PRICING WITH SPLINES

C. GOURIEROUX [1] and A. MONFORT [2]

May 17, 2001

[1] CREST and CEPREMAP
[2] CNAM and CREST

1. Introduction

The standard for option pricing models is the Black-Scholes approach, [Black, Scholes (1973)], which assumes i.i.d. gaussian geometric stock returns, continuous trading and derives an analytical formula for pricing european calls from the arbitrage free constraints. The derivative prices and the associated risk neutral probability basically depend on the underlying historical volatility (and not on the historical mean). However it is known that the Black-Scholes specification is misspecified both for return dynamics and cross-sectional links between the prices of derivatives with different characteristics. Typically the implied Black-Scholes volatility surfaces are not flat and vary with the day and the environment.

Different solutions have been proposed in the literature to reduce the misspecification errors that is to get call prices with independent variations with respect to the strike, the maturity and the date.

A first direction consists in extending the dynamic model for the underlying asset price and in deriving the new corresponding valuation formulas. For instance Merton (1973) assumes a non constant volatility, which is a deterministic function of time and allows to reproduce the term structure of implied Black-Scholes volatilities. This approach is extended by Dupire (1994), who considers a risk neutral[3] volatility, which is a deterministic function of the time and asset price, in order to reproduce the observed smiles. Hull, White (1987), Hull (1989) Chesney, Scott (1989), Melino, Turnbull (1990), Stein, Stein (1991), Heston (1993), Ball, Roma (1994) introduce a stochastic volatility, which depends on an additional non traded random factor. In this incomplete market framework the model has to be completed by specifying the risk premium corresponding to this unobservable risk. In the same spirit incompleteness can also be introduced by means of jump processes [see e.g. Merton (1976), Ball, Torous (1985), Bates (1996)]. These models are generally written in continuous time, and provide coherent specifications for analysing return dynamics and cross-sectional derivative pricing.

Alternatively the practitioners prefer a general to specific approach. For each date they study how the call prices depend on the strike and the maturity. They can consider directly the price surface or equivalent summary statistics. Standard ones are 1) the state price density which provides the Arrow-Debreu prices and is deduced from the second order derivative of the

[3] In the diffusion framework the historical and risk neutral volatilities coincide.

call price with respect to the strike [see Breeden, Litzenberger (1978), Banz, Miller (1978)]; 2) the surface of Black-Scholes implied volatilities obtained by inverting the Black-Scholes formula with respect to the volatility ; 3) the local volatility introduced by Dupire (1994), which is deduced from partial derivatives of the call price with respect to strike and time to maturity and can be computed daily. Once a type of summary has been selected, they try to smooth and structure the surfaces. Then, in a second step, they can introduce a dynamics on the cross-sectional structures.

The surfaces are often smoothed by a nonparametric approach. For instance Dumas, Fleming, Whaley (1998) consider polynomial approximations of the local volatility surface. Ait-Sahalia (1996), Ait-Sahalia, Lo (1995) apply kernel smoothing to observed call prices and deduce the state price density as a by-product.[4] Other authors propose direct approximations of the state price density. For instance Bahra (1996), Campa, Chang, Reider (1997), Melick, Thomas (1997) introduce mixture of distribution whereas Jarrow, Ruud (1982), Madan, Milne (1994), Abken, Madan, Ramamurtie (1996) consider expansions of the underlying risk neutral density by means of Hermite polynomials. In the latter approach, it is possible to estimate date by date parameters measuring the weights of polynomials of degree one, two, three, four... in this expansion. They are generally interpreted as implied mean, volatility, skewness and kurtosis. Then in the second step, it is possible to plot these parameters with respect to time and to analyse their dynamics. However the polynomial approximations of the risk neutral density can provide negative values, which is not compatible with the no arbitrage condition.

In this paper we develop a similar approach based on spline approximations, which ensures the nonnegativity of the stochastic discount factor. Moreover it is possible to ensure the coherency between the historical and risk neutral distributions.

We derive new parametric and nonparametric derivative pricing formulas in a discrete time framework. In this framework the markets are incomplete and there are multiple choices for the risk-neutral distribution. We restrict the choice by imposing an exponential-affine stochastic discount factor [Gourieroux, Monfort (2001)]. This allows the use of the Esscher transformation to pass from the historical distribution to the risk-neutral one [see

[4] However their approach assumes that the call prices depend in a deterministic way of the asset price.

e.g. Gerber, Shiu (1994), Buhlman et alii (1996), Shiryaev (1999), Darolles, Gourieroux, Jasiak (2001)]. To simplify the computations, we consider an approximation of the historical conditional p.d.f. by means of exponential splines of order one. Then by writing the arbitrage free restrictions, we derive the exponential spline representation of the conditional risk-neutral distribution.

