Fitting volatility skews and smiles with analytical stock-price models *

Damiano Brigo Fabio Mercurio

Product and Business Development Group
Banca IMI, SanPaolo IMI Group
Corso Matteotti, 6
20121 Milano, Italy
Fax: +39 02 7601 9324
E-mail: brigo@bancaimi.it fmercurio@bancaimi.it

Abstract

We propose two different classes of analytical models for the dynamics of an asset price that respectively lead to skews and smiles in the term structure of implied volatilities. Both classes are based on an explicit SDE which admits a unique strong solution whose marginal density is also provided. We then consider some particular examples in each class and explicitly calculate European option prices implied by these models. We also hint at the implementation of efficient procedures for pricing exotic derivatives.

Keywords

Stock-Price Dynamics, Risk–Neutral Density, Analytical Models, Explicit Option Pricing, Mixture of lognormals, Volatility Skew, Volatility Smile, Fokker–Planck equation.

1 Introduction

It is widely known that the Black and Scholes (1973) model can not consistently price all European options that are quoted in one specific market. The assumption of a constant

*We are grateful to Aleardo Adotti, head of the Product and Business Development Group at Banca IMI for encouraging us in the prosecution of the most speculative side of research in mathematical finance. We are also grateful to Chris Rogers and Giulio Sartorelli for their suggestions and remarks.

volatility that should be used to price any derivative security with the same underlying asset fails to hold true in practice. Indeed, it is commonly observed in real markets that the term structure of implied volatilities features some particular shapes that are termed “skews” and “smiles”. The term skew is used to indicate those structures where, for a fixed maturity, low-strikes implied-volatilities are higher than high-strikes implied-volatilities. The term smile is used instead to denote those structures where, again for a fixed maturity, the volatility has a minimum value around the underlying forward price.

If the implied volatilities were different along the time-to-maturity dimension only, while being equal for different strikes at any fixed maturity, a simple extension of the Black-Scholes model would exactly retrieve the market option prices. It would be in fact sufficient to introduce a time-dependent (deterministic) volatility σ_t in the Black-Scholes dynamics for the stock price

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t,$$

and, given the N maturities $T_1, \dots, T_N$, to recursively solve

$$\int_0^{T_i} \sigma_t^2 dt = v_i^2 T_i,$$

where v_i is the implied volatility for the maturity T_i.[1] Unfortunately, real financial markets display more complex volatility structures, so that the extended Black-Scholes model does not lead to a satisfactory fitting of market data. This issue can then be tackled by introducing a more articulated form of the volatility coefficient in the stock-price dynamics. This is the approach we follow in this paper. We in fact propose two different classes of stock-price models by specifying the stock price dynamics under the risk-neutral measure. The volatility σ_t we introduce is in both cases a function of time t and the stock price S_t at the same time. By doing so, we are able to construct two different classes of models that lead respectively to a skew and a smile in the term structure of implied volatilities.

Many researchers have tried to address the problem of a good, possibly exact, fitting of market option data. We now briefly review the major approaches that have been proposed.

A first approach is based on assuming an *alternative explicit dynamics* for the stock-price process that immediately leads to volatility smiles or skews. In general this approach does not provide sufficient flexibility to properly calibrate the whole volatility surface. An example is the general CEV process being analysed by Cox (1975) and Cox and Ross (1976). A general class of processes is due to Carr et al. (1999). The first class of models we propose also fall into this *”alternative explicit dynamics”* cathegory, and while it adds flexibility with respect to the previous known examples, it does not completely solve the flexibility issue.

A second approach is based on the assumption of a *continuum of traded strikes* and goes back to Breeden and Litzenberger (1978). Successive developments are due to Dupire

[1] Of course, in principle we might have imaginary values for σ_t. However, if the term structure of implied volatilities is sufficiently smooth, this problem is not encountered.

(1994, 1997), Derman and Kani (1994, 1998) and Dempster and Richard (1999) who derive an explicit expression for the Black-Scholes volatility as a function of strike and maturity. This approach has the major drawback that one needs to smoothly interpolate option prices between consecutive strikes in order to be able to differentiate them twice with respect to the strike. Explicit expessions for the risk-neutral stock price dynamics are also derived by Avellaneda et al. (1997) by minimizing the relative entropy to a prior distribution, and by Brown and Randall (1999) by assuming a quite flexible analytical function describing the volatility surface.

Another approach, pioneered by Rubinstein (1994), consists in finding the risk-neutral probabilities in a binomial/trinomial model for the stock price that lead to a best fitting of market option prices due to some smoothness criterion. We refer to this approach as to the *lattice* approach. Further examples are in Jackwerth and Rubinstein (1996) and Andersen and Brotherton-Ratcliffe (1997) who use instead finite-difference grids. A different lattice approach is due to Britten-Jones and Neuberger (1999).

In general the problem of finding a risk-neutral distribution which consistently prices all quoted options is largely undetermined. Indeed, a continuum of values can be basically chosen in order to fit a finite number of market prices. A possible solution is then given by assuming a particular *parametric risk-neutral distribution* depending on several, possibly time-dependent, parameters and then use such parameters for the volatility calibration. An example of this approach is the work by Shimko (1993). But the question remains of finding an asset price dynamics consistent with the chosen parametric form of the risk–neutral density. The second class we propose addresses this question by finding a dynamics leading to a parametric risk–neutral distribution which is flexible enough for all practical purposes. The resulting process combines therefore the *parametric risk-neutral distribution* approach with the *alternative dynamics approach*, providing explicit dynamics leading to flexible parametric risk–neutral densities.

The major challenge that our second class of models is fit to face is the introduction of a risk-neutral distribution that leads i) to *analytical formulas* for European options, so that the calibration to market data and the computation of Greeks can be extremely rapid, ii) to *explicit asset-price dynamics*, so that exotic claims can be priced through a Monte Carlo simulation and iii) to *recombining lattices*, so that instruments with early-exercise features can be valued via backward calculation in the tree.[2]

The paper is structured as follows. Section 2 proposes a class of analytical asset-price models leading to volatility skews. The cases of a CEV process and a geometric Brownian motion are treated as particular examples. Section 3 proposes a different class of analytical asset-price models leading instead to volatility smiles. The example of a mixture of lognormal distribution is then considered. Section 4 concludes the paper. Some technical results are written in appendices.

[2]Alternative methods of extracting a risk-neutral distribution from option prices are in Malz (1997) and Pirkner et al. (1999). An alternative model with explicit formulas for European option has been proposed by Li (1998).

2 A class of analytical models allowing for volatility skews

Let us denote by S the asset-price process and assume that interest rates are constant through time and equal to $r > 0$ for all maturities. We assume that the asset price under the risk-neutral measure has the following dynamics

$$dS_t = \mu S_t dt + \nu(t, S_t) dW_t, \tag{1}$$

with S_0 given and deterministic and where W is a standard Brownian motion, ν is a smooth function of t and S_t and μ is the risk-neutral drift rate associated to the process S.[3]

We want to determine the diffusion coefficient ν in such a way that S is given by an affine transformation of a diffusion process X whose marginal density is known, so that the marginal density of S is obtained by shifting the known density of this diffusion process. In formulas, we want to find the deterministic functions of time a and b such that we can write

$$S_t = a_t + b_t X_t,$$

where the process X satisfies

$$dX_t = \varphi(t, X_t) dt + \psi(t, X_t) dW_t,$$

with φ and ψ regular functions such that the marginal density of X is known. By Ito's lemma,

$$dS_t = [a'_t + b'_t X_t + b_t \varphi(t, X_t)] dt + b_t \psi(t, X_t) dW_t, \tag{2}$$

with $'$ denoting the time derivative. The risk-neutral drift condition

$$a'_t + b'_t X_t + b_t \varphi(t, X_t) = \mu S_t \tag{3}$$

then defines which functions a and b are feasible to our purpose. Of course, such condition becomes more explicit once a particular process X is selected. To this end, we will consider the case where X is a general CEV process. As we shall see in the sequel, this choice is motivated by the model analytical tractability.

