ECONOMETRIC SPECIFICATIONS OF STOCHASTIC DISCOUNT FACTOR MODELS

C. GOURIEROUX [1] and A. MONFORT[2]

July 19, 2001

We have benefited of helpful comments by O. Scaillet and J. Simonato.

[1]CREST, CEPREMAP and University of Toronto
[2]CNAM and CREST

Abstract
Econometric Specifications of Stochastic Discount Factor Models

We consider the problem of derivative pricing when the stochastic discount factors are exponential-affine functions of underlying factors. In particular we discuss the conditionally gaussian framework and introduce semi-parametric pricing methods for models with path dependent location and scale parameters. This approach is also applied to more complicated frameworks, such as the pricing of a derivative written on an index, when the interest rate is stochastic.

Keywords : Derivative Pricing, Stochastic Discount Factor, Implied Volatility, Variance-gamma Model.

Résumé
Econométrie des modèles à facteurs d'escompte stochastique

Nous considérons le problème de valorisation de produits dérivés, lorsque les facteurs d'escompte stochastiques sont fonctions exponentielles-affines de facteurs sous-jacents. En particulier nous discutons le cas conditionnellement gaussien et développons des méthodes de valorisation semi-paramétriques, pour des modèles où les paramètres d'échelle et de position dépendent de l'historique. Cette approche est également appliquée à des contextes plus complexes, tel la valorisation d'un dérivé sur indice en présence de taux d'intérêt stochastique.

Mots clés : Valorisation, facteur d'escompte stochastique, volatilité implicite, modèle variance-gamma.

1. Introduction

The pricing of derivatives is generally based on continuous time models, which may include latent stochastic factors, or jumps. This approach provides formulas for derivative prices, expressed as either conditional expectations, or solutions of partial differential equations. However the formulas are often difficult to implement, since both the data and the hedging strategies are in discrete time. This explains why standard continuous time models have a rather simple structure [see e.g. Black, Scholes (1973), Hull, White (1987), Melino, Turnbull (1990), Heston (1993)]. In particular the number of underlying factors is generally small, the risk premia associated with these factors are assumed constant or even equal to zero, and the term structure of interest rates is often treated independently of the index derivatives . However there restrictions, that simplify the implementation of the derivative pricing formulas, often induce a poor fit when historical data, or data on derivative prices are considered.

The aim of this paper is to address the problem of derivating pricing using the stochastic discount factor introduced in [Harrison, Kreps (1979), Garman, Ohlson (1980), Hansen, Richard (1987)]. It is known that, if agents make their investments at date t ($t \in \mathbb{N}$) based on an information set J_t, the prices of actively traded assets satisfy a linear valuation formula. More precisely there exists a stochastic discount factor (SDF) $M_{t,t+1}$ function of the updated information set J_{t+1}, such that the price of an asset that provides the payoff g_{t+1} at date $t+1$ is :

$$C_t(g) = E\left[M_{t,t+1} g_{t+1} | J_t\right]. \tag{1.1}$$

Since the market is incomplete in discrete time, there exists a multiplicity of stochastic discount factors that are compatible with the valuation formula (1.1) for the actively traded assets.

In this paper we consider a class of SDF, that are exponential functions of an affine combination of factors. This specification corresponds to the Esscher transform used in insurance [see e.g. Esscher (1932)], and used for derivative pricing [see Buhlman et alli (1996), Shyraev (1999), Gourieroux, Monfort (2001)]. Next we discuss the restrictions implied by the valuation formula (1.1).

In section 2, we review standard pricing formulas derived from either an equilibrium condition [as the consumption based CAPM, or the model with recursive utility], or the condition of no arbitrage in a continuous time framework [see e.g. Hull, White (1987), Heston (1993)]. The aim is to justify the exponential-affine specification of the SDF and to give examples of possible factors. In section 3, we consider a simple framework in which the riskfree rate is constant (equal to zero) and the information set includes the returns on the actively traded assets only. We prove that there exists a single SDF compatible with the exponential-affine specification. Then we discuss in section 4 the conditionally gaussian framework and the semi-parametric models with path dependent location and scale parameters. These examples are used to derive a semi-parametric pricing method and to discuss the patterns of implied (Black-Scholes) volatility surfaces in terms of departures from time independence and conditional normality. Section 5 extends the basic approach to the framework of stochastic interest rates and unobservable factors. As illustration we discuss the conditionally gaussian factors framework and explain how to price a derivative written on an index when the interest rate is stochastic. Statistical inference is discussed in section 6.

2. Examples of stochastic discount factors

The expressions of stochastic discount factors are generally derived under either an equilibrium condition, or the arbitrage free conditions in a continuous time framework. We provide below some examples to show that stochastic discount factors considered in the literature are often exponential-affine functions of underlying state variables.

Example 1 : Consumption based CAPM (CCAPM)

In the standard CCAPM, an agent maximizes his expected intertemporal utility expressed in terms of a physical good. The intertemporal transfers are ensured by means of investments on a financial market. Let us denote by U the utility function, by δ the intertemporal psychological discount factor, and assume the existence of a representative agent. At equilibrium we get the relation :

$$p_t = E_t\left[p_{t+1}\frac{q_t}{q_{t+1}}\delta\frac{\frac{dU}{dc}(C_{t+1})}{\frac{dU}{dc}(C_t)}\right],$$

where p_t is the vector of prices of financial assets, q_t the price of the consumption good and C_t the quantity consumed at date t. Thus, for a power utility function, the SDF associated with the consumption based CAPM is :

$$\begin{aligned} M_{t,t+1} &= \frac{q_t}{q_{t+1}}\delta\left(\frac{C_{t+1}}{C_t}\right)^{\gamma} \\ &= \exp\left[\log\delta - \log\frac{q_{t+1}}{q_t} + \gamma\log\frac{C_{t+1}}{C_t}\right]. \end{aligned}$$

It is an exponential-affine function of the two state variables, which are the inflation rate and the rate of change in consumption. It does not depend on the asset returns. However the asset returns, that can be considered as additional state variables, are generally included in the available information set, and thus also influence the derivative prices.

Example 2 : Recursive utility

Similar results can be derived, when the representative agent maximizes a recursive utility [see Epstein-Zin (1991) and Weil (1989)]. If the utility is a power function and the aggregator admits a Cobb-Douglas form, the SDF is given by :

$$\begin{aligned} M_{t,t+1} &= \delta^{\alpha/\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-(1-\rho)\alpha/\rho}\left(\frac{W_t}{W_{t+1}}\right)^{1-\alpha/\rho}\left(\frac{q_{t+1}}{q_t}\right)^{-\alpha/\rho} \\ &= \exp\left\{\frac{\alpha}{\rho}\log\delta - \frac{(1-\rho)\alpha}{\rho}\log\frac{C_{t+1}}{C_t} - (1-\alpha/\rho)\log\frac{W_{t+1}}{W_t} - \frac{\alpha}{\rho}\log\frac{q_{t+1}}{q_t}\right\}. \end{aligned}$$

The exponential-affine function now involves a third state variable $\log\frac{W_{t+1}}{W_t}$, that represents the return on the market portfolio.

Example 3 : Stochastic volatility model

In the literature standard stochastic volatility models [Hull, White (1987), Heston (1993)] are of the type :

$$
\begin{cases} dS_t = \mu S_t dt + \sigma_t S_t dW_t^S, \\ \\ df(\sigma_t) = a(\sigma_t)dt + b(\sigma_t)dW_t^\sigma, \end{cases}
$$

where S_t is the asset price, σ_t the stochastic volatility and $(W_t^S), (W_t^\sigma)$ two independent brownian motions. For a constant riskfree rate r, the discount factor is deduced from Girsanov's theorem and given by :

$$
\begin{aligned}
M_{t,t+1} &= \exp(-r) \exp \left\{ -(\mu - r) \int_t^{t+1} \frac{dW_\tau^S}{\sigma_\tau} - \frac{1}{2}(\mu - r)^2 \int_t^{t+1} \frac{d\tau}{\sigma_\tau^2} \right\} \\
& \exp \left\{ - \int_t^{t+1} \nu_\tau dW_\tau^\sigma - \frac{1}{2} \int_t^{t+1} \nu_\tau^2 d\tau \right\},
\end{aligned}
$$

where ν_τ is the risk premium associated with the stochastic volatility. This premium can be selected arbitrarily as a function of the past due to the incomplete market framework. By approximating the integrals by their discrete time counterpart, we note that :

$$
M_{t,t+1} \simeq \exp(-r) \exp \left\{ -(\mu - r) \frac{\varepsilon_{t+1}^S}{\sigma_t} - \frac{1}{2}(\mu - r)^2 \frac{1}{\sigma_t^2} - \nu_t \varepsilon_{t+1}^\sigma - \frac{1}{2} \nu_t^2 \right\}.
$$

It is an exponential transformation of an affine function of the innovations $\varepsilon_{t+1}^S, \varepsilon_{t+1}^\sigma$ corresponding to both the return and volatility processes, with path dependent coefficients.

The examples above show :

i) the convenience of exponential-affine specifications, which ensure both the positivity and the tractability of the SDF ;

ii) the various candidates for state variables, including financial returns, innovations on financial return or on volatilities, rate of change in consumption, inflation rate, etc. These state variables can be observable, or latent;

iii) the multiplicity of specifications in the incomplete market framework due to the arbitrary choice of the risk premium for the non traded factors.

In the next section, we adopt the approach, in which we specify a priori the admissible forms of the SDF. Then this set is reduced by taking into account the arbitrage free restrictions.

3. SDF modelling : the principle

To present the modelling principle, we first consider the framework of a riskfree asset with zero riskfree rate [3] and several risky assets with prices $p_{j,t}, j = 1, \ldots, J$ and (geometric) returns : $r_{j,t+1} = \log(p_{j,t+1}/p_{j,t})$. We assume that these assets are actively traded on the markets, and that the different prices are observable for the investor.

3.1 The historical distribution

The (conditional) historical distribution of the return $r_{t+1} = (r_{1,t+1}, \ldots, r_{J,t+1})'$ is defined by means of its (conditional) Laplace transform, (also called moment generating function) supposed to belong to a parametric set :

$$\begin{aligned} & E[\exp u' r_{t+1} | \underline{r_t}] \\ = \quad & \exp \psi_t(u; \theta), \qquad \text{(say)}. \end{aligned}$$

The Laplace transform is defined on a convex set, that depends on the tails of the conditional distribution. We assume below that this convex set is not reduced to one point located at the origin.