The plan of the paper is the following. In section 2, we recall the principle of exponential-affine pricing. Then this approach is applied to a skewed Laplace conditional historical distribution of geometric return and extended to exponential-affine splines. The example of the conditional Laplace distribution is interesting as an introductory case for the exponential-splines. It is also important, since we derive analytical pricing formulas for the european calls. This formula is a direct competitor of the standard Black-Scholes, and involves two types of parameters, which allows to capture location and tail effects. The extension to the multiperiod framework is presented in section 3. We introduce a Markov chain specification for describing the dynamics of the different spline regimes and derive the change of probability at any maturity. Section 4 concludes.

2. The two period framework

In this subsection we consider the two period framework. We denote by r the riskfree rate between the dates t and $t+1$ and by $y = y_{t+1} = \log(S_{t+1}/S_t)$ the geometric return on the risky asset with price S_t. We first recall the principle of exponential-affine pricing [see Gourieroux-Monfort (2001)]. Then this approach is applied to skewed a Laplace conditional historical distribution of geometric return and extended to exponential-affine splines.

2.1 Exponential-affine pricing

Let us introduce the truncated Laplace transform (or moment generating function) of the conditional distribution of the geometric return. It is defined by :

$$\psi(u, \gamma) = E[\exp(uy)\mathbb{1}_{y>\gamma}], \tag{2.1}$$

where the notation means :

$$\psi(u, \gamma) = E(\exp\{u \log(S_{t+1}/S_t)\}\mathbb{1}_{\log(S_{t+1}/S_t)>\gamma}|I_t),$$

I_t is the information available at time t for the investor and the path dependence of ψ is not mentioned for notational simplicity. .

The derivative asset, whose payoff $g(y)(= g(y_{t+1}))$ is written on the geometric return of the underlying asset, can be priced by means of a stochastic discount factor model [see e.g. Hansen, Richard (1987), Campbell, Lo, McKinlay, (1997) chapter 8, Cochrane (2001), Gourieroux, Jasiak (2001), chapter 13]. The derivative price at date t is :

$$C(g) = E[Mg(y)], \tag{2.2}$$

where M denotes the stochastic discount factor. In an exponential-affine framework the form of the stochastic discount factor is restricted to [5] :

$$M = \exp[\alpha y + \beta]. \tag{2.3}$$

[5] As above the time index has been omitted for convenience. More explicit equations would be : $C_t(g) = E[M_{t,t+1}g(y_{t+1})|I_t]$, where : $M_{t,t+1} = \exp(\alpha_t y_{t+1} + \beta_t)$ is the stochastic discount factor for the period $t, t+1$. The coefficients α_t, β_t and the derivative price $C_t(g)$ are I_t-measurable, whereas the stochastic discount factor $M_{t,t+1}$ is I_{t+1}-measurable.

It is exponential-affine with respect to the geometric return $y(= y_{t+1})$.[6] This special pattern of the stochastic discount factor corresponds to a valuation formula in a two period price exchange economy under preference restrictions [see e.g. Breeden, Litzenberger (1978), Huang, Litzenberger (1988)]. The exponential-affine forms correspond to power utility functions. The arbitrage-free constraints are derived by applying the pricing formula to the zero-coupon bond with payoff 1 and to the risky asset with payoff $\exp y = S_{t+1}/S_t$. These constraints are :

$$\begin{cases} E[M \exp r] = 1, \\ \\ E[M \exp y] = 1. \end{cases}$$

They provide the values of the risk correcting factors α, β by solving the system below [7], which depends on the untruncated Laplace transform :

$$\begin{cases} \exp(\beta + r)\psi(\alpha, -\infty) = 1, \\ \\ \exp(\beta)\psi(\alpha + 1, -\infty) = 1. \end{cases} \tag{2.4}$$

Then the price of an european call written on $\exp y$, with (moneyness) strike k and maturity one, is easily deduced. It is given by[8] :

$$\begin{aligned} C(k) &= E[M(\exp y - k)^+] \\ &= E[\exp(\alpha y + \beta)[\exp y - k]\mathbb{1}_{y > \log k}], \\ C(k) &= \exp(\beta)[\psi(\alpha + 1, \log k) - k\psi(\alpha, \log k)], \end{aligned} \tag{2.5}$$

where α, β are the solutions of system (2.4).