Let us assume that X follows the general CEV process[4]

$$dX_t = \gamma_t X_t dt + \eta_t X_t^{\rho} dW_t, \tag{4}$$

---

[3]For example, if the asset is a asset paying a continuous dividend yield q, then $\mu = r - q$. If the asset is instead an exchange rate, then $\mu = r - r_f$, where r_f is the (assumed constant) risk-free rate for the foreign currency.

[4]See also Cox (1975) and Cox and Ross (1976).

where γ_t and η_t are deterministic functions of time and ρ is any real constant in $[0, 1]$.

Under the model (4), the dynamics (2) are

$$dS_t = \left[a'_t - a_t\left(\frac{b'_t}{b_t} + \gamma_t\right) + \left(\frac{b'_t}{b_t} + \gamma_t\right) S_t\right] dt + b_t\eta_t \left(\frac{S_t - a_t}{b_t}\right)^\rho dW_t, \tag{5}$$

so that condition (3) becomes equivalent to the following system

$$\begin{cases} a'_t - a_t\left(\dfrac{b'_t}{b_t} + \gamma_t\right) = 0 \\ \dfrac{b'_t}{b_t} + \gamma_t = \mu \end{cases}$$

whose solution is

$$\begin{cases} a_t = a_0 e^{\mu t} \\ b_t = b_0 e^{\int_0^t (\mu - \gamma_u) du}, \end{cases}$$

for any real constants a_0 and b_0. From (5), it is then easy to see that, with no loss of generality, we can set $b_t = 1$ for each t, so that the required asset-price dynamics is

$$dS_t = \mu S_t dt + \eta_t (S_t - a_t)^\rho dW_t. \tag{6}$$

Indeed, the effect of a more general b can be absorbed into η.

Remark 2.1. *We started from a basic model consisting of the general time–varying–coefficients formulation of the CEV model, which will be particularly useful in the time–homogeneous case due the analytical tractability of the CEV model. However, the procedure can be used to shift any other distribution coming from a pre–selected dynamics. We could for example shift the hyperbolic diffusion model of Bibby and Sørensen (1997), or the dynamics suggested by Platen (1999), although the benefits of such a procedure are less clear when analytical tractability of the basic model is missing.*

2.1 The shifted CEV process with deterministic coefficients

We now assume that η_t is constant and equal to η for each t. In fact, restricting our attention to the case with constant coefficients allows for the derivation of explicit marginal densities and analytical formulas. We also assume that $\rho \neq 1$, meaning that we are excluding the lognormal case. Notice that it might seem necessary to assume $\mu \neq 0$ in order to ensure existence of the coefficients below. However, if $\mu = 0$ the below formulas (and in particular k) still hold by substituting the relevant quantities with their limits for $\mu \to 0$. We then have the following.

Proposition 2.2. *If we set $\alpha := a_0$, the SDE (6) admits a unique strong solution that is given by*

$$S_t = P_t + \alpha e^{\mu t}, \quad t \geq 0, \tag{7}$$

where P is the CEV process

$$dP_t = \mu P_t dt + \eta P_t^\rho dW_t. \tag{8}$$

Moreover, the continuous part of the density function of S_T conditional on S_t, $t < T$, is

$$p_{S_T|S_t}(x) = 2(1-\rho)k^{1/(2-2\rho)}(uw^{1-4\rho})^{1/(4-4\rho)}e^{-u-w}I_{1/(2-2\rho)}(2\sqrt{uw}), \tag{9}$$

where

$$\begin{aligned} k &= \frac{\mu}{\eta^2(1-\rho)[e^{2\mu(1-\rho)(T-t)}-1]} \\ u &= k(S_t - \alpha e^{\mu t})^{2(1-\rho)}e^{2\mu(1-\rho)(T-t)} \\ w &= k(x-\alpha e^{\mu T})^{2(1-\rho)} \end{aligned} \tag{10}$$

and I_q denotes the modified Bessel function of the first kind of order q. Denoting by $g(y,z) = \frac{e^{-z}z^{y-1}}{\Gamma(y)}$ the gamma density function and by $G(y,x) = \int_x^{+\infty} g(y,z)dz$ the complementary gamma distribution, the probability that $S_T = \alpha e^{\mu T}$ conditional on S_t is $G\Big(\frac{1}{2(1-\rho)}, u\Big)$.

Proof. We just have to notice that $b_t = 1$ for each t implies that $\gamma_t = \mu$ for each t, so that (8) holds with $X = P$ and hence (7) follows. The density (9) is then simply obtained by shifting the density of the CEV process (8), which for instance can be found in Schroder (1989), by the relevant quantity at each time and by writing $P_t = S_t - \alpha e^{\mu t}$. $\square$

It is well known that by modeling the asset-price dynamics with the process (8) we can derive explicit formulas for European options on the asset. The same analytical feature applies to the process (7) as well. Indeed, let us consider a European option with maturity T, strike K and written on the asset. The arbitrage-free option price under the model (7) is given in the following.

Proposition 2.3. *Under the assumption that $\alpha e^{\mu T} < K$, the European call option value at any time $t < T$ is*

$$\begin{aligned} \mathcal{C}_t = {} & (S_t - \alpha e^{\mu t})e^{(\mu-r)(T-t)} \sum_{n=0}^{+\infty} g(n+1,u) G\left(n+1+\frac{1}{2(1-\rho)}, k(K-\alpha e^{\mu T})^{2(1-\rho)}\right) \\ & - (K-\alpha e^{\mu T})e^{-r(T-t)} \sum_{n=0}^{+\infty} g\left(n+1+\frac{1}{2(1-\rho)}, u\right) G\big(n+1, k(K-\alpha e^{\mu T})^{2(1-\rho)}\big). \end{aligned} \tag{11}$$

where k and u are defined as in (10).

Proof. Denoting by E_t the time t-conditional expectation under the risk-neutral measure, we have

$$\mathcal{C}_t = e^{-r(T-t)}E_t\left\{(S_T-K)^+\right\} = e^{-r(T-t)}E_t\left\{[P_T - (K-\alpha e^{\mu T})]^+\right\}.$$

Since P is a CEV process with the proper risk-neutral drift, the last discounted expectation is simply given by the CEV option price formula[5] with the strike being equal to $K - \alpha e^{\mu T}$. Formula (11) is then obtained by remembering that $P_t = S_t - \alpha e^{\mu t}$. $\square$

Remark 2.4. *The assumption $\alpha e^{\mu T} < K$ is rather natural since, otherwise, the payoff would lose its optionality. Indeed, if $\alpha e^{\mu T} \geq K$ we obtain at time* 0

$$E_0[P_T - (K - \alpha e^{\mu T})]^+ = E_0[P_T - (K - \alpha e^{\mu T})] = e^{\mu T} S_0 - K \ .$$

In such a case the shift is so large as to render the payoff linear again, which is of course not desirable.

The option price (11) leads to skews in the implied volatility structure. This is intuitive since already the basic CEV process, corresponding to $\alpha = 0$, shows such property. However, we now have three parameters, α, η and ρ, that can be employed for a better fitting of the market volatility structure. An example of the volatility structure that can be implied by the model (7) is shown in Figure 1. Besides pricing European options analytically,

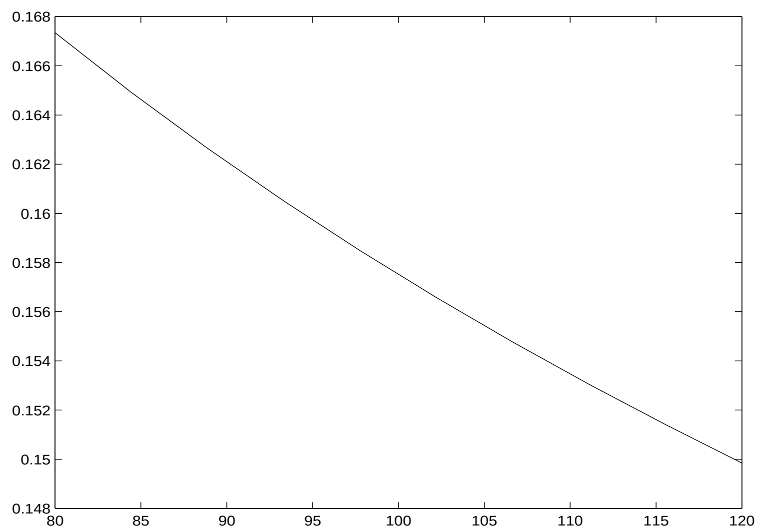

Figure 1: Volatility structure implied by the option prices (11) at time $t = 0$, where we set $\mu = r = 0.035$, $T = 1$, $\alpha = -10$, $\eta = 1.5$, $\rho = 0.5$ and $S_0 = 100$.

the model (7) leads to fast numerical procedures for pricing exotic derivatives. In fact, the known transition density (9) renders the implementation of Monte Carlo procedures easier and more efficient when pricing a large variety of path-dependent claims. Moreover, binomial or trinomial trees for S can be constructed by first building the tree for P and then by properly displacing the corresponding nodes.