3.2 The stochastic discount factor

We assume a priori that the SDF can be written under an exponential-affine form :

$$M_{t,t+1} = \exp\left(\alpha_t' r_{t+1} + \beta_t\right), \tag{3.1}$$

with $r_{j,t+1}, j = 1, \ldots, J$ as the J state variables and coefficients α_t and β_t, that can be path dependent, that is function of the past : $\underline{r_t} = (r_t, r_{t-1}, \ldots)$.

[3]If the future evolution of the riskfree rate is known at date t, it is possible to get a zero riskfree rate by a deterministic change of numeraire. The case of a predetermined stochastic interest rate is considered in the next section.

By writing the pricing formula for the riskfree asset and the J risky assets, we get $J+1$ restrictions on the relationship between the SDF and the historical distribution. More precisely the constraints induced by the arbitrage free conditions are :

$$\begin{cases} E\left(M_{t,t+1}|\underline{r_t}\right) = 1, \\ \\ E\left[M_{t,t+1}\dfrac{p_{j,t+1}}{p_{j,t}}|\underline{r_t}\right] = E\left[M_{t,t+1}\exp r_{j,t+1}|\underline{r_t}\right] = 1, j = 1,\ldots,J, \end{cases}$$

$$\Longleftrightarrow \begin{cases} E\left[\exp\left(\alpha_t' r_{t+1} + \beta_t\right)|\underline{r_t}\right] = 1, \\ \\ E\left[\exp\left(\alpha_t' r_{t+1} + e_j' r_{t+1} + \beta_t\right)|\underline{r_t}\right] = 1, j = 1,\ldots,J, \end{cases}$$

where $e_j = (0,\ldots 0,1,0,\ldots 0)'$, with 1 as component of order j,

$$\Longleftrightarrow \begin{cases} \exp\left[\psi_t(\alpha_t;\theta) + \beta_t\right] = 1, \\ \\ \exp\left[\psi_t(\alpha_t + e_j;\theta) + \beta_t\right] = 1, \forall j = 1,\ldots,J, \end{cases}$$

$$\Longleftrightarrow \begin{cases} \beta_t = -\psi_t(\alpha_t;\theta), \\ \\ \psi_t(\alpha_t + e_j;\theta) - \psi_t(\alpha_t;\theta) = 0, \forall j = 1,\ldots,J. \end{cases}$$

This system generally admits a unique solution :

$$\begin{cases} \alpha_t = \alpha(\underline{r_t};\theta), \\ \\ \beta_t = \beta(\underline{r_t};\theta) = -\psi_t[\alpha(\underline{r_t};\theta),\theta], \text{ say.} \end{cases}$$

Then we deduce a unique form of the SDF (3.1) that satisfies the condition of no arbitrage. The associated risk neutral distribution Q admits the conditional Laplace transform :

$$\begin{aligned}
&\overset{Q}{E}\left[\exp u' r_{t+1} | \underline{r_t}\right] \\
&= E\left[\exp(\alpha_t' r_{t+1} + \beta_t) \exp u' r_{t+1} | \underline{r_t}\right] \\
&= \exp\left[\psi_t(\alpha_t + u; \theta) + \beta_t\right] \\
&= \exp\left[\psi_t(\alpha_t + u; \theta) - \psi_t(\alpha_t; \theta)\right].
\end{aligned}$$

This result is summarized in the proposition below.

Proposition 1 : For a zero riskfree rate and an exponential-affine SDF with state variable r_{t+1}, there exists in general a unique admissible risk-neutral distribution. Its Laplace transform is given by :

$$\overset{Q}{E}\left[\exp u' r_{t+1} | \underline{r_t}\right] = \exp\left[\psi_t(\alpha_t + u; \theta) - \psi_t(\alpha_t; \theta)\right],$$

where α_t is the solution of :

$$\psi_t(\alpha_t + e_j; \theta) = \psi_t(\alpha_t; \theta), j = 1, \ldots, J,$$

and ψ_t is the conditional historical Log-Laplace transform.

In discrete time we are in an incomplete market framework. Therefore there exists an infinity of admissible risk neutral distributions. The uniqueness condition in Proposition 3.1 is due to the constrained exponential-affine form of the SDF [4]. [5]

We will explain in the next section how to introduce a multiplicity of SDF by means of unobservable stochastic factors with parametrized risk premia.

By using the SDF $M_{t,t+1} = M_{t,t+1}(\underline{r}_{t+1}; \theta)$, we can propose a price for any

[4]Some other constraints could have been imposed on $M_{t,t+1}$. For instance [Hansen, Jagannathan (1997), Cochrane (2000)] assume an affine form $M_{t,t+1} = \alpha_t' r_{t+1} + \beta_t$. This specification does not ensure the positivity of the underlying state prices.

[5]The dynamics of the return under the historical and risk neutral distributions can be very different. For instance they can feature stationarity or nonstationarity under the risk neutral distribution whereas the historical world is stationary. This question is difficult to study in the general framework of this paper [see the discussion in Gourieroux-Monfort (2001)b for affine models of interest rates].

derivative written on r_t [6]. Let us denote by $g(\underline{r_{t+h}}; t+h), h = 1, \ldots, H$ the payoffs provided at $t+h, h = 1, \ldots, H$. A derivative price is :

$$C_t(g) = \sum_{h=1}^{H} E\left[M_{t,t+1}(\underline{r_{t+1}}; \theta) \ldots M_{t,t+h}(\underline{r_{t+h}}; \theta) g(\underline{r_{t+h}}; t+h) | \underline{r_t}\right]. \qquad (3.2)$$

In practice this price cannot be computed analytically and is approximated by simulations. These simulations can be performed under the historical probability, or under a modified probability [see Gourieroux, Jasiak (2001), chapter 13, section 5.2, for a discussion].

3.3 Information and time aggregation

The assumption of exponential-affine SDF depends on the selected information set and time horizon. To illustrate this dependence we consider below two examples.

i) If we are interested on derivatives based on a subset of risky assets with returns r^*_{t+1}, the basic pricing formula (1.1) applied to $g(r^*_{t+1})$ can be written as :

$$\begin{aligned} C_t(g) &= E[M_{t,t+1} g(r^*_{t+1}) | \underline{r_t}] \\ &= E[M^*_{t,t+1} g(r^*_{t+1}) | \underline{r_t}], \end{aligned}$$

where $M^*_{t,t+1} = E[M_{t,t+1} | r^*_{t+1}, \underline{r_t}]$

This modified SDF does not admit in general an exponential-affine expression $M^*_{t,t+1} = \exp(\alpha^{*'}_t r^*_{t+1} + \beta^*_t)$, even if the initial SDF does.

ii) Moreover the assumption is not stable by a change of time unit. Let us consider the pricing of a european derivative at horizon 2. The price is given by :

[6] On the market the derivatives are usually written on prices and not directly on returns. However, let us consider a european call for instance, with payoff $(S_{t+1} - K)^+$ at date $t+1$. This derivative is S_t times the derivative with payoff $(\exp r_{t+1} - k_t)^+$, where $k_t = K/S_t$ is the moneyness strike. Thus we are just assuming a preliminary transformation of the payoff.

$$\begin{aligned} C_t(g) &= E[M_{t,t+1}M_{t+1,t+2}g(r_{t+2})|\underline{r_t}] \\ &= E[E[M_{t,t+1}, M_{t+1,t+2}|r_{t+2}, \underline{r_t}]|\underline{r_t}]. \end{aligned}$$

Neither $M_{t,t+1}M_{t+1,t+2}$, nor $E[M_{t,t+1}M_{t+1,t+2}|r_{t+2}, \underline{r_t}]$ is exponential-affine in general even if $M_{t,t+1}$ is. [7]

Therefore in practice it will be necessary to check for the right information set and horizon, before applying the approach of exponential-affine SDF. Various test procedures are described in section 6.

4. Examples

4.1 The conditionally gaussian framework

If the conditional historical distribution of r_{t+1} given $\underline{r_t}$ is gaussian, with mean m_t and variance-covariance matrix Σ_t, the conditional log-Laplace transform is given by :

$$\psi_t(u;\theta) = u'm_t(\theta) + \frac{1}{2}u'\Sigma_t(\theta)u. \tag{4.1}$$

The risk correction term α_t satisfies :

$$\begin{aligned} & (\alpha_t' + e_j')m_t(\theta) + \frac{1}{2}(\alpha_t' + e_j')\Sigma_t(\theta)(\alpha_t + e_j) \\ & -\alpha_t'm_t(\theta) - \frac{1}{2}\alpha_t'\Sigma_t(\theta)\alpha_t = 0, \forall j = 1, \ldots, J, \\ \iff\ & e_j'm_t(\theta) + e_j'\Sigma_t(\theta)\alpha_t + \frac{1}{2}e_j'\Sigma_t(\theta)e_j = 0, \forall j = 1, \ldots J, \\ \iff\ & m_t(\theta) + \Sigma_t(\theta)\alpha_t + \frac{1}{2}\, vdiag\ \Sigma_t(\theta) = 0, \end{aligned}$$

where $vdiag\ \Sigma_t(\theta)$ is the vector, whose components are the diagonal elements of $\Sigma_t(\theta)$. We deduce that :

[7] In the standard continuous time stochastic volatility model (see example 3), the SDF is exponential-affine at infinitesimal horizon, and is not exponential-affine for more realistic horizons.

$$\alpha_t = -\Sigma_t(\theta)^{-1}[m_t(\theta) + \frac{1}{2}\ vdiag\ \Sigma_t(\theta)]. \tag{4.2}$$

Then the conditional log-Laplace transform of the risk-neutral distribution given in Proposition 1 is :

$$\begin{aligned} & \psi_t(\alpha_t + u; \theta) - \psi_t(\alpha_t; \theta) \\ = \quad & u' \left[m_t(\theta) + \Sigma_t(\theta)\alpha_t \right] + \frac{1}{2} u' \Sigma_t(\theta) u \\ = \quad & -\frac{1}{2} u'\ vdiag\ \Sigma_t(\theta) + \frac{1}{2} u' \Sigma_t(\theta) u. \end{aligned} \tag{4.3}$$

As usual the risk-neutral distribution is also conditionally gaussian, with the same variance-covariance matrix as the conditional historical distribution, and with a conditional mean function of $\Sigma_t(\theta)$, which is introduced to ensure the arbitrage-free condition. These formulas can be applied to a large class of models including for instance the conditionally gaussian multivariate ARCH models [see e.g. Duan (1995)]. In particular they do not require a Markov condition for (r_{t+1}), that is the fact that m_t and Σ_t depend on the past through r_t only.