2.2 Pricing with Laplace distributions

[6] The stochastic discount factor is in general not exponential-affine with respect to the current and lagged values of the return ; indeed they influence the change of probability by means of α and β.

[7] When the time index is taken into account, the solutions α and β are generally path dependent, like function ψ.

[8] Note that a call written on S_{t+1} with payoff $(S_{t+1} - kS_t)^+$, where k is the moneyness strike, is a multiple of the call written on $\exp y$ with payoff $(S_{t+1}/S_t - k)^+ = (\exp y - k)^+$.

As an illustration let us consider a geometric return, whose conditional historical distribution is a skewed Laplace distribution denoted by $\mathcal{L}(b_0, b_1, c)$. The p.d.f is given by :

$$\begin{aligned} p(y) &= \frac{b_0 b_1}{b_0 + b_1} \exp[b_0(y-c)], \text{ if } y \leq c, \\ & \frac{b_0 b_1}{b_0 + b_1} \exp[-b_1(y-c)], \text{ if } y \geq c, \end{aligned}$$

where b_0 and b_1 are strictly positive and c is a location parameter. The parameter c defines the mode of the distribution, whereas b_0 and b_1 characterize the left and right exponential tails, respectively. The mean of the distribution is : $m = c + \frac{1}{b_1} - \frac{1}{b_o}$, and the variance is : $\sigma^2 = \frac{1}{b_0^2} + \frac{1}{b_1^2}$. Note that b_0, b_1, c can be path dependent. This type of distribution fits the conditional distribution of observed returns better than the gaussian distribution. It admits fatter tails, which decrease at an exponential rate and a sharp peak at the mode, which balances the tail effect. By applying the general approach described in subsection 2.1, we get the pricing formulas below.

Proposition 1 : If the conditional historical distribution is a skewed Laplace distribution $\mathcal{L}(b_0, b_1, c)$ with $b_0 + b_1 > 1$, and if the stochastic discount factor is exponential-affine :

i) the conditional risk-neutral distribution is unique and corresponds to the skewed Laplace distribution $\mathcal{L}(b_0 + \alpha, b_1 - \alpha, c)$, with p.d.f. :

$$\begin{aligned} \pi(y) &= \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1} \exp[(b_0+\alpha)(y-c)], \text{ if } y \leq c, \\ & \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1} \exp[-(b_1-\alpha)(y-c)], \text{ if } y \geq c, \end{aligned}$$

where α is the solution of :

$$\exp(c-r)(b_0+\alpha)(b_1-\alpha) = (b_0+\alpha+1)(b_1-\alpha-1),$$

such that : $-b_0 < \alpha < b_1 - 1$.

The risk neutral distribution depends on b_0, b_1 through $b_0 + b_1$, only.

ii) The price of the call written on $\exp y$ with payoff $(\exp y - k)^+$ is :

$$C(k) = C_1(k) = \frac{b_0 + \alpha + 1}{(b_0 + b_1)(b_1 - \alpha)} \exp[-(b_1 - \alpha - 1)(\log k - c)], \text{ if } \log k \geq c,$$

$$C(k) = C_2(k) = 1 - k\exp(-r) + \frac{b_1 - \alpha - 1}{(b_0 + b_1)(b_0 + \alpha)} \exp[(b_0 + \alpha + 1)(\log k - c)], \text{ if : } \log k \leq c.$$

iii) By the put-call parity relationship, the put prices are :
$P(k) = P_1(k) = 1 - k\exp(-r) + \dfrac{b_0 + \alpha + 1}{(b_0 + b_1)(b_1 - \alpha)} \exp[-(b_1 - \alpha - 1)(\log k - c)],$
if $\log k \geq c$,

$$P(k) = P_2(k) = \frac{b_1 - \alpha - 1}{(b_0 + b_1)(b_0 + \alpha)} \exp[(b_0 + \alpha + 1)(\log k - c)], \text{ if } \log k \leq c.$$

Proof : See appendix 1.

The condition $-b_0 < \alpha < b_1 - 1$ ensures the existence of the stock price. It is easily checked that there is a unique solution for α, which belongs to the interval $(-b_0, b_1 - 1)$, if and only if $b_0 + b_1 > 1$, i.e. if the tails are in average sufficiently thin.

