \quad .$$

Pseudo risk aversion correction for worst case expected utility:

$$\alpha_T^{*wc} = \alpha_{F_0}^* - \frac{\lambda c_1 \zeta\left(\frac{A_0}{A_0 C_0 - B_0^2}\right)}{T^{\frac{1}{2}}\left\|\Sigma_0^{-1}\left(\mu_0 - \frac{B_0}{A_0}1\right)\right\|}\left(1 - \frac{tr\left(\Phi_0 \Sigma_0^{-1} \Pi_0\right)}{T}\left(\frac{A_0}{A_0 C_0 - B_0^2}\right)\right)$$

8 Conclusions

- We considered pseudo risk aversion corrections attempting to take ER and MR jointly into account

- These corrections are based on robust location and scale estimates of the given opportunity set

- The loss reduction induced by the robust pseudo risk aversion corrected strategies is substantial

- Taking either ER or MR separately into account does not generally produce portfolios with a satisfactory performance under local deviations from normality and in small samples