[5] As seen, for instance, in Beckers (1980).

2.2 The shifted-lognormal case

The second case we consider is that of a basic process X following the geometric Brownian motion[6]

$$dX_t = \mu X_t dt + \beta_t X_t dW_t, \tag{12}$$

under the risk-neutral measure, so that the asset price $S_t = X_t + \alpha e^{\mu t}$ evolves under such measure according to

$$dS_t = \mu S_t dt + \beta_t (S_t - \alpha e^{\mu t}) dW_t. \tag{13}$$

We then have the following.

Proposition 2.5. *The asset price S can be explicitly written as*

$$S_t = \alpha e^{\mu t} + (S_0 - \alpha) e^{\int_0^t \left(\mu - \frac{1}{2}\beta_u^2\right) du + \int_0^t \beta_u dW_u}. \tag{14}$$

Moreover, the distribution of the asset price S_T, conditional on S_t, $t < T$, is a shifted lognormal distribution with density

$$p_{S_T|S_t}(x) = \frac{1}{(x - \alpha e^{\mu T}) V_t \sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{\ln(x - \alpha e^{\mu T}) - M_t}{V_t} \right)^2 \right\}, \quad x > \alpha e^{\mu T}, \tag{15}$$

where

$$\begin{aligned} M_t &= \ln(S_t - \alpha e^{\mu t}) + \int_t^T \left(\mu - \frac{1}{2}\beta_u^2 \right) du \\ V_t^2 &= \int_t^T \beta_u^2 du. \end{aligned} \tag{16}$$

Proof. The explicit form (14) is obtained by integration of the SDE (12) and by remembering the previous general results. The density (15) is then obtained by suitably shifting the lognormal density of X and by writing $X_t = S_t - \alpha e^{\mu t}$. □

As before, the knowledge of the risk-neutral distribution of the process S allows us to explicitly derive formulas for European options. This is accomplished in the following.

Proposition 2.6. *Consider a European option with maturity T, strike K and written on the asset. Then, under the assumption that $\alpha e^{\mu T} < K$, the option value at any time $t < T$ is given by*

$$\begin{aligned} O_t = \omega \Bigg[&(S_t - \alpha e^{\mu t}) e^{(\mu - r)(T-t)} \Phi\left(\omega \frac{\ln \frac{S_t - \alpha e^{\mu t}}{K - \alpha e^{\mu T}} + \int_t^T \left(\mu + \frac{1}{2}\beta_u^2\right) du}{\sqrt{\int_t^T \beta_u^2 du}} \right) \\ &- (K - \alpha e^{\mu T}) e^{-r(T-t)} \Phi\left(\omega \frac{\ln \frac{S_t - \alpha e^{\mu t}}{K - \alpha e^{\mu T}} + \int_t^T \left(\mu - \frac{1}{2}\beta_u^2\right) du}{\sqrt{\int_t^T \beta_u^2 du}} \right) \Bigg], \end{aligned} \tag{17}$$

[6] A displaced diffusion model has been first considered by Rubinstein (1983).

where $\omega = 1$ for a call and $\omega = -1$ for a put and with Φ denoting the standard normal cumulative distribution function.

Proof. As before, let us denote by E_t the time t-conditional expectation under the risk-neutral measure. Noting that $\ln(X_T)$ conditional on X_t is normally distributed with mean M_t and variance V_t^2, we have

$$\begin{aligned}
O_t &= e^{-r(T-t)}E_t\left\{[\omega(S_T - K)]^+\right\} = e^{-rT}E_t\left\{[\omega X_T - \omega(K - \alpha e^{\mu T})]^+\right\} \\
&= e^{-r(T-t)}\int_{-\infty}^{+\infty}[\omega e^x - \omega(K - \alpha e^{\mu T})]^+ \frac{1}{V_t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-M_t}{V_t}\right)^2}dx \\
&= e^{-r(T-t)}\omega\left[e^{M_t+\frac{1}{2}V_t^2}\Phi\left(\omega\frac{M_t - \ln(K - \alpha e^{\mu T}) + V_t^2}{V_t}\right)\right. \\
&\qquad \left. -(K - \alpha e^{\mu T})\Phi\left(\omega\frac{M_t - \ln(K - \alpha e^{\mu T})}{V_t}\right)\right],
\end{aligned}$$

which immediately leads to (17). □

Notice that the option price (17) is nothing but the Black-Scholes price with the asset spot price and the strike being respectively replaced with $S_t - \alpha e^{\mu t}$ and $K - \alpha e^{\mu T}$ and where the implied volatility is $\sqrt{\frac{1}{T-t}\int_t^T \beta_u^2 du}$.

Model (13) is a simple analytical extension of the Black-Scholes model allowing for skews in the option implied volatility. An example of the skewed volatility structure that is implied by (13) is provided in Figure 2. Appendix A shows that for strikes close to the forward asset price, the implied volatility structure is indeed monotone in the strike. Similarly to the model (7), therefore, model (13) can be used to fit market volatility structures that are skewed. In this case, however, we have one parameter less than in model (7), since ρ is set to one.

As to pricing of exotic derivatives, the decomposition (7) with $P = X$ given by (12) allows for a fast Monte Carlo generation of asset-price scenarios, through the generation of paths of the geometric Brownian motion (12). Binomial or trinomial trees for S can be constructed by starting from the basic ones for X and displacing the corresponding nodes by the relevant (deterministic) quantity at each time step.

3 A class of analytical models allowing for volatility smiles

In this section we propose a new class of analytical models for the asset-price dynamics that are capable to fit a more general volatility structures. The diffusion processes we obtain follow from assuming a particular risk-neutral distribution for the asset price S. Precisely, we assume that the marginal density of S under the risk-neutral measure is the weighted average of known densities of some given diffusion processes.

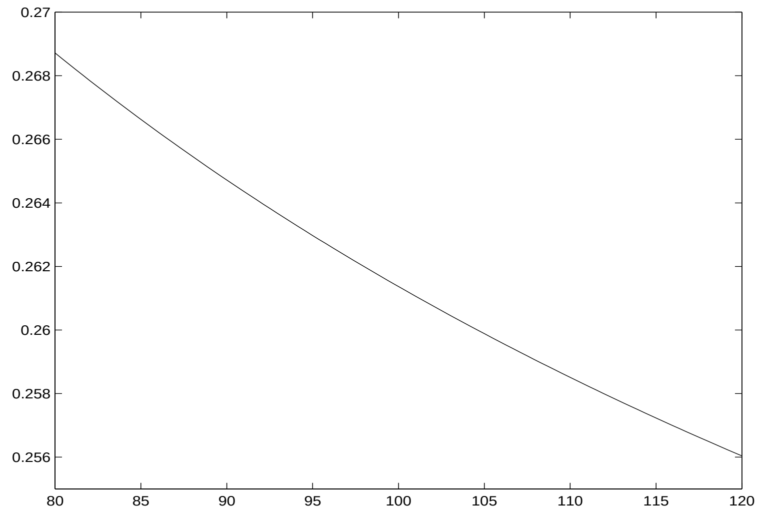

Figure 2: Volatility structure implied by the option prices (17) at time $t = 0$, where we set $\mu = r = 0.035$, $T = 1$, $\alpha = -30$, $\beta_t = 0.2$ for each t and $S_0 = 100$.