Remark 1 : The price of a european call written on S_{t+1} with strike K is given by : $C^* = S_t C(K/S_t)$. Generally C^*/S_t is not an homogenous function of K/S_t, since the coefficients b_0, b_1, c can be path dependent [see Garcia, Renault (1998) for a discussion of the link between homogeneity and leverage effect].

i) Value of the call and moneyness strike

We get an explicit formula for the price of the call written on $\exp y$. It is easily checked that this price is a differentiable function of k, which decreases from 1 to 0, is convex and such that the elasticity of the call price [the put price, respectively] with respect to the moneyness strike is constant for $k \geq \exp c$ [$k \leq \exp c$, respectively].

Remark 2 : When the parameters b_0, b_1, c are path independent, the elasticity of the call price C^* with respect to S_t is :

$$\begin{aligned}\frac{\partial \log C^*}{\partial \log S_t} &= 1 + \frac{\partial \log C(K/S_t)}{\partial \log S_t} \\ &= 1 + \frac{\partial \log C}{\partial \log k}(K/S_t).\frac{\partial \log(K/S_t)}{\partial \log S_t} \\ &= 1 - \frac{\partial \log C}{\partial \log k}(K/S_t).\end{aligned}$$

Therefore the condition of constant elasticity of C with respect to the moneyness strike for large k is equivalent to the condition of constant elasticity of C^* with respect to the current stock price.

In particular the call prices satisfy simple deterministic relationships. If k, k_1, k_2 are moneyness strikes larger than $\exp c$, we get :

$$\log C(k) = \log C(k_1) + \frac{\log k - \log k_1}{\log k_2 - \log k_1}[\log C(k_2) - \log C(k_1)].$$

Remark 3 : This constraint is also valid when the derivatives are written on S_{t+1}. With obvious notations, the relation becomes :

$$\log C^*(K) = \log C^*(K_1) + \frac{\log K - logK_1}{\log K_2 - \log K_1}\{\log C^*(K_2) - \log C^*(K_1)\}.$$

ii) Implied Black-Scholes Volatility

The pricing formula given in Proposition 1 can be compared to the standard Black-Scholes formula. We immediately note that it depends generally on two independent parameters, i.e. $b_0 + b_1$ and c, instead of only one σ in the standard Black-Scholes. Thus the Laplace pricing formula allows for implied location or tail effects. These features are easily observed on Figures 1 and 2, which provide the Black-Scholes implied volatilities for different sets of parameters b_0, b_1, c, and $r = 0$. The Laplace model is appropriate for recovering the so-called smile, smirk and sneer effects.

[**Insert Figure 1 : Black-Scholes implied volatilities with c varying, $b_0 + b_1 = 2$ fixed**].

[Insert Figure 2 : Black-Scholes implied volatilities with $b_0 + b_1$ varying, $c = 1$ fixed].

It is especially important to correct for the risk in the Laplace framework. Indeed the payoff $\exp y$ of the underlying asset may be non integrable with respect to the conditional historical Laplace distribution [9]. Indeed if $b_1 < 1$, the payoff $\exp y$ is not integrable with respect to the conditional historical Laplace distribution, whereas it is integrable with respect to the conditional risk-neutral Laplace distribution, since $b_1 - \alpha > 1$. An effect of the risk correction by α is to reduce the tails in order to ensure this integrability and the existence of a finite stock price .

iii) Value of the call and historical parameters

The patterns of the call prices as functions of c and $b_0 + b_1$ are provided in Figures 3 and 4.

[Insert Figure 3 : Call price as a function of c]

It is always difficult to understand how the call price depends on a location parameter, that is the mean in the standard Black-Scholes model and the mode c in the Laplace framework. This feature is clearly observed, when we consider the underlying stock with cash-flow $\exp y$. When the location parameter tends to $+\infty$ (resp. $-\infty$), the cash-flow tends to $+\infty$ (resp. 0), but the price remains constant equal to one. In fact when the location parameter tends to infinity the historical distribution tends to a point mass at infinity, whereas the risk neutral distribution may tend to a limit which does not correspond to this point mass. Typically for $y = -\infty, \exp y = 0$ and we expect a price for $\exp y$ equal to zero, whereas it is equal to one. Contrary to the Black-Scholes case in which the call price is independent of the mean, we observe a dependence in the Laplace framework. The symmetric pattern observed in figure 3 is due to the special choice $k = 1, r = 0$, which implies $1 - k \exp -r = 0$ and identical call and put prices.[10]

[9] Note that $\exp y$ is conditionally not integrable, if and only if the conditional expectation $E_t(S_{t+1})$ does not exist. In such a framework, the standard mean-variance management cannot be applied.

