Perturbative Approaches for Robust Intertemporal Optimal Portfolio Selection

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ECAS Course, Lugano, October 7-13, 2001

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2 Introduction, Motivation

- Only a few intertemporal optimal portfolio problems can be solved explicitly (cf. Kim and Omberg (1996), Chacko and Viceira (1999))

- Existence of closed-form solutions depends on assumptions regarding agent's utility functions, the opportunity set dynamics, intermediate consumption, aversion to model misspecification....

- Perturbation Theory (PT) is based on approximation methods by which approximate analytical expressions can be achieved

- Kogan and Uppal (2000): PT is a powerful approximation method for financial optimal decision making also

2.1 Preferences for Robustness (I)

- Agents have a reference model in mind which describes the approximate probabilistic features of some underlying state variables process; AHS (2000), Maenhout (1999), Lei (2001), Trojani and Vanini (2001), Uppal and Wang (2001)

- Agents believe in the possibility that the benchmark model could be slightly misspecified

- Model deviations that are particularly different from the reference model are penalized in their impact on the final decision

- The entity of the penalization is parameterized by a parameter that is interpreted as the strength of a preference for robustness

2.2 Preferences for Robustness (II)

- Differences through the way by which model deviations are penalized
  - AHS (2000): penalizes deviations proportionally to their relative entropy w.r.t. the reference model
  - Meanhout (1999): penalizes relative entropy, however in a way that is scaled by the current level of indirect utility
  - AHS (1998), Lei (2001), Trojani and Vanini (2001): put a maximal bound on the relative entropy "distance" of a relevant candidate misspecification
- The first two approaches produce second order, the third first order risk aversions

3 Merton's Model

- $\left(Z_t^P\right)$, $\left(Z_t^X\right)$ standard BM in R with covariance ρ

- Price, state and wealth dynamics

$$\begin{aligned} dB_t &= r_t B_t dt \quad , \quad dP_t = \alpha_t P_t dt + \sigma_t P_t dZ_t^P \\ dX_t &= \zeta_t dt + \xi_t dZ_t^X \\ dW_t &= \left[w_t W_t \left(\alpha_t - r_t\right) + \left(r_t W_t - C_t\right)\right] dt + w_t W_t \sigma_t dZ_t^P \end{aligned}$$

- Utility and implied objective function

$$u\left(C_t\right) = \frac{C_t^\gamma - 1}{\gamma} \quad , \quad V_\gamma\left(W, X\right) = E\left[\int_0^\infty e^{-\delta t} u\left(C_t\right) dt\right]$$

3.1 Vector Notation

- Covariance matrix Σ_t of $(dX_t, dW_t)'$ can be factorized as $\Sigma_t = \Lambda_t \Lambda_t'$, where

$$\Lambda_t = \begin{bmatrix} \xi_t & 0 \\ \rho w_t W_t \sigma_t & \left(1-\rho^2\right)^{\frac{1}{2}} w_t W_t \sigma_t \end{bmatrix}$$

- Orthogonalization $Z_t = \left(Z_t^X, Z_t^{X^\perp}\right)'$, $Z_t^P = \rho Z_t^X + \left(1-\rho^2\right)^{\frac{1}{2}} Z_t^{X^\perp}$, gives for the vector valued state variable $Y_t = (X_t, W_t)'$

$$dY_t = \mu_t dt + \Lambda_t dZ_t \quad ,$$

where $\mu_t = (\zeta_t, w_t W_t (\alpha_t - r_t) + (r_t W_t - C_t))'$

3.2 Optimization Problem

1. HJB equation $\left(c := \frac{C}{W}\right)$

$$
\begin{aligned}
0 &= \sup_{c,w} \{u(cW) - \delta J + \mathcal{A}_W J + \mathcal{A}_X J + wW\rho\sigma\xi J_{WX}\} \\
\mathcal{A}_W &= (r + w(\alpha - r) - C)\, W \frac{\partial}{\partial W} + \frac{1}{2} w^2 \sigma^2 W^2 \frac{\partial^2}{\partial W^2} \\
\mathcal{A}_X &= \zeta \frac{\partial}{\partial X} + \frac{1}{2}\xi^2 \frac{\partial^2}{\partial X^2}
\end{aligned}
$$

2. Homogeneous functional form for candidate solution:

$$
J(W, X) = \frac{1}{\delta} \frac{\left(e^{g(\gamma, X)} W\right)^{\gamma} - 1}{\gamma}
$$