Let us then consider N diffusion processes with risk-neutral dynamics given by

$$dS_t^i = \mu S_t^i dt + v_i(t, S_t^i) dW_t, \quad i = 1, \ldots, N, \tag{18}$$

with initial value S_0^i, where W is a standard Brownian motion under the risk-neutral measure Q and $v_i(t, y)$'s are real functions satisfying regularity conditions to ensure existence and uniqueness of the solution to the SDE (18). In particular we assume that, for a suitable $L_i > 0$, the following linear-growth condition holds:

$$v_i^2(t, y) \leq L_i(1 + y^2) \;\; \text{uniformly in } t. \tag{19}$$

For each t, we denote by $p_t^i(\cdot)$ the density function of S_t^i, i.e., $p_t^i(y) = d(Q\{S_t^i \leq y\})/dy$, where, in particular, $p_0^i(y)$ is the δ-Dirac function centered in S_0^i.

Let us now assume that the dynamics of the asset price S under the risk-neutral measure Q is given by

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t, \tag{20}$$

where $\sigma(\cdot, \cdot)$ satisfies, for a suitable positive constant L, the linear-growth condition

$$\sigma^2(t, y) y^2 \leq L(1 + y^2) \;\; \text{uniformly in } t\ . \tag{21}$$

The growth condition ensures existence of a strong solution. Moreover, we assume σ satisfies some further condition assuring uniqueness of the solution.[7]

[7]The classical example is the local Lipschitz condition. In this section, however, we need not to write it explicitly.

The problem we want to address is the derivation of the "local volatility" $\sigma(t, S_t)$ such that the risk-neutral density of S satisfies

$$p_t(y) := \frac{d}{dy} Q\{S_t \le y\} = \sum_{i=1}^N \lambda_i \frac{d}{dy} Q\{S_t^i \le y\} = \sum_{i=1}^N \lambda_i p_t^i(y), \tag{22}$$

where each S_0^i is set to S_0, and λ_i's are strictly positive constants such that $\sum_{i=1}^N \lambda_i = 1$. Indeed, $p_t(\cdot)$ is a proper risk-neutral density function since, by definition,

$$\int_0^{+\infty} y p_t(y) dy = \sum_{i=1}^N \lambda_i \int_0^{+\infty} y p_t^i(y) dy = \sum_{i=1}^N \lambda_i S_0 e^{\mu t} = S_0 e^{\mu t}.$$

Remark 3.1. *Notice that in the last calculation we were able to recover the proper risk–neutral expectation thanks to our assumption that all processes (18) share the same drift parameter μ. However, the role of the processes S^i is merely instrumental, and there is no need to assume their drift to be of the form μS_t^i if not for simplifying calculations. In particular, what matters in obtaining the right expectation as in the last formula above is the marginal distribution p_i of the S^i's. We could generate alternative dynamics leading to the same marginal densities as in the S_i's by selecting arbitrary diffusion coefficients and then by defining appropriate drifts according to the results in Brigo and Mercurio (1999a, 1999b), although computations would generally become involved.*

As already noticed by many authors,[8] the above problem is essentially the reverse to that of finding the marginal density function of the solution of an SDE when the coefficients are known. In particular, $\sigma(t, S_t)$ can be found by solving the Fokker-Planck equation

$$\frac{\partial}{\partial t} p_t(y) = -\frac{\partial}{\partial y}\left(\mu y p_t(y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,y) y^2 p_t(y)\right), \tag{23}$$

given that each density $p_t^i(y)$ satisfies itself the Fokker-Planck equation

$$\frac{\partial}{\partial t} p_t^i(y) = -\frac{\partial}{\partial y}\left(\mu y p_t^i(y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(v_i^2(t,y) p_t^i(y)\right). \tag{24}$$

Applying the definition (22) and the linearity of the derivative operator, (23) can be written as

$$\sum_{i=1}^N \lambda_i \frac{\partial}{\partial t} p_t^i(y) = \sum_{i=1}^N \lambda_i \left[-\frac{\partial}{\partial y}\left(\mu y p_t^i(y)\right)\right] + \sum_{i=1}^N \lambda_i \left[\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,y) y^2 p_t^i(y)\right)\right],$$

that by substituting from (24) becomes

$$\sum_{i=1}^N \lambda_i \left[\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(v_i^2(t,y) p_t^i(y)\right)\right] = \sum_{i=1}^N \lambda_i \left[\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,y) y^2 p_t^i(y)\right)\right].$$

---

[8]See for instance Dupire (1997).

Using again the linearity of the second order derivative operator,

$$\frac{\partial^2}{\partial y^2}\left[\sum_{i=1}^{N}\lambda_i v_i^2(t,y)p_t^i(y)\right] = \frac{\partial^2}{\partial y^2}\left[\sigma^2(t,y)y^2\sum_{i=1}^{N}\lambda_i p_t^i(y)\right] .$$

If we look at this last equation as to a second order differential equation for $\sigma(t,\cdot)$, we find easily its general solution

$$\sigma^2(t,y)y^2\sum_{i=1}^{N}\lambda_i p_t^i(y) = \sum_{i=1}^{N}\lambda_i v_i^2(t,y)p_t^i(y) + A_t y + B_t, \tag{25}$$

with A and B suitable real functions of time. The regularity conditions (19) and (21) imply that the LHS of the equation has zero limit for $y \to \infty$.[9] As a consequence, the RHS must have a zero limit as well. This holds if and only if $A_t = B_t = 0$, for each t. We therefore obtain that the expression for $\sigma(t,y)$ that is consistent with the marginal density (22) and with the regularity constraint (21) is, for $(t,y) > (0,0)$,

$$\sigma(t,y) = \sqrt{\frac{\sum_{i=1}^{N}\lambda_i v_i^2(t,y)p_t^i(y)}{\sum_{i=1}^{N}\lambda_i y^2 p_t^i(y)}}. \tag{26}$$

Indeed, notice that by setting

$$\Lambda_i(t,y) := \frac{\lambda_i p_t^i(y)}{\sum_{i=1}^{N}\lambda_i p_t^i(y)} \tag{27}$$

for each $i = 1,\ldots,N$ and $(t,y) > (0,0)$, we can write

$$\sigma^2(t,y) = \sum_{i=1}^{N}\Lambda_i(t,y)\frac{v_i^2(t,y)}{y^2}, \tag{28}$$

so that the square of the volatility σ can be written as a (stochastic) convex combination of the squared volatilities of the basic processes (18). In fact, for each (t,y), $\Lambda_i(t,y) \geq 0$ for each i and $\sum_{i=1}^{N}\Lambda_i(t,y) = 1$. Moreover, by (19) and setting $L := \max_{i=1,\ldots,N} L_i$, the condition (21) is fulfilled since

$$\sigma^2(t,y)y^2 = \sum_{i=1}^{N}\Lambda_i(t,y)v_i^2(t,y) \leq \sum_{i=1}^{N}\Lambda_i(t,y)L_i(1+y^2) \leq L(1+y^2).$$

The function σ may be then extended to the semi-axes $\{(t,0) : t > 0\}$ and $\{(0,y) : y > 0\}$ according to the specific choice of the basic densities $p_t^i(\cdot)$.

---

[9] Notice in fact that the linear-growth condition (19) implies that each $p_t^i(\cdot)$ has a second moment, hence that $\lim_{y\to\infty} y^2 p_t^i(y) = 0$, for each i and t.

Formula (26) leads to the following SDE for the asset price under the risk–neutral measure Q:

$$dS_t = \mu S_t dt + \sqrt{\frac{\sum_{i=1}^N \lambda_i v_i^2(t, S_t) p_t^i(S_t)}{\sum_{i=1}^N \lambda_i S_t^2 p_t^i(S_t)}} S_t dW_t. \tag{29}$$

This SDE, however, must be regarded as defining some candidate dynamics that leads to the marginal density (22). Indeed, if σ is bounded, then

$$E\left\{\int_0^t \sigma^2(u, S_u)du\right\} < \infty,$$

so that the SDE

$$d\ln(S_t) = \left[\mu - \frac{1}{2}\sigma^2(t, S_t)\right] dt + \sigma(t, S_t)dW_t$$

is well defined, and so is (29). But the conditions we have imposed so far are not sufficient to grant the uniqueness of the strong solution, so that a verification must be done on a case-by-case basis.