[10] It is easily checked that the correcting factor $\alpha = \alpha[b_0 + b_1, \exp(c - r)]$ satisfies : $\alpha[b_0 + b_1, \exp(r - c)] = b_1 - b_0 + 1 - \alpha[b_0 + b_1, \exp(c - r)]$.

[Insert Figure 4 : Call price as a function of $b_0 + b_1$]

When $b_0 + b_1 = 1$, we get $b_0 + \alpha = 0, b_1 - \alpha = 1$ and the call price is equal to one. When $b_0 + b_1 \to +\infty$, there exists an underlying historical distribution such that the variance tends to zero and the stock geometric return is constant equal to the riskfree rate. Then $C(k) = \exp -r(\exp r - k)^+ = [1 - k\exp -r]^+$, where $\exp -r$ is introduced for discounting.

iv) A particular case

Finally let us consider the case $c = r$, where the mode of the historical distribution corresponds to the riskfree return. The risk correcting factor α is the solution of :

$$(b_0 + \alpha)(b_1 - \alpha) = (b_0 + \alpha + 1)(b_1 - \alpha - 1)$$

$$\iff \quad \alpha = \frac{b_1 - b_0}{2} - \frac{1}{2}.$$

By replacing in the expression of the call-prices, we get :

$$\begin{cases} C_1(k) = \dfrac{1}{2\bar{b}} \exp[-(\bar{b} - 1/2)(\log k - r)], \text{ if } \log k \geq r, \\ \\ C_2(k) = 1 - k\exp(-r) + \dfrac{1}{2\bar{b}} \exp[(\bar{b} + 1/2)(\log k - r)], \text{ if } \log k \leq r. \end{cases}$$

As mentioned above, the pricing formula depends on the single parameter $\bar{b} = \frac{b_0 + b_1}{2}$, which measures the average tail magnitude. This parameter $\bar{b}$ has the same role than the volatility σ in the Black-Scholes model. When $\bar{b}$ increases, the average tail decreases. The derivatives of the call prices with respect to $\bar{b}$, that is the analogues of the standard Black-Scholes vega, are :

$$\frac{\partial C_1}{\partial \bar{b}}(k) = -\frac{1}{2\bar{b}^2} \exp[-(\bar{b} - 1/2)(\log k - r)][1 + \bar{b}(\log k - r)], \text{ if } \log k \geq r,$$

$$\frac{\partial C_2}{\partial \bar{b}}(k) = -\frac{1}{2\bar{b}^2} \exp[(\bar{b} + 1/2)(\log k - r)][1 - \bar{b}(\log k - r)], \text{ if } \log k \leq r.$$

These derivatives are negative, which implies a decreasing relationship between the average tail magnitude $\bar{b}$ and the call price. By inverting the pricing formula, we can define the implied tail magnitude associated with any observed call price. The surface of implied Laplace tail magnitude contains the same information as the call-price surface.

It is interesting to consider the admissible call prices when the historical variance $\sigma^2 = \frac{1}{b_0^2} + \frac{1}{b_1^2}$ is known. Since the price is a monotonous function of $\bar{b}$, we get an interval of admissible prices, whose bounds are obtained for the values of b_0, b_1, which optimize $b_0 + b_1$ submitted to $\sigma^2 = \frac{1}{b_0^2} + \frac{1}{b_1^2}$. We easily deduce this interval, for instance when $\log k \geq r$. We get :

$$C_1(k) \in [0, \frac{\sigma}{2\sqrt{2}} \exp(-[\frac{\sqrt{2}}{\sigma} - \frac{1}{2}](\log k - r))], \text{ if } \sigma < 2\sqrt{2},$$

$$C_1(k) \in [0, 1], \text{ if } \sigma \geq 2\sqrt{2}.$$

The interval increases with σ, and is equal to $[0, 1]$ in the limiting case $\sigma = 2\sqrt{2}$. The latter interval is the largest one compatible with the free arbitrage inequalities, since the constraints $0 \leq (\exp y - k)^+ \leq \exp y, \forall k$, imply $0 \leq C(k) \leq 1$.