4.2 Variance-gamma model

This model has been introduced in Madan, Seneta (1990), Madan, Milne (1991), Madan, Carr, Chang (1998). There is only one risky asset and the historical log-Laplace transform of its return is :

$$\psi_t(u) = \nu_t \log \left(1 - u m_t - u^2 \frac{\sigma_t^2}{2} \right),$$

where $\nu_t > 0$; it corresponds to a time deformed gaussian model. The correcting factor is :

$$\alpha_t = -\frac{1}{\sigma_t^2} \left(m_t + \frac{\sigma_t^2}{2} \right),$$

whereas the risk-neutral log-Laplace transform is :

$$\psi_t^Q(u) = \nu_t \log \left(1 - u m_t^* - u^2 \frac{\sigma_t^{*2}}{2} \right),$$

where : $m_t^* = \dfrac{m_t + \alpha_t \sigma_t^2}{1 - \alpha_t m_t - \alpha_t^2 \dfrac{\sigma_t^2}{2}}, \sigma_t^{*2} = \dfrac{\sigma_t^2 \alpha_t^2}{1 - \alpha_t m_t - \dfrac{\alpha_t^2 \sigma_t^2}{2}}.$

Thus the (conditional) parameter ν_t, that determines the time deformation is unchanged, whereas m_t and σ_t^2 are modified. It is easy to check that the Laplace transforms of the historical and risk neutral distributions are both defined on non degenerate intervals.

4.3 Path dependent location and scale parameters

In this subsection we consider a single risky asset ($J = 1$) and assume that the return satisfies :

$$r_{t+1} = m_t + \sigma_t \varepsilon_{t+1}, \sigma_t > 0, \tag{4.4}$$

where m_t and σ_t are the location and scale parameters respectively, that may depend on lagged values of the return and (ε_t) is a sequence of i.i.d. variables with Laplace transform :

$$E(\exp u \varepsilon_{t+1}) = \exp \psi(u), \qquad \text{(say)}. \tag{4.5}$$

This model is convenient to show the consequence of the conditional normality assumption. Indeed it includes the case of gaussian errors, but may also allow for other types of conditional distributions with heavier tails, such as Student or Laplace distribution.

i) The stochastic discount factor and the risk neutral distribution.

The conditional log-Laplace transform of r_{t+1} is given by :

$$\psi_t(u) = m_t u + \psi(\sigma_t u).$$

From Proposition 1, the log-Laplace transform of the risk-neutral distribution is given by :

$$\begin{aligned} \psi_t^Q(u) &= \psi_t(\alpha_t + u) - \psi_t(\alpha_t) \\ &= m_t u + \psi[\sigma_t(\alpha_t + u)] - \psi[\sigma_t \alpha_t], \end{aligned}$$

where α_t is solution of :

$$
\begin{aligned}
\psi_t(\alpha_t+1) &= \psi_t(\alpha_t) \\
\Longleftrightarrow -m_t &= \psi[\sigma_t(\alpha_t+1)] - \psi[\sigma_t\alpha_t].
\end{aligned}
\tag{4.6}
$$

Proposition 2 : There exists a unique solution α_t to equation (4.6), if the log-Laplace transform ψ is strictly convex and tends to infinity at the boundaries of its domain.

Proof : Indeed the mapping $\alpha \to \psi[\sigma_t(\alpha+1)] - \psi(\sigma_t\alpha)$ is continuous, strictly increasing, with range $(-\infty, +\infty)$. The result follows directly.

QED

Various examples are provided in Table 1. They include standard models such as the Black-Scholes model with the gaussian error, the binomial tree with dichotomous errors, or the Laplace models [Gourieroux, Monfort (2001)] and a number of other specifications.

Table 1 : Historical and risk neutral distributions

Distribution	p.d.f.	ψ	$\frac{d^2\psi}{du^2}$	α_t	R.N. Distribution ψ_t^Q
$N(0,1)$	$\frac{1}{\sqrt{2\pi}} \exp - \frac{\varepsilon^2}{2}$	$\frac{u^2}{2}$ $u \in I\!R$	$1 > 0$	$-\frac{1}{2} - \frac{m_t}{\sigma_t^2}$	$-\frac{\sigma_t^2}{2} u + \frac{\sigma_t^2}{2} u^2$
symmetrical exponential	$\frac{1}{2} \exp -\lvert x \rvert$	$-\log(1-u^2)$ $u \in [-1,1]$	$2\frac{1+u^2}{1-u^2} > 0$	$-\frac{1}{2}$ if $m_t = 0$ $\sigma_t < 2$	skewed Laplace distribution with p.d.f. : $\pi(y)$ $= \frac{1}{2}(1 - \frac{\sigma_t^2}{4}) \exp(\frac{1}{\sigma_t} - \frac{1}{2})y$, if $y \leq 0$. $= \frac{1}{2}(1 - \frac{\sigma_t^2}{4}) \exp(\frac{1}{\sigma_t} + \frac{1}{2})y$, if $y \geq 0$.
$u_{[-1,1]}$	$\frac{1}{2} 1_{[-1,+1]}(x)$	$\log \frac{shu}{u}$ $u \in I\!R$	$\frac{1}{u^2} - \frac{1}{sh^2 u} > 0$		
$\frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$	$P[\varepsilon = \pm 1] = \frac{1}{2}$	$\log ch(u)$ $u \in I\!R$	$[ch(u)]^{-2} > 0$	(*)	$\log\{p_t \exp u(m_t + \sigma_t)$ $+(1-p_t)\exp u(m_t - \sigma_t)\}$ where : $p_t = \frac{\exp(-\sigma_t) - \exp(-m_t)}{\exp(-\sigma_t) - \exp(\sigma_t)}$
$\sum_{j=1}^{J} p_j \delta_{\varepsilon_j}$	$P[\varepsilon = \varepsilon_j] = p_j$	$\log(\sum_{j=1}^{J} p_j \exp u\varepsilon_j)$			$\log[\sum_{j=1}^{J} p_{j,t} \exp u(m_t + \sigma_t \varepsilon_j)]$ where : $p_{j,t} = \frac{p_j \exp(\sigma_t \alpha_t \varepsilon_j)}{\sum_{j=1}^{J} p_j \exp(\sigma_t \alpha_t \varepsilon_j)}$

(*) $\alpha_t = \frac{1}{2\sigma_t} \log \left\{ \frac{\exp(-\sigma_t) - \exp(-m_t)}{\exp(-m_t) - \exp(\sigma_t)} \right\}$, if $|m_t| < \sigma_t$.

ii) Semi-parametric pricing

The approach above can in particular be applied to an empirical distribution of the residuals from a given model of returns. As an illustration, let us consider a parametric specifications of the location and scale parameters :

$$m_t = m(r_t; \theta), \sigma_t = \sigma(r_t; \theta), \qquad \text{(say)}, \tag{4.7}$$

and let us leave unspecified the distribution of the error term. The available observations on the returns are denoted by $r_1, \ldots, r_T$.

Step 1 : Calibration

The parameter θ can be consistently estimated from historical data by applying a quasi-maximum likelihood method; the estimator is given by :

$$\hat{\theta}_T = \arg\max_\theta \sum_{t=1}^{T} \left\{ -\log \sigma(r_{t-1};\theta) - \frac{1}{2}\frac{[r_t - m(r_{t-1};\theta)]^2}{\sigma^2(r_{t-1};\theta)} \right\}.$$

Step 2 : Estimation of the error distribution

Then we compute the residuals :

$$\hat{\varepsilon}_\tau = \frac{r_\tau - m(r_{\tau-1};\hat{\theta}_T)}{\sigma(r_{\tau-1};\hat{\theta}_T)}, \tau = 2, \dots, T,$$

and the empirical distribution of the residuals :

$$\hat{P} = \frac{1}{T-1}\sum_{\tau=2}^{T} \delta_{\hat{\varepsilon}_\tau},$$

where δ_ε is the point mass at ε.

Its log-Laplace transform is given by :

$$\hat{\psi}_T(u) = \log\left(\frac{1}{T-1}\sum_{\tau=2}^{T} \exp u\hat{\varepsilon}_\tau \right), u \text{ varying},$$

and is a consistent estimator of the unknown log-Laplace transform ψ.

Step 3 : Determination of the risk correction factor

The correcting term α_t associated with the true distribution ψ can be consistently approximated by the correcting term $\hat{\alpha}_t$ associated with $\hat{\psi}_T$. The correcting term $\hat{\alpha}_t$ is the solution of :

$$\sum_{\tau=2}^{T} \exp[\sigma(r_t;\hat{\theta}_T)\hat{\alpha}_t\hat{\varepsilon}_\tau]\{\exp[m(r_t;\hat{\theta}_T) + \sigma(r_t;\hat{\theta}_T)\hat{\varepsilon}_\tau] - 1\} = 0.$$

Step 4 : Determination of the SDF

The true underlying SDF is approximated by :

$$
\begin{aligned}
\hat{M}_{t,t+1} &= \exp\left[\hat{\alpha}_t[r_{t+1} - m(r_t;\hat{\theta}_T)] - \hat{\psi}_T(\hat{\sigma}_t\hat{\alpha}_t)\right] \\
&= \exp\hat{\alpha}_t[r_{t+1} - m(r_t;\hat{\theta}_T)]\left[\frac{1}{T-1}\sum_{\tau=2}^{T}\exp\hat{\sigma}_t\hat{\alpha}_t\hat{\varepsilon}_\tau\right]^{-1}. \\
&= \hat{m}_{t,t+1}(r_{t+1}, r_t), \qquad \text{(say)}.
\end{aligned}
$$

Step 5 : Pricing

Finally the price of a derivative can easily be approximated by simulations based on the empirical distribution. For instance the price of a derivative with residual maturity 1 and payoff $g(r_{t+1})$ is approximated by :

$$
\begin{aligned}
\hat{C}_t(g) &= \hat{E}\left[\hat{m}_{t,t+1}(r_{t+1}, r_t)g(r_{t+1})\right] \\
&= \hat{E}\left\{g[m(r_t;\hat{\theta}_T) + \sigma(r_t;\hat{\theta}_T)\varepsilon_{t+1}]\exp[\hat{\alpha}_t\sigma(r_t;\hat{\theta}_T)\varepsilon_{t+1}]\right\} \\
&\quad \left[\frac{1}{T-1}\sum_{\tau=2}^{T}\exp\hat{\alpha}_t\hat{\sigma}_t\hat{\varepsilon}_\tau\right]^{-1} \\
&= \frac{\sum_{\tau=2}^{T}\exp(\hat{\alpha}_t\hat{\sigma}_t\hat{\varepsilon}_\tau)g(\hat{m}_t + \hat{\sigma}_t\hat{\varepsilon}_\tau)}{\sum_{\tau=2}^{T}\exp(\hat{\alpha}_t\hat{\sigma}_t\hat{\varepsilon}_\tau)},
\end{aligned}
$$

by the last line of Table 1.