Let us now give for granted that the SDE (29) has a unique strong solution. Then, remembering the definition (22), it is straightforward to derive the model option prices in terms of the option prices associated to the basic models (18). Indeed, let us consider a European option with maturity T, strike K and written on the asset. Then, if $\omega = 1$ for a call and $\omega = -1$ for a put, the option value $\mathcal{O}$ at the initial time $t = 0$ is given by

$$\begin{aligned}
\mathcal{O} &= e^{-rT} E^Q \left\{[\omega(S_T - K)]^+\right\} \\
&= e^{-rT} \int_0^{+\infty} [\omega(y - K)]^+ \sum_{i=1}^N \lambda_i p_T^i(y) dy \\
&= \sum_{i=1}^N \lambda_i e^{-rT} \int_0^{+\infty} [\omega(y - K)]^+ p_T^i(y) dy \\
&= \sum_{i=1}^N \lambda_i \mathcal{O}_i,
\end{aligned} \tag{30}$$

where E^Q denotes expectation under Q and $\mathcal{O}_i$ denotes the option price associated to (18).

Remark 3.2. *The above derivation shows that a dynamics leading to a marginal density for the asset price that is the convex combination of basic densities induces the same convex combination among the corresponding option prices. Furthermore, due to the linearity of the derivative operator, the same convex combination applies to all option Greeks. In particular this ensures that starting from analytically tractable basic densities one finds a model which preserves the analytical tractability.*

3.1 The mixture-of-lognormals case

Let us now consider the particular case where the densities $p_t^i(\cdot)$'s are all lognormal. Precisely, we assume that, for each i,

$$v_i(t, y) = \sigma_i(t)y, \tag{31}$$

where all σ_i's are deterministic functions of time that are bounded from above and below by positive constants. Notice that if moreover σ_i's are continuous and we take a finite time–horizon, then boundedness from above is automatic, and the only condition to be required explicitly is boundedness from below by a positive constant. Then, the marginal density of S^i conditional on S_0 is given by

$$\begin{aligned} p_t^i(y) &= \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - \mu t + \tfrac{1}{2}V_i^2(t)\right]^2\right\}, \\ V_i(t) &:= \sqrt{\int_0^t \sigma_i^2(u)du}. \end{aligned} \tag{32}$$

The case where the risk-neutral density is a mixture of lognormal densities has been originally studied by Ritchey (1990)[10] and subsequently used by Melick and Thomas (1997), Bhupinder (1998) and Guo (1998). However, their works are mainly empirical: They simply assumed such risk-neutral density and then studied the resulting fitting to option data.

In this paper, instead, we develop the model from a theoretical point of view and derive the specific asset-price dynamics that implies the chosen distribution. As a consequence, not only do we propose an asset-price model that is capable to fit real option data, but also we can construct efficient procedures for the pricing of exotic derivatives that are path-dependent or have early-exercise features.

Proposition 3.3. *Let us assume that each σ_i is also continuous and that there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \dots, N$. Then, if we set*

$$\sigma(t, y) = \sqrt{\frac{\sum_{i=1}^N \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - \mu t + \frac{1}{2}V_i^2(t)\right]^2\right\}}{\sum_{i=1}^N \lambda_i \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - \mu t + \frac{1}{2}V_i^2(t)\right]^2\right\}}}, \tag{33}$$

for $(t, y) > (0, 0)$ and $\sigma(t, y) = \sigma_0$ for $(t, y) = (0, S_0)$, the SDE (20) has a unique strong solution whose marginal density is given by the mixture of lognormals

$$p_t(y) = \sum_{i=1}^N \lambda_i \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - \mu t + \tfrac{1}{2}V_i^2(t)\right]^2\right\}. \tag{34}$$

[10] Indeed, Ritchey (1990) assumed a mixture of normal densities for the density of the asset log-returns. However, it can be easily shown that this is equivalent to assuming a mixture of lognormal densities for the density of the asset price.

Moreover, for $(t, y) > (0, 0)$, we can write

$$\sigma^2(t, y) = \sum_{i=1}^{N} \Lambda_i(t, y)\sigma_i^2(t), \tag{35}$$

where, for each (t, y) and i, $\Lambda_i(t, y) \geq 0$ and $\sum_{i=1}^{N} \Lambda_i(t, y) = 1$. As a consequence

$$0 < \tilde{\sigma} \leq \sigma(t, y) \leq \hat{\sigma} < +\infty \quad \text{for each } t, y > 0. \tag{36}$$

where

$$\tilde{\sigma} := \inf_{t \geq 0} \left\{ \min_{i=1,\dots,N} \sigma_i(t) \right\},$$
$$\hat{\sigma} := \sup_{t \geq 0} \left\{ \max_{i=1,\dots,N} \sigma_i(t) \right\}.$$

Proof. See Appendix C. □

The function $\sigma(t, y)$ can be extended by continuity to the semi-axes $\{(0, y) : y > 0\}$ and $\{(t, 0) : t \geq 0\}$ by setting $\sigma(0, y) = \sigma_0$ and $\sigma(t, 0) = \sigma^*(t)$, where $\sigma^*(t) := \sigma_{i^*}(t)$ and i^* is such that $V_{i^*}(t) = \max_{i=1,\dots,N} V_i(t)$. In particular, $\sigma(0, 0) = \sigma_0$. Indeed, for every $\bar{y} > 0$ and every $\bar{t} \geq 0$,

$$\lim_{t \to 0} \sigma(t, \bar{y}) = \sigma_0,$$
$$\lim_{y \to 0} \sigma(\bar{t}, y) = \sigma^*(t).$$

The function σ^* can in principle be discontinuous. However, we can easily make it a continuous function by assuming that $\sigma_1(t) \leq \sigma_2(t) \leq \cdots \leq \sigma_N(t)$ for each t, so that $\sigma^*(t) = \sigma_N(t)$.

Remark 3.4. *The above proposition provides us with the analytical expression for the diffusion coefficient in the SDE (20) such that the resulting equation has a unique strong solution whose marginal density is given by (22). Moreover, the square of the "local volatility" $\sigma(t, y)$ can be viewed as a weighted average of the squared "basic volatilities" $\sigma_1^2(t), \dots, \sigma_N^2(t)$, where the weights are all functions of the lognormal marginal densities (32). In particular, the "local volatility" $\sigma(t, y)$ lies in the interval $[\tilde{\sigma}, \hat{\sigma}]$.*[11] *In case $\sigma_1(t) \leq \sigma_2(t) \leq \cdots \leq \sigma_N(t)$ for each t, we can actually prove, for each fixed t, the tighter inequalities*

$$\tilde{\sigma} \leq \min_y \sigma_i(t, y) = \sqrt{\frac{\sum_{i=1}^{N} \frac{\lambda_i \sigma_i^2(t)}{V_i(t)} e^{-\frac{1}{8}V_i^2(t)}}{\sum_{i=1}^{N} \frac{\lambda_i}{V_i(t)} e^{-\frac{1}{8}V_i^2(t)}}} \leq \sigma(t, y) \leq \max_{i=1,\dots,N} \sigma_i(t) = \sigma^*(t) \leq \hat{\sigma}.$$

[11] This property relates our model to that of Avellaneda et al. (1995) who considered a stochastic volatility evolving within a predefined band.

As we have already noticed, the pricing of European options under the asset-price model (20) with (33) is quite straightforward. Indeed, we have the following.