2.3 Pricing with splines

We extend the Laplace distribution by considering a conditional p.d.f., which is specified as an exponential-affine spline :

$$p(y) = \exp[a + yb_0 + \sum_{j=1}^{J} b_j (y - c_j)^+], \tag{2.6}$$

where a is fixed by the unit mass restriction, $c_1 < \ldots < c_J$ defines a partition of $I\!R, b_0 > 0, \sum_{j=0}^{J} b_j < 0$. With the convention $c_0 = -\infty, c_{J+1} = +\infty$, this conditional p.d.f. can also be written as :

$$p(y) = \exp[a - A_j + B_j y], \text{ if } y \in (c_j, c_{j+1}) \text{ for } j = 0, \ldots, J, \tag{2.7}$$

where :

$$
\begin{aligned}
A_j &= \sum_{l=1}^{j} b_l c_l \text{ (with } A_0 = 0), \\
B_j &= \sum_{l=0}^{j} b_l, \\
\exp a &= \left[\sum_{j=0}^{J} \frac{\exp(-A_j)}{B_j} (\exp B_j c_{j+1} - \exp B_j c_j) \right]^{-1}. \qquad (2.8)
\end{aligned}
$$

Thus the conditional historical distribution is a mixture of truncated exponential distributions :

$$
p_j(y) = B_j \frac{\exp B_j y}{\exp B_j c_{j+1} - \exp B_j c_j} \mathbb{1}_{(c_j, c_{j+1})}(y), \qquad (2.9)
$$

with weights :

$$
\begin{aligned}
\pi_j &= \frac{\exp(-A_j)}{B_j} (\exp B_j c_{j+1} - \exp B_j c_j) \\
&\quad [\Sigma_l \frac{\exp(-A_l)}{B_l} [\exp B_l c_{l+1} - \exp B_l c_l)]^{-1}. \qquad (2.10)
\end{aligned}
$$

Proposition 2 : If the conditional historical distribution is specified as an exponential-affine spline and if the stochastic discount factor is exponential-affine :

i) the conditional risk neutral distribution is unique and is an exponential-affine spline :

$$
\begin{aligned}
q(y) &= \exp[a^q + y(b_0 + \alpha) + \sum_{j=1}^{J} b_j (y - c_j)^+] \\
&= \sum_{j=0}^{J} \{\exp[a^q - A_j + (B_j + \alpha) y] \mathbb{1}_{(c_j, c_{j+1})}(y)\},
\end{aligned}
$$

where a^q is fixed by the unit mass restriction and α is solution of :

$$\exp r \sum_{l=0}^{J} \left\{ \frac{\exp(-A_l)}{B_l+\alpha} [\exp[(B_l+\alpha)c_{l+1}] - \exp[(B_l+\alpha)c_l]] \right\}$$

$$= \sum_{l=0}^{J} \left\{ \frac{\exp(-A_l)}{B_l+\alpha+1} [\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]] \right\}.$$

ii) The price of the call is given by :

$$\begin{aligned}
C(k) &= C_j(k) \\
&= [\sum_{l=0}^{J} \frac{\exp(-A_l)}{B_l+\alpha+1} \{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\}]^{-1} \\
& \quad [\frac{\exp((-A_j)}{B_j+\alpha+1} \{\exp[(B_j+\alpha+1)c_{j+1}] - \exp[(B_j+\alpha+1)\log k]\} \\
&- \frac{k\exp(-A_j)}{B_j+\alpha} \{\exp[(B_j+\alpha)c_{j+1}] - \exp[(B_j+\alpha)c_j]\} \\
&+ \sum_{l=j+1}^{J} \frac{\exp(-A_l)}{B_l+\alpha+1} \{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\} \\
&- k \sum_{l=1+1}^{J} \frac{\exp(-A_l)}{B_l+\alpha} \{\exp[(B_l+\alpha)c_{l+1}] - \exp[(B_l+\alpha)c_l]\},
\end{aligned}$$

for $\exp c_j \le k \le \exp c_{j+1}$.

Proof : See appendix 2.

In statistical theory the approximations by splines are usually introduced to estimate nonparametrically regression functions. The result of Proposition 2 can be used in a similar way for nonparametric pricing. [11] Indeed any conditional p.d.f. can be approximated as close as possible by an exponential

[11] Other nonparametric pricing methods are discussed in Gourieroux, Monfort (2001), Darolles, Gourieroux, Jasiak (2001).

affine spline, when the partition is increased. The proposition says that this approximation is appropriate for derivative pricing, since it provides compatible approximations for both the historical and risk neutral distributions.[12] These approximations can be used for cross-sectional pricing, that is for pricing at a given date and a given maturity, k varying. [see Aït-Sahalia (1996) for a similar approach]. The implementation is along the following lines :

i) Fix a partition $c_1, \ldots, c_J$;

ii) Estimate the parameters $b_j, j = 0, \ldots, J$ from either the historical distribution, or observed derivative prices [see Gourieroux, Jasiak (2001), for a discussion of these alternative estimation methods].

iii) Reconstitute the estimated historical and risk neutral distributions by replacing $b_j, j = 0, \ldots, J$ by their estimates.