3.3 Perturbative Approach

- First order expansion

$$
\begin{aligned}
J(W,X) &= \frac{1}{\delta}\frac{\left(e^{g(\gamma,X)}W\right)^{\gamma}-1}{\gamma} \underset{\gamma\to 0}{\longrightarrow} \frac{1}{\delta}(\ln(W)+g_0(X)) \\
g(\gamma,X) &= g_0(X)+\gamma g_1(X)+O(\gamma^2)
\end{aligned}
$$

- Approximate optimal Policies

$$
\begin{aligned}
c(X) &= \left(\frac{1}{\delta}e^{\gamma g(\gamma,X)}\right)^{\frac{1}{\gamma-1}} = \delta\left(1-\gamma(g_0(X)-\ln(\delta))\right)+O\left(\gamma^2\right) \\
w(X) &= \frac{1}{1-\gamma}\left(\frac{\alpha-r}{\sigma^2}+\gamma\frac{\partial g(\gamma,X)}{\partial X}\frac{\rho\xi}{\sigma}\right) \\
&= \frac{1}{1-\gamma}\left(\frac{\alpha-r}{\sigma^2}+\gamma\frac{\partial g_0(X)}{\partial X}\frac{\rho\xi}{\sigma}\right)+O\left(\gamma^2\right)
\end{aligned}
$$

3.4 Remarks

- g_1 can be neglected in first order analysis

- g_0 is sufficient to determine the optimal policies up to first order in γ

- g_0 is obtained from the solution of a log utility agent

$$g_0(X) = \ln(\delta) - 1 + E\left\{\int_0^\infty e^{-\delta t}\left[r_t + \frac{1}{2}\left(\frac{\alpha_t - r_t}{\sigma_t}\right)^2\right]dt\right\}$$

  and is typically easier to compute than the function g.

3.5 Summary on Kogan and Uppal (2000) Approach

1. Parameterize the problems under scrutiny and identify a specific parameter value for which the solution is known explicitly

2. Determine a functional form for the solution, such that to first order only the solvable benchmark model enters in the optimality conditions

3. Compute the optimal policies using the functional form of step 2

4. Expand the optimal policies to first order and determine the value function for the explicitly solvable model

4 Introducing Preferences for Robustness

- Step 1: Define the reference model for asset prices and state dynamics

- Step 2: Define the candidate model misspecifications

- Step 3: Define the relevant model misspecifications

- Step 4: Solve a max-min expected utility problem

4.1 Reference Model

- $\left(Z_t^P\right)$, $\left(Z_t^X\right)$ standard BM in R with covariance ρ

- Price, state and wealth dynamics

$$
\begin{aligned}
dB_t &= r_t B_t dt \quad , \\
dP_t &= \alpha_t P_t dt + \sigma_t P_t dZ_t^P \\
dX_t &= \zeta_t dt + \xi_t dZ_t^X \\
dW_t &= \left[w_t W_t \left(\alpha_t - r_t\right) + \left(r_t W_t - C_t\right)\right] dt + w_t W_t \sigma_t dZ_t^P
\end{aligned}
$$

- Orthogonalization $Z_t = \left(Z_t^X, Z_t^{X^\perp}\right)'$, $Z_t^P = \rho Z_t^X + \left(1 - \rho^2\right)^{\frac{1}{2}} Z_t^{X^\perp}$ gives

$$
dY_t = \mu_t dt + \Lambda_t dZ_t \quad ,
$$

4.2 Model Misspecifications

- Model contaminations $\nu = (\nu_t)_{t \geq 0}$ are modelled as absolutely continuous changes of measure:

$$\nu_t = \exp\left(- \int_0^t h_s \cdot dZ_s - \frac{1}{2} \int_0^t |h_s|^2 \, ds \right) \quad ,$$

where $Z_t = \left(Z_t^X, Z_t^{X^\perp} \right)'$ and for a suitable process $(h_s) = \left(h_s^X, h_s^P \right)'$

- By Girsanov Theorem, agents are thus concerned only with misspecifications in the drift of risky assets and state dynamics

4.3 Relative Entropy as a Measure of Model Discrepancy

- Relative entropy at time t

$$I_t(\nu) = E\left(\nu_t \ln(\nu_t)\right) \qquad ,$$

- Continuous-time relative entropy

$$\frac{d}{dt} I_t(\nu) = \frac{1}{2} h_t' h_t \qquad .$$

4.4 Max Min Expected Utility Problem

- Optimization problem of a robust agent

$$J\left(W,X\right)=\left\{\begin{array}{c}\sup_{C,w}\inf_{h}E^{h}\left[\int_{0}^{\infty}e^{-\delta t}\frac{C_{t}^{\gamma}-1}{\gamma}dt\right]\\ \frac{1}{2}h'h\leq\eta\end{array}\right. .$$