When the derivative to be priced corresponds to a maturity H much larger than 1, different steps of the algorithm above have to be modified. For instance the correcting factors α and the SDF's have to be computed for the dates $t, t+1, \ldots t+H-1$, and the pricing formula modified according to (3.2). Finally the expectation in the price formula has to be approximated by an empirical average over simulated paths $r_{t+1}, \ldots, r_{t+H-1}$ drawn from the empirical distribution. Indeed an average performed on all admissible empirical paths would involve a number of points increasing exponentially with the maturity H.

iii) Implied volatility surfaces

In practice the implied Black-Scholes volatilities depend on the moneyness strike and residual maturity contrary to what is assumed in the Black-Scholes model. They can feature smiles, asymmetries or sneers. These patterns are due to the misspecification of the Black-Scholes model, which assumes i.i.d. gaussian returns. The aim of this section is to consider conditionally heteroscedastic models of the type :

$$r_{t+1} = \sigma(r_t)\varepsilon_{t+1},$$

and to describe how the implied volatility surface depends on the functional form of the volatility and on the error distribution. We consider the following four experiments :

experiment (1) : ARCH model

$$\sigma(r_t) = (10^{-3} + 0.95 r_t^2)^{1/2}, \varepsilon_{t+1} \sim N(0,1)$$

experiment (2) : Symmetrical piecewise linear volatility

$$\sigma(r_t) = (0.1\ \mathbb{1}_{r_t>0} + 0.01\ \mathbb{1}_{r_t<0})^{1/2}, \varepsilon_t \sim N(0,1)$$

experiment (3) : Asymmetrical piecewise linear volatility

$$\sigma(r_t) = (0.005\ \mathbb{1}_{|r_t|>0.05} + 0.001\ \mathbb{1}_{|r_t|<0.05})^{1/2}, \varepsilon_t \sim N(0,1)$$

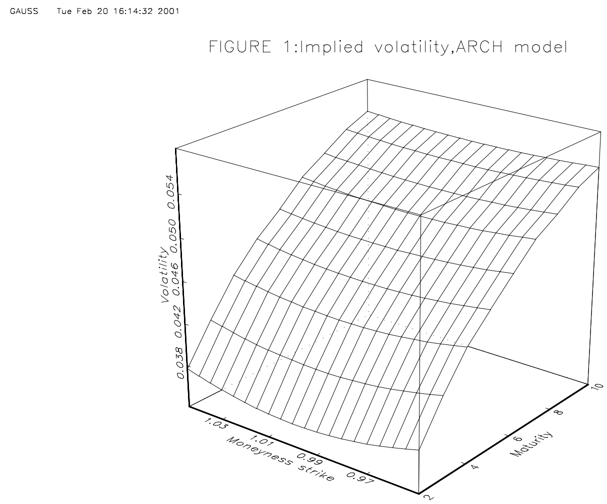
experiment (4) : $\sigma(r_t) = 0.1, \varepsilon_{t+1}$ i.i.d. symmetrical exponential.

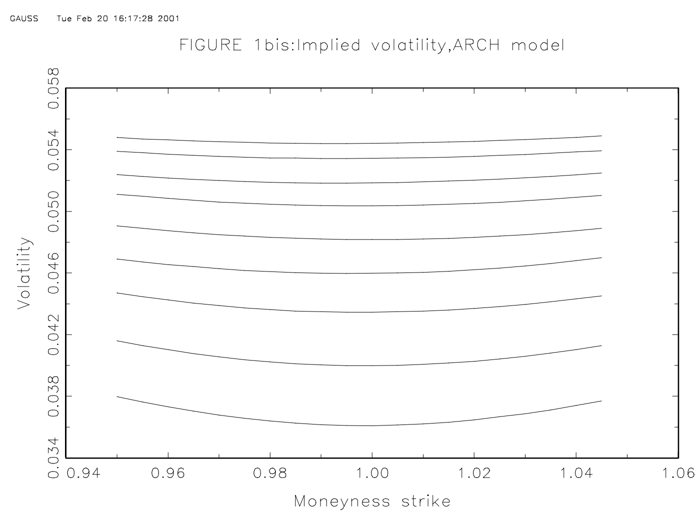
As expected, we observe smile or sneer effects. The importance of smile effects is a function of how the volatility depends on the past and on the magnitude of the tails of the error distribution. A sneer effect is obtained by introducing either an asymmetric volatility function, or a skewed distribution of the error. Finally the smile and sneer effects diminish, when maturity increases. This is a consequence of the central limit theorem, which can be applied to the return between t and $t+H$, given by :

$$r_{t,t+H} = r_{t+1} + r_{t+2} + \ldots + r_{t+H},$$

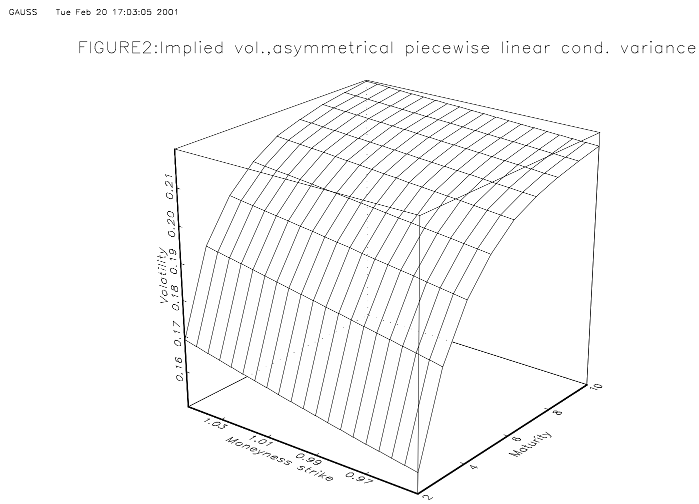
for return processes, that are stationary under the risk neutral distribution.

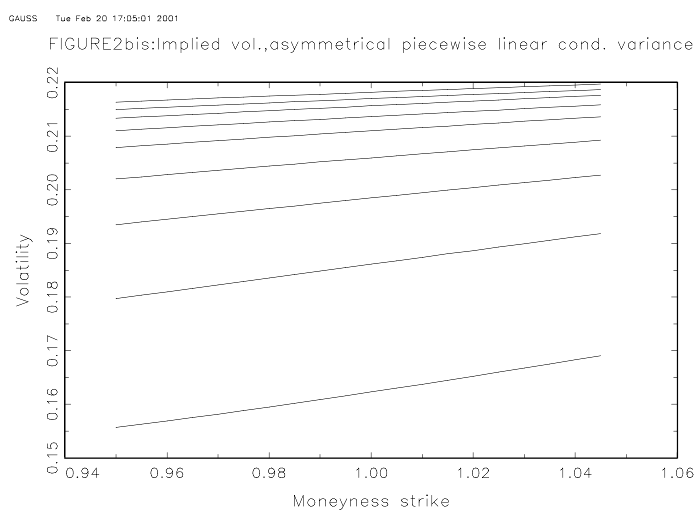
[Insert Figure 1 and 1 bis : Black-Scholes implied volatilities, ARCH model]



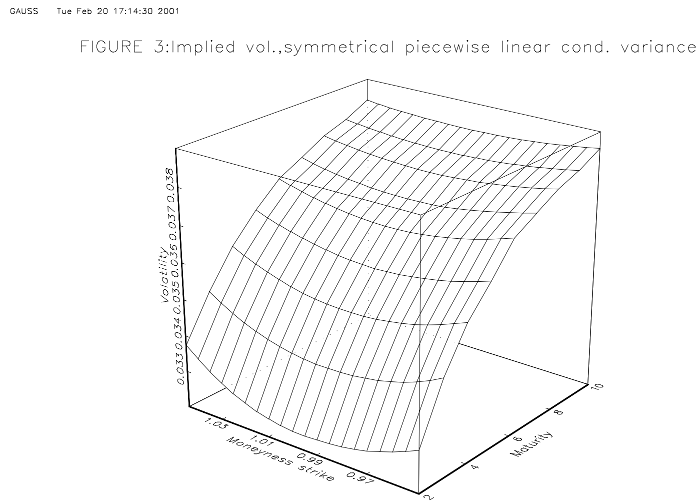


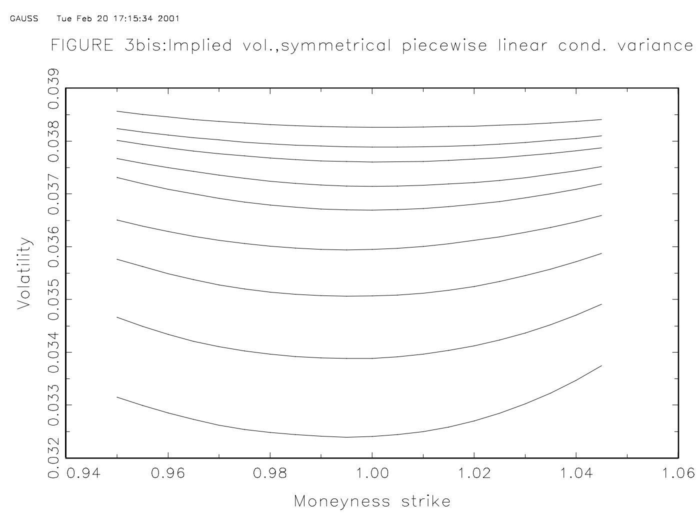
[**Insert Figure 2 and 2 bis : Black-Scholes implied volatilities, asymmetrical piecewise linear conditional variance**]