Proposition 3.5. *Consider a European option with maturity T, strike K and written on the asset. Then, the option value at the initial time $t = 0$ is given by the following convex combination of Black-Scholes prices*

$$\mathcal{O} = \omega \sum_{i=1}^{N} \lambda_i \left[S_0 e^{(\mu-r)T} \Phi\!\left(\omega \frac{\ln \frac{S_0}{K} + \left(\mu + \frac{1}{2}\eta_i^2\right) T}{\eta_i \sqrt{T}} \right) - K e^{-rT} \Phi\!\left(\omega \frac{\ln \frac{S_0}{K} + \left(\mu - \frac{1}{2}\eta_i^2\right) T}{\eta_i \sqrt{T}} \right) \right], \tag{37}$$

where $\omega = 1$ for a call and $\omega = -1$ for a put and

$$\eta_i := \frac{V_i(T)}{\sqrt{T}} = \sqrt{\frac{\int_0^T \sigma_i^2(t)dt}{T}}. \tag{38}$$

Proof. We just have to apply (30) and notice that $\mathcal{O}_i = e^{-rT} \int_0^{+\infty} [\omega(y-K)]^+ p_T^i(y)dy$ is nothing but the Black-Scholes call/put price corresponding to the volatility η_i. □

The option price (37) leads to smiles in the implied volatility structure. An example of the shape that can be reproduced in shown in Figure 3. Indeed, the volatility implied by the option prices (37) has a minimum exactly at a strike equal to the forward asset price $S_0 e^{\mu T}$. This is formally proven in Appendix B.

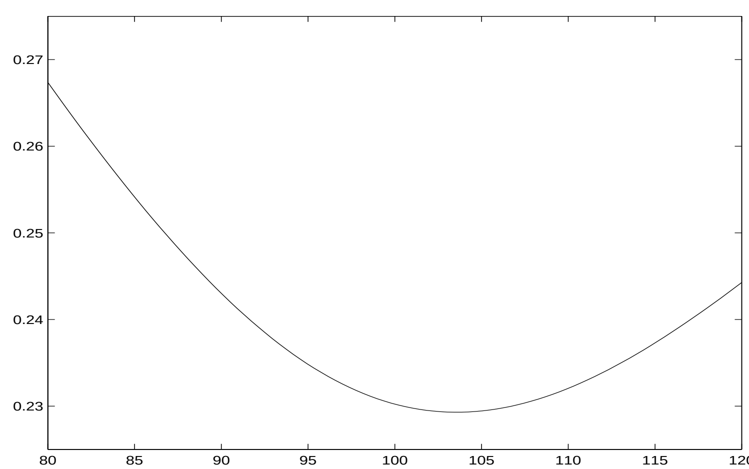

Figure 3: Volatility structure implied by the option prices (37), where we set $\mu = r = 0.035$, $T = 1$, $N = 3$, $(\sigma_1, \sigma_2, \sigma_3) = (0.5, 0.1, 0.2)$, $(\lambda_1, \lambda_2, \lambda_3) = (0.2, 0.3, 0.5)$ and $S_0 = 100$.

Like the models falling into the class considered in the previous section, also these second-class models are quite appealing when pricing exotic derivatives. Notice, indeed, that having explicit dynamics implies that the asset-price paths can be simulated by discretising the associated SDE with a numerical scheme. Hence we can use Monte Carlo procedures to price path-depending derivatives. Claims with early-exercise features can be priced with grids or lattices that can be constructed given the explicit form of the asset-price dynamics.

4 Conclusions and suggestions for further research

We have considered two different classes of asset-price models allowing respectively for skews and smiles in the implied volatility structure. The first class is based on shifting the marginal distribution of a known diffusion process for which there exist analytical formulas for European options. As a result, the processes in this class have known dynamics and transition densities, which also lead to explicit European option prices. The cases of a CEV process with constant coefficients and a general geometric Brownian motion are considered as major applications.

The second class is instead based on asset-price processes whose marginal density is given by the mixture of some suitably chosen densities. In particular, if the basic densities are associated to specific (risk-neutral) asset-price dynamics that implies an analytical option price, so does their mixture. Indeed, we have derived the diffusion process followed by the asset price under the risk-neutral measure such that its marginal density is given by the chosen mixture. As a major example, we have considered the case where the process density is a mixture of lognormal densities. The use of a mixture of lognormal densities for fitting the market volatility smile is not new in the financial literature. However, it is now clear how to relate such asset-price density to some specific dynamics. This is very important, because it allows the pricing and hedging of general derivatives either through a Monte Carlo procedure or through a lattice implementation.

The classes proposed in this paper are defined in a quite general way, so that many more examples could be considered and studied. For instance, processes in the first class could act as bricks for building a second-class process. Vice versa, we could take a process in the second class and obtaining a first-class process by shifting its dynamics. In particular, the risk-neutral asset-price process

$$dS_t = \mu S_t dt + \sqrt{\frac{\sum_{i=1}^N \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln \frac{S_t - \alpha e^{rt}}{S_0} - \mu t + \frac{1}{2}V_i^2(t)\right]^2\right\}}{\sum_{i=1}^N \lambda_i \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln \frac{S_t - \alpha e^{rt}}{S_0} - \mu t + \frac{1}{2}V_i^2(t)\right]^2\right\}}}(S_t - \alpha e^{rt})dW_t,$$

with $\alpha \neq 0$, has a marginal density that is given by shifting a mixture of lognormal densities. The corresponding option prices lead to an implied volatility structure whose minimum point is shifted from the asset forward price. We are thus able to obtain more flexible structures to better fit the real market volatility data.

The study of the goodness of the fit to market data for the models we have considered is an interesting area for future research.

A final remark concerns the type of financial market being considered in this paper. Our asset, in fact, can be viewed as an equity stock or index or as an exchange rate. However, our treatment perfectly works also for forward rates, as soon as we think of them as special assets. In a such a case, we just have to replace the risk-neutral measure with the corresponding forward measure and remember that, under its canonical measure, a forward rate has null drift, i.e., $\mu = 0$.

Appendix A

Under the simplification that $\mu = r$, we prove here that the ATM-forward volatility that is implied by the model (13) has a non-zero derivative with respect to the strike. The Black-Scholes volatility σ that is implied by the call option price (17), with $\omega = 1$, is implicitly defined by

$$C^{BS}(S_0, K, \sigma) = C^{BS}(S_0 - \alpha, K - \alpha e^{rT}, v), \tag{39}$$

where $v := \sqrt{\frac{1}{T-t}\int_t^T \beta_u^2 du}$, and $C^{BS}(S, K, \sigma)$ is the Black-Scholes price when the underlying asset price is S, the strike is K and the volatility is σ, with the option maturity being equal to T. Notice that the equation (39) always has a unique solution $\sigma = \sigma(K)$ since $\alpha < Ke^{-rT}$.

Since $\frac{\partial C^{BS}}{\partial \sigma} \neq 0$, by Dini's theorem

$$\begin{aligned}\frac{d\sigma}{dK}(K) &= \frac{\frac{\partial C^{BS}}{\partial K}(S_0 - \alpha, K - \alpha e^{rT}, v) - \frac{\partial C^{BS}}{\partial K}(S_0, K, \sigma(K))}{\frac{\partial C^{BS}}{\partial \sigma}(S_0, K, \sigma(K))} \\ &= -\frac{e^{-rT}\left[\Phi\left(\frac{\ln\frac{S_0-\alpha}{K-\alpha e^{rT}} + \left(r - \frac{1}{2}v^2\right)T}{v\sqrt{T}}\right) - \Phi\left(\frac{\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2(K)\right)T}{\sigma(K)\sqrt{T}}\right)\right]}{S_0\sqrt{T}\Phi'\left(\frac{\ln\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2(K)\right)T}{\sigma(K)\sqrt{T}}\right)},\end{aligned}$$

where Φ' denote the standard normal density function. In particular, for an ATM-forward strike $\bar{K} = S_0 e^{rT}$,

$$\frac{d\sigma}{dK}(\bar{K}) = -\frac{e^{-rT}\left[\Phi\left(-\frac{1}{2}v\sqrt{T}\right) - \Phi\left(-\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)\right]}{S_0\sqrt{T}\Phi'\left(\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)}, \tag{40}$$

while (39) becomes

$$2S_0\Phi\left(\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right) = 2(S_0 - \alpha)\Phi\left(\frac{1}{2}v\sqrt{T}\right) + \alpha,$$

or equivalently

$$2S_0\left[\Phi\left(\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)-\Phi\left(\frac{1}{2}v\sqrt{T}\right)\right]=\alpha\left[1-2\Phi\left(\frac{1}{2}v\sqrt{T}\right)\right].$$

Hence $\alpha > 0$ implies that $\sigma(\bar{K}) < v$,[12] and, by (40), that $\frac{d\sigma}{dK}(\bar{K}) > 0$, with the sign of the derivative that is preserved in a neighborhood of $\bar{K}$. Conversely, $\alpha < 0$ implies that $\sigma(\bar{K}) > v$, and that $\frac{d\sigma}{dK}(\bar{K}) < 0$.