- A preference for robustness is modelled by a bound η on the rate at which continuous time relative entropy can increase over time

- Larger η's represent larger preferences for robustness

4.4.1 Worst Case Scenario

- Infimization w.r.t. h yields the implied worst case model (cf. also AHS (1998)

$$h = -\left(\frac{2\eta}{\Gamma(w)}\right)^{\frac{1}{2}} \begin{pmatrix} \xi J_X + \rho w W \sigma J_W \\ \left(1-\rho^2\right)^{\frac{1}{2}} w W \sigma J_W \end{pmatrix} \quad ,$$

where

$$\Gamma(w) = w^2 W^2 \sigma^2 J_W^2 + \xi^2 J_X^2 + 2 w W \rho \xi \sigma J_W J_X$$

4.4.2 Single-Agent Optimization Problem

- HJB equation ($c := \frac{C}{W}$)

$$0 = \sup_{c,w} \left\{ u\left(cW\right) - \delta J + A_W J + A_X J + wW \rho\sigma\xi J_{WX} - \left(2\eta\Gamma\left(w\right)\right)^{\frac{1}{2}} \right\}$$

- Homogeneous functional form

$$J(W, X) = \frac{1}{\delta} \frac{\left(e^{g(\gamma,\eta,X)} W\right)^{\gamma} - 1}{\gamma}$$

4.4.3 Robust Optimal Policies

- Optimal consumption and risky asset allocation

$$
\begin{aligned}
c &= \left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma-1}}, \\
w &= \frac{1}{1-\left(\frac{2\eta}{\Gamma}\right)^{\frac{1}{2}}\frac{J_W^2}{J_{WW}}}\cdot\frac{1}{(1-\gamma)}\left(\frac{\alpha-r}{\sigma^2}+\left(\gamma\frac{\partial g}{\partial X}-\left(\frac{2\eta}{\Gamma}\right)^{\frac{1}{2}}J_X\right)\frac{\rho\xi}{\sigma}\right) .
\end{aligned}
$$

- The functional form of c is the same as in the non robust case

- w is characterized by the solution of an implicit equation through the function $\Gamma(w)$

5 Perturbative Optimal Policies

- First order expansion in γ and $\eta^{\frac{1}{2}}$:

$$w(X) = \frac{\alpha - r}{\sigma^2} + \gamma w_1(X) + (2\eta)^{\frac{1}{2}} w_2(X) + O^2\left(\gamma, \eta^{\frac{1}{2}}\right) \quad ,$$

$$g(X) = g_0(X) + \gamma g_1(X) + (2\eta)^{\frac{1}{2}} g_2(X) + O^2\left(\gamma, \eta^{\frac{1}{2}}\right) \quad .$$

- Optimal policies, $G_0(X) = \left(\frac{\alpha - r}{\sigma}\right)^2 + \left(\xi \frac{\partial g_0}{\partial X}\right)^2 + 2\left(\frac{\alpha - r}{\sigma}\right) \rho \xi \frac{\partial g_0}{\partial X}$:

$$c(X) = \delta\left(1 - \gamma\left(g_0(X) - \ln(\delta)\right)\right) \quad ,$$

$$w(X) = \frac{1}{1 - \left(\gamma - \left(\frac{2\eta}{G_0}\right)^{\frac{1}{2}}\right)} \left(\frac{\alpha - r}{\sigma^2} + \left(\gamma - \left(\frac{2\eta}{G_0}\right)^{\frac{1}{2}}\right) \frac{\partial g_0}{\partial X} \frac{\rho \xi}{\sigma}\right)$$

5.1 Remark 1

- To $\gamma, \eta^{\frac{1}{2}}-$first order , robustness influences both the myopic and the hedging demand for risky assets

$$
\begin{aligned}
w(X) &= \underbrace{\frac{1}{1-\left(\gamma-\left(\frac{2\eta}{G_0(X)}\right)^{\frac{1}{2}}\right)}\left(\frac{\alpha-r}{\sigma^2}\right)}_{\text{MD}} \\
&\quad+\underbrace{\frac{\gamma-\left(\frac{2\eta}{G_0(X)}\right)^{\frac{1}{2}}}{1-\left(\gamma-\left(\frac{2\eta}{G_0(X)}\right)^{\frac{1}{2}}\right)}\frac{\partial g_0}{\partial X}\cdot\frac{\rho\xi}{\sigma}}_{\text{HD}}
\end{aligned}
$$

5.2 Remark 2

- The robust risky allocation is the portfolio strategy of an investor with a state dependent effective risk aversion $1 - \left(\gamma - \left(\frac{2\eta}{G_0(X)}\right)^{\frac{1}{2}}\right)$

- The state dependent effective risk aversion correction depends on the state X only through the "risk factors" $\phi = \frac{\alpha - r}{\sigma}$ and $\psi = \xi \frac{\partial g_0}{\partial X}$

- The largest relative portofolio corrections are realized when $\phi, \psi \to 0$, in a neighborhood of the origin in (ϕ, ψ)-space

- Robustness affects optimal portfolios precisely when the standard myopic and intertemporal demands for risky assets are small, that is when risk exposure is low.