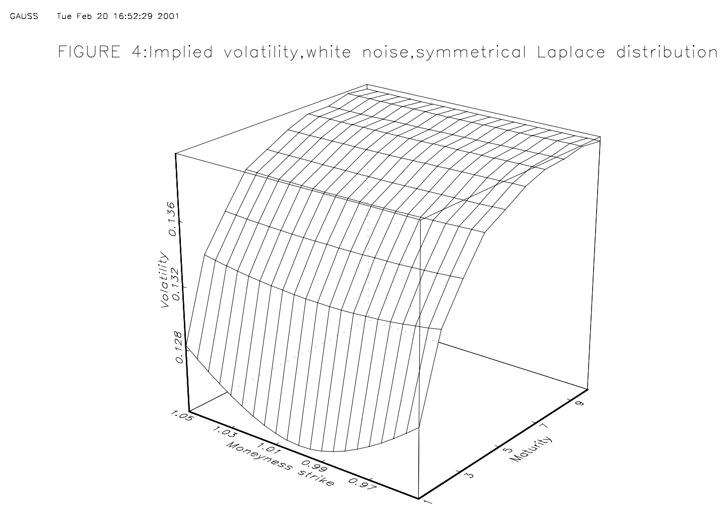


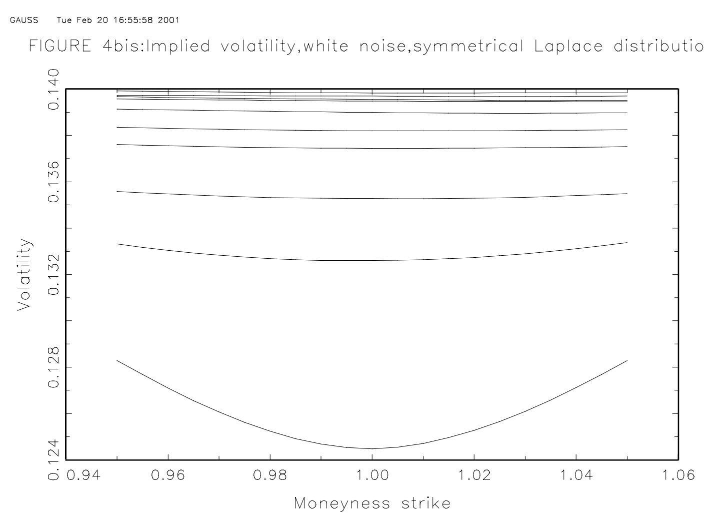
[Insert Figure 3 and 3 bis : Black-Scholes implied volatilities, symmetrical piecewise linear conditional variance]





[Insert Figure 4 and 4 bis : Black-Scholes implied volatilities, symmetrical exponential]





5. General specification

In this section the basic approach to SDF modelling is extended in two directions. First, the information used by the agents to fix their asset demands does not only include the asset returns, but also variables from the real sector of the economy and additional factors. These factors are not directly observed by the econometrician. Second we introduce a time varying riskfree rate r_{t+1}^f. This rate is predetermined, that is known at time t, but assumed stochastic and in particular unknown at time $t-1$. This allows for a joint analysis of the derivatives written on the risky assets and on the short term interest rate. In particular we are interested in the analysis of the term structure of interest rates and in the effect of the stochastic interest rate on the price of a european call with maturity H written on a stock index, for instance [8].

5.1 The investor's information and the SDF

The investors' information at date t is denoted by J_t. It includes the data used by the investors, when they rebalance their portfolios, submit their orders and decide the volumes to be traded. This information includes :

i) the current riskfree rate and its lagged values, $\underline{r_{t+1}^f}$, say;

ii) the lagged returns on J risky assets, $\underline{r_t}$, say ;

iii) the lagged changes of real economic variables, $\underline{x_t}$, say, that may be macrovariables as the GNP, the retail price index..;

iv) the values of additional factors, $\underline{f_t}$, say.

At this stage the additional factors cannot be interpreted. However they will be partly recovered through their effect on the prices of basic and derivative assets, as discussed below.

[8] The standard financial literature treats separately the analysis of the term structure and the pricing of options written on a stock. Typically the term structure is constructed using only the information contained in the past interest rates, whereas the standard option pricing formula assumes that the interest rate r_{t+1}^f is deterministic, that is known in advance for all future dates.

Thus the information is :

$$J_t = \left(\underline{r^f_{t+1}}, \underline{r_t}, \underline{x_t}, \underline{f_t}\right). \tag{5.1}$$

The sizes of the different vectors are $1, J, L$ and K, respectively.

In the sequel we are interested in pricing derivatives written on the basic riskfree and risky assets [9]. If the derivative provides a payoff $g\left(\underline{r^f_{t+2}}, \underline{r_{t+1}}\right)$ at date $t+1$, its price at date t is given by [10]. :

$$C_t(g) = E\left[M_{t,t+1}\, g\left(\underline{r^f_{t+2}}, \underline{r_{t+1}}\right)|J_t\right], \tag{5.2}$$

where the stochastic discount factor $M_{t,t+1}$ depends on J_{t+1}. As before the SDF is constrained to be exponential-affine :

$$M_{t,t+1} = \exp\left(\alpha_{o,t} r^f_{t+2} + \alpha'_t r_{t+1} + \gamma'_t x_{t+1} + \delta'_t f_{t+1} + \beta_t\right), \tag{5.3}$$

where the different coefficients depend on J_t. We denote :

$$F_{t+1} = \left[r^f_{t+2}, x'_{t+1}, f'_{t+1}\right]', \Delta_t = (\alpha_{o,t}, \gamma'_t, \delta'_t)',$$

such that the SDF becomes :

$$M_{t,t+1} = \exp\left[\alpha'_t r_{t+1} + \Delta'_t F_{t+1} + \beta_t\right]. \tag{5.4}$$

5.2 The historical distribution

This distribution has to be considered for all variables in the investor's information set. The conditional Laplace transform of $(r'_{t+1}, F'_{t+1})'$ given J_t is :

[9]It would have been possible to also include real variables in the contractual payoff as for Treasury Bonds indexed on inflation.

[10]The role of conditioning information is important in derivative pricing (see e.g. Hansen, Richard (1987)). For instance the pricing formula (4.2) can also be written as :

$$\begin{aligned} C_t(g) &= E\left[E(M_{t,t+1}|r^f_{t+2}, r_{t+1}, J_t) g(\underline{r^f_{t+2}}, \underline{r_{t+1}})|J_t\right] \\ &= E\left[M^*_{t,t+1} g(\underline{r^f_{t+2}}, \underline{r_{t+1}})|J_t\right]. \end{aligned}$$

Clearly $M^*_{t,t+1}$ is another admissible SDF, but it does not admit an exponential-affine expression with respect to r_{t+1}, r^f_{t+2}, once the factors x_{t+1}, f_{t+1} have been integrated out. Thus the constraint of exponential-affine function is associated with the basic traders' information J_{t+1}.

$$E\left[\exp(u' r_{t+1} + v' F_{t+1}) | J_t\right] = \exp \psi_t(u, v; \theta). \tag{5.5}$$

It depends on the lagged values of the asset returns $\underline{r_t}$, of the real variables $\underline{x_t}$, of the unobservable factors $\underline{f_t}$ and on the current and lagged values of the riskfree rate $\underline{r^f_{t+1}}$. θ is the vector of parameters that characterizes this conditional distribution.

It has to be noted that the riskfree rate has a particular status. It is not only predetermined, but also constrained to be positive, whereas r_t and x_t are generally of any sign due to their interpretations as changes (see the example of CCAPM for the real variables) .

5.3 Constraints on the S.D.F.

By writing the pricing conditions for the riskfree asset and the J risky assets, we get the equations :

$$\begin{cases} E\left[M_{t,t+1} \exp r^f_{t+1} | J_t\right] = 1, \\ \\ E\left[M_{t,t+1} \exp r_{j,t+1} | J_t\right] = 1, \forall j = 1, \ldots, J, \end{cases}$$

$$\iff \begin{cases} \psi_t(\alpha_t, \Delta_t; \theta) + \beta_t + r^f_{t+1} = 0, \\ \\ \psi_t(\alpha_t + e_j, \Delta_t; \theta) + \beta_t = 0, \forall j = 1, \ldots, J, \end{cases}$$

the solutions of which are :

$$\begin{cases} \alpha_t = \alpha(\Delta_t, \underline{r_t}, \underline{F_t}; \theta), \\ \\ \beta_t = \beta(\Delta_t, \underline{r_t}, \underline{F_t}; \theta). \end{cases} \tag{5.6}$$

Since the additional variables do not correspond to returns on assets traded at t, their coefficient Δ_t can be fixed arbitrarily as a function of the information set. Thus the undeterminacy of the risk correction Δ_t still exists, despite the a priori constraint on the specification of the SDF. The results are summarized in the proposition below.

Proposition 3 : For an exponential-affine SDF with state variables r_{t+1}, F_{t+1}, there exists a multiplicity of admissible risk-neutral distributions. Their conditional Laplace transforms are given by :

$$\overset{Q}{E}\left[\exp(u'r_{t+1}+v'F_{t+1})|J_t\right]$$

$$= E[M_{t,t+1}\exp r^f_{t+1}\exp(u'r_{t+1}+v'F_{t+1})|J_t]$$

$$= \exp\left\{\psi_t(\alpha_t+u,\Delta_t+v;\theta)-\psi_t(\alpha_t,\Delta_t;\theta)\right\},$$

where α_t is the solution of :

$$\psi_t(\alpha_t+e_j,\Delta_t;\theta)-\psi_t(\alpha_t,\Delta_t;\theta)-r^f_{t+1}=0,\forall j=1,\ldots,J,$$

and the risk correction Δ_t can be chosen arbitrarily.

The dimension of incompleteness is equal to the dimension of Δ_t. It corresponds to the number of real economy state variables (x_t) plus the future prices of traded assets (r^f_{t+2}) introduced in the SDF.

5.4 The conditionally gaussian framework.

Since the riskfree interest rate r^f_{t+1} admits positive values, the conditionally gaussian framework can only be applied if the future path of the riskfree rate is completely known at date t. Thus this subsection extends subsection 4.1, when there are additional unobservable factors.

When $\left(r'_{t+1},F'_{t+1}\right)'$ are conditionally gaussian under the historical distribution, the log-Laplace transform is :

$$\psi_t(u,v;\theta)=(u',v')m_t(\theta)+\frac{1}{2}(u',v')\Sigma(\theta)\begin{pmatrix}u\\v\end{pmatrix}.$$

The log-Laplace transforms of the risk-neutral distributions are :

$$\begin{aligned}\psi_t^Q(u,v;\theta) &= \psi_t(\alpha_t+u,\Delta_t+v;\theta)-\psi_t(\alpha_t,\Delta_t;\theta)\\ &= (u',v')\left[m_t(\theta)+\Sigma_t(\theta)\begin{pmatrix}\alpha_t\\\Delta_t\end{pmatrix}\right]+\frac{1}{2}(u',v')\Sigma_t(\theta)\begin{pmatrix}u\\v\end{pmatrix}.\end{aligned}$$

They correspond to gaussian distributions with the same conditional variance-covariance matrix as the historical distribution and a conditional mean corrected for the risk. The expression of this mean depends on the solution α_t of the pricing restrictions.