Appendix B

Under the simplification that $\mu = r$, we prove here that the volatility implied by the option price (37) has a derivative with respect to the strike that is zero at the forward asset price S_0e^{rT}. In the call option case, such a volatility σ is implicitly defined by

$$C^{BS}(S_0,K,\sigma)=\sum_{i=1}^{N}\lambda_i C^{BS}(S_0,K,\eta_i), \tag{41}$$

where the option maturities are all equal to T and η_i is defined by (38). Also in this case, we notice that the equation (41) always has a unique solution $\sigma = \sigma(K)$.

Since $\frac{\partial C^{BS}}{\partial\sigma} \neq 0$, by Dini's theorem

$$\begin{aligned}\frac{d\sigma}{dK}(K)&=\frac{\sum_{i=1}^{N}\lambda_i\frac{\partial C^{BS}}{\partial K}(S_0,K,\eta_i)-\frac{\partial C^{BS}}{\partial K}(S_0,K,\sigma(K))}{\frac{\partial C^{BS}}{\partial\sigma}(S_0,K,\sigma(K))}\\&=-\frac{e^{-rT}\left[\sum_{i=1}^{N}\lambda_i\Phi\left(\frac{\ln\frac{S_0}{K}+\left(r-\frac{1}{2}\eta_i^2\right)T}{\eta_i\sqrt{T}}\right)-\Phi\left(\frac{\ln\frac{S_0}{K}+\left(r-\frac{1}{2}\sigma^2(K)\right)T}{\sigma(K)\sqrt{T}}\right)\right]}{S_0\sqrt{T}\Phi'\left(\frac{\ln\frac{S_0}{K}+\left(r+\frac{1}{2}\sigma^2(K)\right)T}{\sigma(K)\sqrt{T}}\right)},\end{aligned}$$

which, for an ATM-forward strike $\bar{K} = S_0e^{rT}$, becomes

$$\frac{d\sigma}{dK}(\bar{K})=-\frac{e^{-rT}\left[\sum_{i=1}^{N}\lambda_i\Phi\left(-\frac{1}{2}\eta_i\sqrt{T}\right)-\Phi\left(-\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)\right]}{S_0\sqrt{T}\Phi'\left(\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)}.$$

Then, we have that $\frac{d\sigma}{dK}(\bar{K}) = 0$, since for $K = \bar{K}$ (41) reduces to

$$\Phi\left(-\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)=\sum_{i=1}^{N}\lambda_i\Phi\left(-\frac{1}{2}\eta_i\sqrt{T}\right). \tag{42}$$

[12]Remember we have assumed $\alpha \neq 0$.

The proof that $K = \bar{K}$ is a mimimum point is completed by showing that $\frac{d^2\sigma}{dK^2}(\bar{K}) > 0$. Indeed, we have that

$$\frac{d^2\sigma}{dK^2}(\bar{K}) = \frac{\sum_{i=1}^N \lambda_i \frac{\partial^2 C^{BS}}{\partial K^2}(S_0, \bar{K}, \eta_i) - \frac{\partial^2 C^{BS}}{\partial K^2}(S_0, \bar{K}, \sigma(\bar{K}))}{\frac{\partial C^{BS}}{\partial \sigma}(S_0, \bar{K}, \sigma(\bar{K}))}$$
$$= \frac{e^{-rT}\left[\sum_{i=1}^N \lambda_i \frac{\Phi'\left(-\frac{1}{2}\eta_i\sqrt{T}\right)}{\bar{K}\eta_i\sqrt{T}} - \frac{\Phi'\left(-\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)}{\bar{K}\sigma(\bar{K})\sqrt{T}}\right]}{S_0\sqrt{T}\Phi'\left(\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)},$$

so that $\frac{d^2\sigma}{dK^2}(\bar{K}) > 0$ if and only if

$$\sum_{i=1}^N \lambda_i \frac{\Phi'\left(-\frac{1}{2}\eta_i\sqrt{T}\right)}{\bar{K}\eta_i\sqrt{T}} > \frac{\Phi'\left(-\frac{1}{2}\sigma(\bar{K})\sqrt{T}\right)}{\bar{K}\sigma(\bar{K})\sqrt{T}}. \tag{43}$$

Now, setting $x_i := -\frac{1}{2}\eta_i\sqrt{T}$ and $\bar{x} := -\frac{1}{2}\sigma(\bar{K})\sqrt{T}$, (42) and (43) become

$$\Phi(\bar{x}) = \sum_{i=1}^N \lambda_i \Phi(x_i)$$
$$\frac{\Phi'(\bar{x})}{\bar{x}} > \sum_{i=1}^N \lambda_i \frac{\Phi'(x_i)}{x_i}.$$

To prove the last inequality, we look for a real constant ρ such that

$$\rho\Big[\Phi(\bar{x}) - \Phi(x_i)\Big] \le \frac{\Phi'(\bar{x})}{\bar{x}} - \frac{\Phi'(x_i)}{x_i}, \quad \forall i = 1, \ldots, n. \tag{44}$$

Indeed, the existence of such ρ implies that, multiplying both members of (44) by λ_i and summing over $i = 1, \ldots, n$,

$$0 = \rho\Big[\Phi(\bar{x}) - \sum_{i=1}^N \lambda_i \Phi(x_i)\Big] \le \frac{\Phi'(\bar{x})}{\bar{x}} - \sum_{i=1}^N \lambda_i \frac{\Phi'(x_i)}{x_i}, \tag{45}$$

with the inequality being actually strict, as we shall prove in the sequel.

Assuming, without lack of generality, that $x_1 < x_2 < \cdots < x_s \le \bar{x} \le x_{s+1} < \cdots < x_N < 0$,[13] since Φ is increasing, (44) is equivalent to the following system

$$\begin{cases} \rho \le \dfrac{\frac{\Phi'(\bar{x})}{\bar{x}} - \frac{\Phi'(x_i)}{x_i}}{\Phi(\bar{x}) - \Phi(x_i)}, & \forall i = 1, \ldots, s, \\ \rho \ge \dfrac{\frac{\Phi'(\bar{x})}{\bar{x}} - \frac{\Phi'(x_i)}{x_i}}{\Phi(\bar{x}) - \Phi(x_i)}, & \forall i = s+1, \ldots, N, \end{cases} \tag{46}$$

[13]Since Φ is increasing, $\Phi(x_1) < \sum_{i=1}^N \lambda_i \Phi(x_i) < \Phi(x_N)$. Hence $x_1 < \bar{x} < x_N$, since Φ^{-1} is increasing too.

where we understand that if $\bar{x} = x_s$ (resp. $\bar{x} = x_{s+1}$), the s-th (resp. $(s+1)$-th) inequality does not appear in the system, since the corresponding inequality in (44) is automatically verified. This system has a solution ρ if and only if

$$\min_{i=1,\ldots,s} \frac{\frac{\Phi'(\bar{x})}{\bar{x}} - \frac{\Phi'(x_i)}{x_i}}{\Phi(\bar{x}) - \Phi(x_i)} \geq \max_{j=s+1,\ldots,N} \frac{\frac{\Phi'(\bar{x})}{\bar{x}} - \frac{\Phi'(x_j)}{x_j}}{\Phi(\bar{x}) - \Phi(x_j)}.$$

But this inequality holds true since the function

$$z \longmapsto \frac{\frac{\Phi'(\bar{x})}{\bar{x}} - \frac{\Phi'(z)}{z}}{\Phi(\bar{x}) - \Phi(z)} \tag{47}$$

is strictly decreasing for each $\bar{x} < 0$ and z in $(-\infty, 0)$, as it can be easiliy proven.