6 Some Explicit Computations

- Version of Kim and Omberg's (1996) model allowing for intermediate consumption

$$\begin{aligned} dB_t &= rB_t dt \quad , \\ dP_t &= \alpha_t P_t dt + \sigma P_t dZ_t^P \quad , \\ dX_t &= \lambda \left(\overline{X} - X_t \right) dt + \xi dZ_t^X \quad , \end{aligned}$$

where r, σ, ξ, λ, $\overline{X} > 0$, and $\alpha_t = r + \sigma X_t$.

- In this model the market price of risk $\frac{\alpha_t - r}{\sigma}$ is an Ornstein Uhlembeck process

6.1 Computation of g_0

- Solution

$$g_0 = a_0 + a_1 X + \frac{1}{2} a_2 X^2 \quad ,$$

$$\begin{aligned} a_0 &= \ln(\delta) - 1 + \frac{r}{\delta} + \frac{\xi^2}{2\delta(\delta+2\lambda)} + \frac{\left(\lambda\overline{X}\right)^2}{\delta(\delta+\lambda)(\delta+2\lambda)} \quad , \\ a_1 &= \frac{\lambda\overline{X}}{(\delta+\lambda)(\delta+2\lambda)} > 0 \quad , \\ a_2 &= \frac{1}{\delta+2\lambda} > 0 \quad . \end{aligned}$$

- In this model the two relevant risk factors $\phi = \frac{\alpha - r}{\sigma}$, $\psi = \xi \frac{\partial g_0}{\partial X}$ are perfectly correlated

6.1.1 Sketch of the Proof

.

$$\begin{aligned} g_0(X) &= \ln(\delta) - 1 + E\left[\int_0^\infty e^{-\delta t}\left(r + \frac{1}{2}\left(\frac{\alpha_t - r}{\sigma}\right)^2\right) dt \middle| X_0 = X\right] \\ &= \ln(\delta) - 1 + \left[\int_0^\infty e^{-\delta t}\left(r + \frac{1}{2}E\left(X_t^2 \middle| X_0 = X\right)\right) dt\right] \quad . \end{aligned}$$

Since

$$\begin{aligned} E\left(X_t^2 \middle| X_0 = X\right) &= Var\left(X_t | X_0 = X\right) + \left(E\left(X_t | X_0 = X\right)\right)^2 \\ &= \xi^2 \cdot \frac{1 - e^{-2\lambda t}}{2\lambda} + \left[e^{-\lambda t}\left(X + \overline{X}\left(e^{\lambda t} - 1\right)\right)\right]^2 \quad , \end{aligned}$$

a final integration gives the result.

6.2 Optimal Policies

Proposition 6.2 In the given model the following first order optimal policies hold true for a robust agent

$$
\begin{aligned}
c(X) &= \delta\left(1-\gamma\left(a_0+a_1X+\frac{1}{2}a_2X^2-\ln(\delta)\right)\right) \\
w(X) &= \frac{1}{1-\left(\gamma-\left(\frac{2\eta}{G_0}\right)^{\frac{1}{2}}\right)}\left(\frac{X}{\sigma}+\left(\gamma-\left(\frac{2\eta}{G_0}\right)^{\frac{1}{2}}\right)\frac{\rho\xi\left(a_1+a_2X\right)}{\sigma}\right)
\end{aligned}
$$

where

$$
G_0=\left(\xi^2a_2^2+2\rho\xi a_2+1\right)X^2+2a_1\xi\left(\rho+\xi a_2\right)X+\xi^2a_1^2
$$

6.3 Conclusions, Further Research

- PT provides analytical solutions for consumption/portfolio problems where otherwise only numerical results are available

- Robustness affects basically only the investment side, by reducing both the myopic and the intertemporal exposure to risky assets

- CR induces significant portfolio modifications already for low risk exposures

- Work in progress: Relation between MR and ER in the intertemporal setting, impact of the investment horizon (see also Barberis (1999))