Let us decompose the conditional mean and variance as :

$$m_t = \begin{pmatrix} m_{r,t} \\ \\ m_{F,t} \end{pmatrix}, \Sigma_t = \begin{pmatrix} \Sigma_{rr,t} & \Sigma_{rF,t} \\ \\ \Sigma_{Fr,t} & \Sigma_{FF,t} \end{pmatrix}.$$

The restrictions of proposition 3 are :

$$(e_j', 0)\left[m_t + \Sigma_t \begin{pmatrix} \alpha_t \\ \Delta_t \end{pmatrix}\right] + \frac{1}{2}(e_j', 0)\Sigma_t \begin{pmatrix} e_j \\ 0 \end{pmatrix} - r_{t+1}^f = 0, \forall j = 1, \ldots, J,$$

$$\begin{aligned} &\iff \quad e_j' m_{r,t} + e_j'\Sigma_{rr,t}\alpha_t + e_j'\Sigma_{rF,t}\Delta_t + \frac{1}{2}e_j'\Sigma_{rr,t}e_j - r_{t+1}^f = 0, \forall j, \\ &\iff \quad m_{r,t} + \Sigma_{rr,t}\alpha_t + \Sigma_{rF,t}\Delta_t + \frac{1}{2}\ vdiag\ \Sigma_{rr,t} - r_{t+1}^f e = 0, \end{aligned}$$

where $e = (1, \ldots, 1)'$.

We deduce that :

$$\alpha_t = \Sigma_{rr,t}^{-1}\left[m_{r,t} - r_{t+1}^f e + \frac{1}{2}\ vdiag\ \Sigma_{rr,t} + \Sigma_{rF,t}\Delta_t\right], \tag{5.7}$$

which extends equation (3.4).

Thus the conditional mean of the risk-neutral distribution is :

$$\begin{aligned} & m_t + \Sigma_t \begin{pmatrix} \alpha_t \\ \Delta_t \end{pmatrix} \\ = & \begin{pmatrix} m_{r,t} + \Sigma_{rr,t}\alpha_t + \Sigma_{r,F,t}\Delta_t \\ \\ m_{F,t} + \Sigma_{Fr,t}\alpha_t + \Sigma_{FF,t}\Delta_t \end{pmatrix} \\ = & \begin{bmatrix} r_{t+1}^f e - \frac{1}{2}\ vdiag\ \Sigma_{rr,t} \\ \\ m_{F,t} - \Sigma_{Fr,t}\Sigma_{rr,t}^{-1}\left(m_{r,t} - r_{t+1}^f e + \frac{1}{2}\ vdiag\ \Sigma_{rr,t}\right) + \left(\Sigma_{FF,t} - \Sigma_{Fr,t}\Sigma_{rr,t}^{-1}\Sigma_{rF,t}\right)\Delta_t \end{bmatrix} \end{aligned}$$

It is easy to recognize the residual conditional mean $m_{F,t} - \Sigma_{Fr,t}\Sigma_{rr,t}^{-1}m_{r,t}$ and variance $\Sigma_{FF,t} - \Sigma_{F,r,t}\Sigma_{rr,t}^{-1}\Sigma_{rF,t}$. The correction for risk is performed on the returns of traded assets, and there is no correction at all on the residual part, i.e. on the component of F_{t+1} which is not conditionally linked with r_{t+1}. This result can be considered as a direct extension of the pricing formula derived by Stapleton, Subrahmanyam (1984).

Example 4 : Stochastic volatility model

It is interesting to discuss the definition of the factors in the framework of the discretized stochastic volatility model :

$$\begin{cases} S_{t+1} - S_t = \mu S_t + \sigma_t S_t \varepsilon_{t+1}^S, \\ \\ f(\sigma_{t+1}) - f(\sigma_t) = a(\sigma_t) + b(\sigma_t)\varepsilon_{t+1}^{\sigma}. \end{cases}$$

A possible expression of the SDF is given in example 3 as an exponential-affine function of ε_{t+1}^S and $\varepsilon_{t+1}^{\sigma}$. However this S.D.F. can also be expressed as an exponential-affine function of $S_{t+1}, f(\sigma_{t+1})$, or of $(S_{t+1} - S_t)/S_t$, $(f(\sigma_{t+1} - f(\sigma_t))/f(\sigma_t)$ with path dependent coefficients. Therefore the factors are not uniquely defined. In the example, we can choose as factors either the innovations, or the levels of variables, or the associated returns. In particular by selecting the innovation processes, we can assume some white noise properties of the factors under the historical distribution.

5.5 Models with stochastic interest rate

We first consider a model without an unobservable factor, where the short term zero-coupon bond is the only tradeable asset. By selecting a compound autoregressive model for the historical distribution of the riskfree rate (Darolles, Gourieroux, Jasiak (2001)), we get a direct extension of the Cox-Ingersoll-Ross model (Cox, Ingersoll, Ross (1985)). Then we discuss the pricing of an index derivative, when the interest rate is stochastic.

i) Model with a stochastic interest rate only

When the short term zero-coupon bond is the only tradable asset and the SDF is given by :

$$M_{t,t+1} = \exp(\alpha_o r_{t+2}^f + \beta_t), \tag{5.8}$$

the arbitrage free condition becomes :

$$\beta_t = -r^f_{t+1} - \psi_t(\alpha_o), \tag{5.9}$$

where : $\psi_t(u) = E[\exp u r^f_{t+2} | \underline{r^f_{t+1}}]$.

As an illustration let us assume that the conditional distribution of r^f_{t+2} is a noncentered gamma distribution [11] with log-Laplace transform :

$$\psi_t(u) = -\nu \log(1-uc) + \frac{\rho u}{1-uc} r^f_{t+1}, \tag{5.10}$$

where ν, c, ρ are positive parameters. We deduce that :

$$\begin{aligned}
\beta_t &= -r^f_{t+1} - \psi_t(\alpha_o) \\
&= -r^f_{t+1} + \nu \log(1-\alpha_o c) - \frac{\rho\alpha_o}{1-\alpha_o c} r^f_{t+1}, \\
M_{t,t+1} &= \exp\left[\alpha_o r^f_{t+2} - \left(1 + \frac{\rho\alpha_o}{1-\alpha_o c}\right) r^f_{t+1} + \nu \log(1-\alpha_o c)\right].
\end{aligned}$$

Finally the log-Laplace transform of the risk-neutral distribution is given by :

$$\begin{aligned}
& \overset{Q}{E}_t\,[\exp u r^f_{t+2}] \\
=\ & \exp[\psi_t(u+\alpha_o) - \psi_t(\alpha_o)] \\
=\ & \exp[-\nu \log\left(\frac{1-(u+\alpha_o)c}{1-\alpha_o c}\right) + \frac{\rho(u+\alpha_o)}{1-(u+\alpha_o)c} r^f_{t+1} - \frac{\rho\alpha_o}{1-\alpha_o c} r^f_{t+1}] \\
=\ & \exp[-\nu \log(1-uc^*) + \frac{\rho^* u}{1-uc^*} r^f_{t+1}],
\end{aligned}$$

[11] The variable r^f_{t+2} follows the gamma distribution with degrees of freedom ν, scale parameter c and noncentrality parameter $\rho r^f_{t+1}/c$, if and only if :

r^f_{t+2}/c follows a gamma distribution $\gamma(\nu + Z_t)$, where Z_t is drawn in the Poisson distribution $\mathcal{P}[\rho r^f_{t+1}/c]$. It is easily checked that this conditional distribution corresponds to a discretized version of the Cox, Ingersoll, Ross model, if $0 < \rho < 1$, (see e.g Gourieroux, Jasiak (2000)), and that its log-Laplace transform is given by (4.10).

where : $c^* = c/(1-\alpha_o c), \rho^* = \rho/(1-\alpha_o c)^2$.

For this risk neutral distribution to be defined, the risk correcting factor α_o has to be chosen such that $1-\alpha_o c > 0 \Longleftrightarrow \alpha_o < 1/c$.

Thus the risk-neutral conditional distributions belong to the same family of noncentered gamma distributions as the historical conditional distribution. The degrees of freedom are the same for the historical and risk-neutral distributions, whereas the parameters c^* and ρ^* depend on the risk correction α_o, associated with the future short term interest rate. The standard Cox, Ingersoll, Ross, model corresponds to a zero risk premium $\alpha_o = 0$.

Let us now consider the derivatives with exponential payoffs $\exp(ur^f_{t+2}), u$ varying. Since the Laplace transform characterizes the distribution, these derivatives define a generating system for all european derivatives.

The price $C_t(u,1)$ of the european derivative providing the payoff $\exp(ur^f_{t+2})$ at date $t+1$ is :

$$C_t(u,1) = \exp(-r^f_{t+1}) \overset{Q}{E_t} (\exp ur^f_{t+2}).$$

This price depends on r^f_{t+1} and α_o :

$$C_t(u,1) = \gamma(\alpha_o, r^f_{t+1}; u) \qquad \text{(say)}.$$

It is easy to check that we get a semi-interval of admissible prices, when α_o varies : $(\gamma^*_t(u), +\infty)$, where : $\gamma^*_t(u) = \min_{\alpha_o < 1/c} \gamma(\alpha_o, r^f_{t+1}; u)$.

Explicit formulas can also be derived for pricing european derivatives at any horizon. We still consider the generating system of derivatives with exponential payoffs.

Proposition 4 : Let us denote by $C_t(u,h)$ the price at t of the european derivative, that provides the payoff $\exp(ur^f_{t+h+1})$ at $t+h$. We get :

$$C_t(u,h) = \exp[a(h,u)r^f_{t+1} + b(h,u)],$$

where the functions a and b satisfy the recursive equations :

$$a(h,u) = -1 - \frac{\rho\alpha_0}{1-\alpha_0 c} + \rho\frac{\alpha_0 + a(h-1,u)}{1-[\alpha_0 + a(h-1,u)]c},$$

$$b(h,u) = \nu \log(1-\alpha_o c) - \nu \log[1-[\alpha_o + a(h-1,u)]c] + b(h-1,u),$$

for $h \geq 2$.