3. The multiperiod framework

The aim of this section is to link the pricing formulas for different dates and various maturities. The dynamics is introduced in the conditional Laplace distributions [resp. exponential-affine splines] by means of the different types of parameters, b_0, b_1 and c [resp. b_j and c], which can be path dependent. It is easily checked that the Laplace family of distributions is generally not stable by time aggregation [13].[14] In the subsections below, we introduce a simple dynamics, where the effect of the past is assumed to be well summarized by the regime indicator giving the interval $(-\infty, c)$ or (c, ∞), [resp. (c_j, c_{j+1})] which contains the lagged value. We first describe the extension in the special case of the conditional Laplace distribution considered in subsection 2.2

[12] Clearly the conditional historical distribution can also be approximated by exponential spline of larger degree such as quadratic, or cubic spline. However for degree striclty larger than one, the corresponding approximation of the risk neutral distribution no more belongs to the class. Similarly the Hermite polynomial approximation proposed by Madan, Milne (1994) cannot be used in a coherent way for both the historical and risk neutral densities.

[13] A similar remark applies to the conditionally gaussian model, which underlies the Black-Scholes formula. The gaussian family of distributions is stable by time aggregation, if the conditional mean is affine and the conditional variance is constant. For more general gaussian specification in discrete time with path dependent mean and volatility, the computation of derivative prices at large maturities require intensive simulation techniques.

[14] See However Dewald, Lewis (1985) for an autoregressive model, which involves Laplace distributions.

before considering more general affine-exponential splines.

3.1 Dynamic Laplace model

Let us consider the framework of subsection 2.2 and introduce the dynamics. We assume a path independent location parameter c and define the regime indicator by :

$$z_t = \begin{cases} 1, & \text{if } y_t \geq c, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Moreover we assume that the conditional distribution of the geometric return y_{t+1} given the past $y_t, y_{t-1}, \dots$ is a skewed Laplace distribution, whose parameters depend on the past through the regime only. If we denote by $p(y_{t+1}|\underline{y_t})$ the conditional p.d.f of y_{t+1}, and $p(y; c, b_0, b_1)$ a Laplace p.d.f. with parameters c, b_0, b_1, we get :

$$\begin{aligned} p(y_{t+1}|\underline{y_t}) &= p(y_{t+1}; c, b_{00}, b_{10}), \text{ if } z_t = 0, \\ \\ & p(y_{t+1}; c, b_{01}, b_{11}), \text{ if } z_t = 1. \end{aligned}$$

It is easily checked that the qualitative process (z_t) defines a Markov chain with transition matrix :

$$\Pi = \begin{pmatrix} \Pi_{00} & \Pi_{01} \\ \\ \Pi_{10} & \Pi_{11} \end{pmatrix} = \begin{pmatrix} \dfrac{b_{10}}{b_{00}+b_{10}} & \dfrac{b_{00}}{b_{00}+b_{10}} \\ \\ \dfrac{b_{11}}{b_{01}+b_{11}} & \dfrac{b_{01}}{b_{01}+b_{11}} \end{pmatrix}, \tag{3.2}$$

where : $\Pi_{ij} = P[z_{t+1} = j | z_t = i]$.

Moreover the conditional historical distribution h steps ahead is :

$$p(y_{t+h}|\underline{y_t})$$

$$= p(y_{t+h}; c, b_{00}, b_{10})\Pi_{00}^{(h-1)} + p(y_{t+h}; c, b_{01}, b_{11})\Pi_{01}^{(h-1)}, \text{ if } z_t = 0,$$

$$p(y_{t+h}|\underline{y_t})$$

$$= p(y_{t+h}; c, b_{00}, b_{10})\Pi_{10}^{(h-1)} + p(y_{t+h}; c, b_{01}, b_{11})\Pi_{11}^{(h-1)}, \text{ if } z_t = 1,$$

where $\Pi_{i,j}^{(h-1)}$ is the element (i,j) of the matrix Π^{h-1}.