Finally, the inequality in (45) is actually strict since if the equality held true, all the inequalities in (44) would be equalities as well. Hence

$$\rho = \frac{\frac{\Phi'(\bar{x})}{\bar{x}} - \frac{\Phi'(x_1)}{x_1}}{\Phi(\bar{x}) - \Phi(x_1)} = \frac{\frac{\Phi'(\bar{x})}{\bar{x}} - \frac{\Phi'(x_n)}{x_n}}{\Phi(\bar{x}) - \Phi(x_n)},$$

which is false since $x_1 < \bar{x} < x_n$ and the function (47) is strictly decreasing in $(-\infty, 0)$.

Appendix C: Proof of Proposition 3.3

The formula (33) for $(t, y) > (0, 0)$ is trivially obtained from (26) by using (31) and (32). Equation (35), which immediately derives from (28), implies that

$$\min_{i=1,\ldots,N} \sigma_i(t) \leq \sigma(t, y) \leq \max_{i=1,\ldots,N} \sigma_i(t),$$

for each $(t, y) > (0, 0)$, so that the value $\sigma(0, S_0)$ is set by continuity. The inequalities (36) immediately follow from (35) and the boundedness of each function σ_i. If we write $S_t = \exp(Z_t)$, where

$$dZ_t = \left[\mu - \frac{1}{2}\sigma^2\left(t, e^{Z_t}\right)\right] dt + \sigma\left(t, e^{Z_t}\right) dW_t, \tag{48}$$

the SDE (20) then admits a unique strong solution since the SDE (48) admits a unique strong solution. In fact, its coefficients are bounded and hence statisfy the usual linear-growth condition. Moreover, setting $u(t, z) := \sigma(t, e^z)$, we have

$$\frac{\partial u^2}{\partial z}(t, z) = (\ln(S_0) + \mu t - z) \frac{\sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j A_i A_j \frac{\sigma_i^2(t)}{V_i(t)V_j(t)} \left(\frac{1}{V_i^2(t)} - \frac{1}{V_j^2(t)}\right)}{\sum_{i=1}^N \sum_{j=1}^N \lambda_i \lambda_j A_i A_j \frac{1}{V_i(t)V_j(t)}},$$

where

$$A_i := \exp\left[-\frac{1}{2V_i^2(t)}\left(z - \ln(S_0) - \mu t + \frac{1}{2}V_i^2(t)\right)^2\right],$$

so that $\frac{\partial u^2}{\partial z}(t,z)$ is well defined and continuous for each $(t,z) \in (0,M] \times (-\infty,+\infty)$, $M > 0$, due to the continuity of each σ_i and V_i, and

$$\lim_{t\to 0} \frac{\partial u^2}{\partial z}(t,z) = 0,$$

since u is constant for $t \in [0,\epsilon]$. Therefore, $\frac{\partial u^2}{\partial z}(t,z)$ is bounded on each compact set $[0,M] \times [-M,M]$, and so is $\frac{\partial u}{\partial z}(t,z) = \frac{1}{2u(t,z)}\frac{\partial u^2}{\partial z}(t,z)$ since σ is bounded from below. Hence, the function u is locally Lipschitz in the sense of Theorem 12.1 in Section V.12 of Rogers and Williams (1996). In view of this theorem, the SDE (48) then admits a unique strong solution.

Finally, the density (34) follows from the construction procedure developed at the beginning of Section 3.

References

[1] Andersen, L. and Brotherton-Ratcliffe, R. (1997) The Equity Option Volatility Smile: An Implicit Finite-Difference Approach. *Journal of Computational Finance* 1, 5-38.

[2] Avellaneda, M., Friedman, C., Holmes, R. and Samperi D. (1997) Calibrating Volatility Surfaces via Relative-Entropy Minimization. Preprint. Courant Institute of Mathematical Sciences. New York University.

[3] Avellaneda, M., Levy, A., and Paras, A. (1995) Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities. *Applied Mathematical Finance* 2, 73-88.

[4] Beckers, S. (1980) The Constant Elasticity of Variance Model and its Implications for Option Pricing. *The Journal of Finance* 35, 661-673.

[5] Bibby, B.M., and Sørensen M. (1997) A hyperbolic diffusion model for stock prices. *Finance and Stochastics* 1, 25-42.

[6] Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81, 637-659.

[7] Breeden, D.T. and Litzenberger, R.H. (1978) Prices of State-Contingent Claims Implicit in Option Prices. *Journal of Business* 51, 621-651.

[8] Brigo, D., and Mercurio, F. (1999a). Expensive Markovianity and time–holes in Black and Scholes. In: *Proceedings of the conference MATHEMATICAL THEORY OF NETWORKS AND SYSTEMS, Padova, Italy, July 6-10, 1998.* Il Poligrafo, Padova.

[9] Brigo, D., and Mercurio, F. (1999b). Option pricing impact of alternative continuous time dynamics for discretely observed stock prices, to appear in *Finance and Stochastics.*

[10] Britten-Jones, M. and Neuberger, A. (1999) Option Prices, Implied Price Processes and Stochastic Volatility. Preprint. London Business School.

[11] Brown, G. and Randall, C. (1999) If the Skew Fits *Risk* April, 62-67.

[12] Bhupinder, B. (1998) Implied Risk-Neutral Probability Density Functions from Option Prices: A Central Bank Perspective. In *Forecasting Volatility in the Financial Markets*, 137-167. Edited by Knight, J., and Satchell, S. Butterworth Heinemann. Oxford.

[13] Carr, P., Tari, M. and Zariphopoulou T. (1999) Closed Form Option Valuation with Smiles. Preprint. NationsBanc Montgomery Securities.

[14] Cox, J. (1975) Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. Working paper. Stanford University.

[15] Cox, J. and Ross S. (1976) The Valuation of Options for Alternative Stochastic Processes. *Journal of Financial Economics* 3, 145-166.

[16] Dempster, M.A.H. and Richard, D.G. (1999) Pricing Exotic American Options Fitting the Volatility Smile. Preprint. Central for Financial Research. University of Cambridge.

[17] Derman, E. and Kani, I. (1994) Riding on a Smile. *Risk* February, 32-39.

[18] Derman, E. and Kani, I. (1998) Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility. *International Journal of Theoretical and Applied Finance* 1, 61-110.

[19] Dupire, B. (1994) Pricing with a Smile. *Risk* January, 18-20.

[20] Guo, C. (1998) Option Pricing with Heterogeneous Expectations. *The Financial Review* 33, 81-92.

[21] Jackwerth, J.C. and Rubinstein, M. (1996) Recovering Probability Distributions from Option Prices. *Journal of Finance* 51, 1611-1631.

[22] Li, A. (1998) A one Factor Volatility Smile Model with Closed Form Solutions for European Options. Forthcoming in *European Financial Management.*

[23] Malz, A.M. (1997) Option-Implied Probability Distributions and Currency Excess Returns. Preprint. Federal Reserve Bank of New York.

[24] Melick, W.R., and Thomas, C.P. (1997) Recovering an Asset's Implied PDF from Option Prices: An Application to Crude Oil During the Gulf Crisis. *Journal of Financial and Quantitative Analysis* 32, 91-115.

[25] Pirkner, C.D., Weigend, A.S. and Zimmermann H. (1999) Extracting Risk-Neutral Densities from Option Prices Using Mixture Binomial Trees. Preprint. Stern School of Business.

[26] Platen, E. (1999) A complete market model. Preprint, University of Sidney.

[27] Ritchey, R.J. (1990) Call Option Valuation for Discrete Normal Mixtures. *Journal of Financial Research* 13, 285-296.

[28] Rogers, L.C.G. and Williams, D. (1996) Diffusions, Markov Processes and Martingales. Volume 2. John Wiley & Sons. Chichester.

[29] Rubinstein, M. (1983) Displaced Diffusion Option Pricing. *Journal of Finance* 38, 213-217.

[30] Rubinstein, M. (1994) Implied Binomial Trees. *Journal of Finance* 49, 771-818.

[31] Shimko, D. (1993) Bounds of Probability. *Risk* April, 33-37.

[32] Schroder, M. (1989) Computing the Constant Elasticity of Variance Option Pricing Formula. *The Journal of Finance* 44, 211-219.