Proof : We get :

$$\begin{aligned}
C_t(u,h) &= E_t[M_{t,t+1}C_{t+1}(u,h-1)] \\
&= E_t\Big\{\exp[\alpha_o r^f_{t+2} - \left(1+\frac{\rho\alpha_o}{1-\alpha_o}\right) r^f_{t+1} + \nu\log(1-\alpha_o c) \\
&\quad + a(h-1,u) r^f_{t+2} + b(h-1,u)]\Big\} \\
&= \exp\left\{-\left(1+\frac{\rho\alpha_o}{1-\alpha_o c}\right) r^f_{t+1} + \nu\log(1-\alpha_o c) + b(h-1,u)\right\} \\
&\quad E_t\left\{\exp[\alpha_o + a(h-1,u)] r^f_{t+2}\right\} \\
&= \exp\Big\{-\left(1+\frac{\rho\alpha_o}{1-\alpha_o c}\right) r^f_{t+1} + \nu\log(1-\alpha_o c) - \nu\log[1-(\alpha_o + a(h-1,u))c] \\
&\quad + b(h-1,u) + \rho\frac{\alpha_o + a(h-1,u)}{1-(\alpha_o + a(h-1,u))c} r^f_{t+1}\Big\}.
\end{aligned}$$

The result follows by identification.

QED

The nonlinear recursive equation does not depend on the argument u, that is on the derivative to be priced [12]. This argument has an effect through the initial condition only, since : $a(1,u) = -1 + \frac{\rho^* u}{1-uc^*}$, $b(1,u) = -\nu\log(1-uc^*)$. Once the functions a and b have been computed, the risk neutral distribution at horizon h admits the log-Laplace transform :

[12] This property is analogous to a standard property for continuous time models, saying that the price of a european derivative satisfies a partial differential equation, which is independent of the payoff.

$$
\begin{aligned}
\psi^Q_{t,h}(u) &= \log C_t(u,h) - \log C_t(0,h) \\
&= [a(h,u) - a(h,0)]r^f_{t+1} + b(h,u) - b(h,0).
\end{aligned}
$$

It is possible to get the explicit expressions of the coefficient $a(h,u)$ (and $b(h,u)$). Indeed the series $a(h,u)$ satisfies a rational recursive equation, which is equivalent to :

$$
\frac{a(h,u)-\gamma_1}{a(h,u)-\gamma_2} = \frac{1+\gamma_1}{\gamma_1}\frac{\gamma_2}{1+\gamma_2}\frac{a(h-1,u)-\gamma_1}{a(h-1,u)-\gamma_2},
$$

where γ_1 and γ_2 are distinct real roots of the second degree polynomial : $c^*\gamma^2 + \gamma[\rho^* + c^* - 1] - 1 = 0$.

Thus we get :

$$
\frac{a(h,u)-\gamma_1}{a(h,u)-\gamma_2} = \left[\frac{1+\gamma_1}{\gamma_1}\frac{\gamma_2}{1+\gamma_2}\right]^{h-1} \frac{a(1,u)-\gamma_1}{a(1,u)-\gamma_2}.
$$

ii) Risky asset and stochastic interest rate

The approach above can be extended to the framework of a risky asset with return r_{t+1} and a stochastic interest rate r^f_{t+1}. A simple specification of the historical distribution assumes independence between the riskfree rate process (r^f_{t+1}) and the excess return process $(r^*_{t+1} = r_{t+1} - r^f_{t+1})$. Then by selecting appropriate distributions for both components, we can take into account the positivity constraint on the riskfree rate.

As an illustration, we consider a gaussian autoregressive model for (r^*_{t+1}) and an autoregressive gamma model for (r^f_{t+1}). Thus the conditional log-Laplace transform is :

$$
\begin{aligned}
\psi_t(u,v) &= \log E[\exp(ur_{t+1} + vr^f_{t+2})|J_t] \\
&= \log E(\exp ur^*_{t+1}|\underline{r^*_t}) + ur^f_{t+1} + \log E(\exp vr^f_{t+2}|\underline{r^f_{t+1}}) \\
&= u(ar^*_t + b) + \frac{\sigma^2 u^2}{2} + ur^f_{t+1} - \nu\log(1-vc) + \frac{\rho v}{1-vc}r^f_{t+1} \\
&= \psi_{1t}(u) + ur^f_{t+1} + \psi_{2t}(v), \qquad \text{(say)}, \qquad (5.11)
\end{aligned}
$$

where $ar_t^* + b$ denotes the one-step ahead prediction of the excess return.

The stochastic discount factor is set as :

$$\begin{aligned} M_{t,t+1} &= \exp[\alpha_o r_{t+2}^f + \alpha_t r_{t+1}^* + \beta_t] \\ &= \exp[\alpha_o r_{t+2}^f + \alpha_t (r_{t+1} - r_{t+1}^f) + \beta_t]. \end{aligned} \tag{5.12}$$

Then the arbitrage free conditions imply :

$$\begin{cases} E_t(M_{t,t+1} \exp r_{t+1}^f) = 1, \\ E_t[M_{t,t+1} \exp(r_{t+1}^* + r_{t+1}^f)] = 1, \end{cases}$$

$$\iff \begin{cases} \psi_{1,t}(\alpha_t) + \psi_{2,t}(\alpha_o) + r_{t+1}^f + \beta_t = 0, \\ \psi_{1,t}(\alpha_t + 1) + \psi_{2,t}(\alpha_o) + r_{t+1}^f + \beta_t = 0. \end{cases}$$

Thus the risk correcting factor α_t satisfies :

$$\begin{aligned} & \psi_{1t}(\alpha_t + 1) = \psi_{1t}(\alpha_t) \\ \iff \quad & \alpha_t = -\frac{1}{2} - \frac{ar_t^* + b}{\sigma^2}. \end{aligned} \tag{5.13}$$

The risk-neutral log-Laplace transforms are given by :

$$\psi_t^Q(u, v) = \psi_{1t}(u + \alpha_t) - \psi_{1t}(\alpha_t) + ur_{t+1}^f + \psi_{2t}(v + \alpha_o) - \psi_{2t}(v). \tag{5.14}$$

The processes (r_{t+1}^*) and (r_{t+1}^f) are still independent in the risk neutral world, for any value of α_o. However the joint distribution of (r_{t+1}, r_{t+1}^f) has to be used, for pricing a standard european call, since the payoff depends jointly on r_{t+1}^* and r_{t+1}^f : :

$$(\exp r_{t+1} - k)^+ = [\exp(r_{t+1}^* + r_{t+1}^f) - k]^+.$$

Thus this standard call can be considered as a quanto-option, for which the riskfree rate provides the exchange rate between the money units of dates t and $t+1$, respectively.

6. Statistical inference

The stochastic discount factors are suitable for an analysis based jointly on the prices of basic assets and derivatives. The observations and the econometric specification are described in the first subsection. Then we discuss the estimation methods based on either the basic variables, or both the basic variables and the derivative prices. Finally we introduce different specification tests.

6.1 The econometric model

The observations concern :

i) the basic returns $r_t, r^f_{t+1}, t = 1, \ldots, T$,

ii) the real sector variables $x_t, t = 1, \ldots, T$

iii) the prices $z_{i,t}, i = 1, \ldots, n$ of n derivatives with stationary cash-flows. More precisely $z_{i,t}$ is the price at t of a european derivative providing the cash-flow $g_i(\underline{r^f_{t+H_i+1}}, \underline{r_{t+H_i}})$ at date $t + H_i$.

As usual the econometric specification is based on a latent model, which defines the dynamics of all state variables and the stochastic discount factor. More precisely the parameterized latent model admits three components :

i) the historical distribution, which is characterized by the conditional distribution of r_t, r^f_{t+1}, x_t, f_t given the information set J_{t-1}. The associated p.d.f. is denoted by :

$$\begin{aligned} & l(r_t, r^f_{t+1}, x_t, f_t | \underline{r_{t-1}}, \underline{r^f_t}, \underline{f_{t-1}}; \theta) \\ = \; & l(y_t, f_t | \underline{y_{t-1}}, \underline{f_{t-1}}; \theta), \end{aligned} \tag{6.1}$$

where : $y_t = (r'_t, r^f_{t+1}, x'_t)'$.

ii) The stochastic discount factor constrained by the no arbitrage condition :

$$\begin{aligned} M_{t,t+1} &= \exp\{\alpha(\Delta_t, \underline{r_t}, \underline{F_t}; \theta) r_{t+1} + \Delta'_t F_{t+1} + \beta(\Delta_t, \underline{r_t}, \underline{F_t}; \theta)\} \\ &= m_{t,t+1}(\underline{y_{t+1}}, \underline{f_{t+1}}; \theta, \Delta_t), \text{ (say)}. \end{aligned} \tag{6.2}$$

iii) A parameterized risk premium :

$$\Delta_t = \Delta(\underline{y_t}, \underline{f_t}; \lambda), \tag{6.3}$$

where Δ is a given function and λ is an additional vector of parameters.

Thus the latent model includes two types of parameters, where θ characterizes the historical dynamics, whereas λ allows for the choice of a risk-neutral distribution among the admissible ones.

Under the specification above, the derivative prices are :

$$\begin{aligned} z_{i,t} \;&=\; E\left\{\Pi_{\tau=1}^{H_i} m_{t+\tau-1,t+\tau}\left[\underline{y_{t+\tau}}, \underline{f_{t+\tau}}, \theta, \Delta(\underline{y_{t+\tau-1}}, \underline{f_{t+\tau-1}}; \lambda)\right]\right. \\ &\quad \left. g_i(\underline{r^f_{t+H_i-1}}, \underline{r_{t+H_i}}) | J_t)\right\}, i = 1, \ldots, n. \end{aligned} \tag{6.4}$$

They are known functions of the current and past values of the state variables and of the two types of parameters. We denote this relation by[13] :

$$z_t = q(y_t, f_t; \underline{y_{t-1}}, \underline{f_{t-1}}; \theta, \lambda). \tag{6.5}$$

6.2 Estimation

To avoid deterministic relationships between the observed variables [see the discussion in Gourieroux, Jasiak (2001)], we assume that the number of unobservable factors K is larger than the number n of observed derivative prices :

Assumption : $K \geq n$.

Then the econometric model is a standard nonlinear state space model, where :

i) the nonlinear transition equation is defined by the parametrized conditional distribution :

$$l(y_t, f_t | \underline{y_{t-1}}, \underline{f_{t-1}}; \theta); \tag{6.6}$$

[13] The function q has generally a complicated expression and its values have to be computed numerically.

ii) the nonlinear measurement equation is :

$$\left\{ \begin{array}{lcl} y_t & = & y_t \\ \\ z_t & = & q(y_t, f_t | \underline{y_{t-1}}, \underline{f_{t-1}}; \theta, \lambda). \end{array} \right. \tag{6.7}$$

When $K > n$, the likelihood function has a complicated expression that includes multiple integrals of order $T(K-n)$, due to the $K-n$ factors which cannot be recovered from the observable variables (or equivalently to the dimension of incompleteness). However the model (5.6), (5.7) is easy to simulate for given value of parameters and initial conditions. This allows for simulation based estimation methods, as the maximum simulated likelihood approach (MSL) [see e.g. Gourieroux, Monfort (1996) chap 3, Gourieroux, Jasiak (2001), chap 13].

When $K = n$, the estimation problem becomes simple. Indeed we can invert the measurement equation to express f_t as a function of $y_t, z_t, \underline{y_{t-1}}, \underline{f_{t-1}}$ and by recursive substitution as a function of $y_t, z_t, \underline{y_{t-1}}, \underline{z_{t-1}}$:

$$f_t = \tilde{q}(y_t, z_t | \underline{y_{t-1}}, \underline{z_{t-1}}; \theta, \lambda) = \tilde{q}_t(\theta, \lambda), \text{ say.} \tag{6.8}$$

Then the likelihood function is directly derived by applying the Jacobian formula; for date t, its component is given by :

$$\begin{aligned} & L(y_t, z_t | \underline{y_{t-1}}, \underline{z_{t-1}}; \theta, \lambda) \\ = & |\det \frac{\partial \tilde{q}}{\partial z_t'}(y_t, z_t | \underline{y_{t-1}}, \underline{z_{t-1}}; \theta, \lambda)| l[y_t, \tilde{q}_t(\theta, \lambda) | \underline{y_{t-1}}, \underline{\tilde{q}_{t-1}(\theta, \lambda)}; \theta, \lambda]. \end{aligned} \tag{6.9}$$

This approach has been initially proposed by Renault, Touzi (1996), Pastorello, Renault, Touzi (2000).

6.3 Specification tests

Besides the standard procedures, several specification tests can be performed in this derivative pricing framework. They concern the specification of the risk premia and the compatibility between the historical and risk-neutral densities.

6.3.1 Constant risk premium

We can check the constancy of the parameter λ by considering the alternative hypothesis, where this parameter is time dependent with two regimes :

$$H_1 = \{\lambda_t = \lambda^{\circ}, \text{ if } t \leq T_o, \lambda_t = \lambda^1, \text{ if } t > T_o\}.$$

Under the alternative the price of a derivative depend on both levels λ_0 and λ_1, if the period $\{t, t+H\}$ includes the threshold T_o. The testing procedure can be based on a Lagrange multiplier approach.

6.3.2 Coherency between the historical and risk-neutral distributions.

For expository purpose, the approach is presented for $K = n$, where the information is : $J_t = \left(\underline{y_t}, \underline{z_t}\right)$. The econometric specification can be nested in a more general one, with different parameter levels in the historical distribution and in the SDF. The alternative is defined by :

i) the pdf : $l(y_t, f_t | \underline{y_{t-1}}, \underline{f_{t-1}}; \theta_o)$;

ii) the SDF : $m_{t,t+1}(\underline{y_{t+1}}, \underline{f_{t+1}}; \theta_1, \Delta_t)$;

iii) the risk premium : $\Delta_t = \Delta(\underline{y_t}, \underline{f_t}; \lambda)$.

Thus the nesting model depends on three types of parameters θ_0, θ_1 and λ, and the null hypothesis is $H_o = (\theta_0 = \theta_1)$.

This approach has been first proposed in the econometric literature on the term structure of interest rates [see e.g. De Munnik, Schotman (1994), De Jong (1997), Bams, Schotman (1997), Bams (1998)]. They propose to test the null hypothesis in the following way.

i) Firstly, the parameter θ_o is estimated by using the time series of observed basic variables. The estimator is :

$$\hat{\theta}_o = \arg\max_{\theta_o} \sum_{t=1}^{T} \log l(y_t | \underline{y_{t-1}}; \theta_o).$$

ii) Secondly, θ_1 and λ are jointly estimated by considering the so-called cross-sectional approach, that is by optimizing :

$$(\hat{\theta}_1, \hat{\lambda}) = \arg\max_{\theta_1, \lambda} \sum_{t=1}^{T} \log l(z_t | y_t, \underline{y_{t-1}}, \underline{z_{t-1}}; \hat{\theta}_o, \theta_1, \lambda).$$

ii) Finally the estimates $\hat{\theta}_o$ and $\hat{\theta}_1$ are compared by means of a standard specification test.

R E F E R E N C E S

Ait-Sahalia, Y. (1996) : "Nonparametric Pricing of Interest Rate Derivative Securities", Econometrica, 64, 527-560.

Bams, D. (1998) : "An Empirical Comparison of Time Series and Cross-Sectional Information in the Longstaff-Schwartz Term Structure Models", Maastricht Univ. DP.

Bams, D., and P., Schotman (1997) : "A Panel Data Analysis of Affine Term Structure Models", Maastricht Univ. DP.

Black, F., and M., Scholes (1973) : "The Pricing of Options and Corporate Liabilities", Journal of Political Economy, 81, 637-659.

Breeden, D., Glosten, L., and R., Jagannathan (1989) : "Economic Significance of Predictable Variations in Stock Index Returns ", Journal of Finance, 44, 1177-1189.

Brown, D., and M., Gibbons (1985) : "A Single Econometric Approach for Utility-Based Asset Pricing Models", Journal of Finance, 40, 359-381.

Buhlman, H., Delbaen, F., Embrechts, P., and A., Shyraev (1996) : "No Arbitrage, Change of Measure and Conditional Esscher Transforms in a Semi-Martingale Model of Stock Processes", CWI Quarterly, 9, 291-317.

Campbell, J., Lo, A., and C., McKinlay (1997) : "The Econometrics of Financial Markets", Princeton Univ. Press.

Clark, S. (1993) : "The Valuation Problem in Arbitrage Price Theory", Journal of Mathematical Economics, 22, 463-478.

Cochrane, J. (2001) : "Asset Pricing", Princeton University Press.

Cochrane, J., and L., Hansen (1992) : "Asset Pricing Explorations for Macroeconomics", in NBER Macroeconomics Annual, MIT Press, 115-165.

Cox, J., Ingersoll, J., and S., Ross (1985) : "A Theory of the Term Structure of Interest Rates", Econometrica, 53, 385-408.

Darolles, S., Gourieroux, C., and J., Jasiak (2001) : "Compound Autoregressive Models", CREST, DP.

De Jong, F. (1997) : "Time Series and Cross Section Information in Affine Term Structure Models", DP.

De Munnik, J., and P., Schotman (1994) : "Cross-Sectional Versus Time Series Estimation of Term Structure Models : Empirical Results for the Dutch Bond Market", Journal of Banking and Finance, 18, 997-1025.

Duan, J.C. (1995) : "The GARCH Option Pricing Model", Mathematical Finance, 5, 13-32.

Engle, R. and C., Mustafa (1992) : "Implied ARCH Models from Option Prices", Journal of Econometrics, 52, 289-311.

Epstein, L., and S., Zin (1991) : "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns I : A Theoretical Framework", Econometrica, 57, 937-969.

Esscher, F. (1932) : "On the Probability Function in the Collective Theory of Risk", Skandinavisk Aktuariedskrift, 15, 165-195.

Garman, M., and J., Ohlson (1980) : "Information and the Sequential Valuation of Assets in Arbitrage Free Economics", Journal of Accounting Research, 18, 420-440.

Gourieroux, C., and J., Jasiak (2000) : "Autoregressive Gamma Processes", CREST, DP.

Gourieroux, C., and J., Jasiak (2001) : "Financial Econometrics", Princeton University Press.

Gourieroux, C., and A., Monfort (1996) : "Simulation Based Econometric Methods", Oxford Univ. Press.

Gourieroux, C., and A., Monfort (2001)a : "Pricing with Splines", CREST DP.

Gourieroux, C., and A., Monfort (2001)b : "Affine Models of Interest

Rates", CREST DP.

Hansen, L., Heaton, J. and E., Luttmer (1995) : "Econometric Evaluation of Asset Pricing Models", The Review of Financial Studies, 8, 237-274.

Hansen, L., and R., Jagannathan (1997) : "Assessing Specification Errors in Stochastic Discount Factor Models", Journal of Finance, 52, 557-590.

Hansen, L., and S., Richard (1987) : "The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models", Econometrica, 55, 587-613.

Hansen, L., and K., Singleton (1982) : "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models", Econometrica, 50, 1269-1286.

Harrison, M., and D., Kreps (1979) : "Martingales and Arbitrage in Multiperiod Securities Markets", Journal of Economic Theory, 20, 381-408.

Heston, S. (1993) : "A Closed Form Solution for Options with Stochastic Volatility with Application to Bond and Foreign Currency Options", Review of Financial Studies, 6, 327-343.

Hull, J., and A., White (1987) : "The Pricing of Options on Assets with Stochastic Volatilities", Journal of Finance, 42, 281-300.

Kallsen, J., and M., Taqqu (1998) : "Option Pricing in ARCH Type Models", Mathematical Finance, 58, 13-26.

Madan, D., Carr, P., and E., Chang (1998) : "The Variance Gamma Process and Option Pricing", European Finance Review, 2, 79-105.

Madan, D., and F., Milne (1991) : "Option Pricing with VG Martingale Components", Mathematical Finance, 1, 39-48.

Madan, D., and E., Seneta (1990) : "The Variance-Gamma Model for Share Market Returns", Journal of Business, 63, 511-524.

Melino, A., and S., Turnbull (1990) : "Pricing Foreign Currency Options with Stochastic Volatility", Journal of Econometrics, 45, 239-265.

Noh, J., Engle, R., and A., Kane (1995) :"Forecasting Volatility and Option Prices of the S& P Index", in Engle, R. ed. "ARCH Selected Readings",314-331, Oxford Univ. Press.

Pastorello, S., Renault, E., and N., Touzi (2000) : "Statistical Inference for Random Variance Option Pricing", Journal of Business and Economic Statistics, 18, 358-367.

Renault, E. (1999) :"Dynamic Factor Models in Finance", CORE Lecture, Oxford Univ. Press, forthcoming.

Renault, E., and N., Touzi (1996) : "Option Hedging and Implied Volatilities in a Stochastic Volatility Model", Mathematical Finance, 6, 279-302.

Shyraev, A. (1999) : "Essentials of Stochastic Finance : Facts, Models, Theory", World Scientific Publishing, London.

Stapleton, R., and M., Subrahmanyam (1984) : "The Valuation of Multivariate Contingent Claims in Discrete Time Models", The Journal of Finance, 39, 207-228.

Weil, P. (1989) : "The Equity Premium Puzzle and the Risk-Free Rate Puzzle", Journal of Monetary Economics, 24, 401-421.