The exponential-affine stochastic discount factor for the period $t, t+1$ is : $M_{t,t+1} = \exp(\beta_t + \alpha_t y_{t+1})$, where α_t, β_t depend on the regime prevailing at date t. Thus we get different corrections (α_0, β_0) and (α_1, β_1) according to the regime. The conditional risk neutral distribution h steps ahead is :

$$q(y_{t+h}|\underline{y_t})$$

$$= p(y_{t+h}; c, b_{00} + \alpha_0, b_{10} - \alpha_o)\Pi_{00}^{q^{(h-1)}} + p(y_{t+h}; c, b_{01} + \alpha_1, b_{11} - \alpha_1)\Pi_{01}^{q^{(h-1)}}, \text{ if } z_t = 0,$$

$$= p(y_{t+h}; c; b_{00} + \alpha_0, b_{10} - \alpha_o)\Pi_{10}^{q^{(h-1)}} + p(y_{t+h}; c, b_{01} + \alpha_1, b_{11} - \alpha_1)\Pi_{11}^{q^{(h-1)}}, \text{ if } z_t = 1,$$

where :

$$\Pi^q = \begin{pmatrix} \dfrac{b_{10} - \alpha_0}{b_{00} + b_{10}} & \dfrac{b_{00} + \alpha_0}{b_{00} + b_{10}} \\ \\ \dfrac{b_{11} - \alpha_1}{b_{01} + b_{11}} & \dfrac{b_{01} + \alpha_1}{b_{01} + b_{11}} \end{pmatrix}.$$

We immediately deduce the price of a derivative whose payoff at $t+h$ is $(\exp y_{t+h} - k)^+$. It is given by :

$$C_t(k,h) = \exp[-r(h-1)][C(k; c, b_{00} + \alpha_0, b_{10} - \alpha_0)\Pi_{00}^{q^{(h-1)}} + C(k, c, b_{01} + \alpha_1, b_{11} - \alpha_1)\Pi_{01}^{q^{(h-1)}}], \text{ if } z_t = 0,$$

$$C_t(k,h) = \exp[-r(h-1)][C(k; c, b_{00} + \alpha_0, b_{10} - \alpha_0)\Pi_{10}^{q^{(h-1)}} + C(k, c, b_{01} + \alpha_1, b_{11} - \alpha_1)\Pi_{11}^{q^{(h-1)}}], \text{ if } z_t = 1,$$

where $C(k;c,b_0,b_1)$ is the call price at maturity one associated with the Laplace distribution $p(y;c,b_0,b_1)$. [15]

As an illustration, let us consider the special case $c = r$ and denote : $\bar{b}_0 = \dfrac{b_{00}+b_{10}}{2}, \bar{b}_1 = \dfrac{b_{01}+b_{11}}{2}$. We get :

$$\Pi^q = \begin{pmatrix} \dfrac{1}{2}+\dfrac{1}{4\bar{b}_0} & \dfrac{1}{2}-\dfrac{1}{4\bar{b}_0} \\ \\ \dfrac{1}{2}+\dfrac{1}{4\bar{b}_1} & \dfrac{1}{2}-\dfrac{1}{4\bar{b}_1} \end{pmatrix},$$

and, for $\log k \geq r, y_t \geq r$ for instance :

$$\begin{aligned} C_t(k,h) &= \exp[-r(h-1)][\frac{1}{2\bar{b}_0}\exp[-(\bar{b}_0-1/2)(\log k - r)]\Pi_{00}^{q^{(h-1)}} \\ &+ \frac{1}{2\bar{b}_1}\exp[-(\bar{b}_1-1/2)(\log k - r)]\Pi_{10}^{q^{(h-1)}}]. \end{aligned}$$

This example shows that this special dynamic model may capture different tail magnitudes in the different regimes, which is the analogue of stochastic volatility models.

3.2 Dynamic exponential-affine splines

The approach extends the dynamic Laplace model by introducing a larger number of regimes. The regimes are defined by means of a partition $c_j, j = 0,\ldots,J$, which is assumed path independent. Then the multiregime indicator at date t is :

$$z_t = j, \text{ if } y_t \in [c_j, c_{j+1}], j = 0,\ldots,J. \tag{3.3}$$

If $c = (c_1,\ldots,c_J), b = (b_0,\ldots,b_J)'$ denote the two types of parameters, we assume that the conditional distribution of the geometric return is a an

[15] Note that a call written on S_{t+h} with moneyness strike k is proportional to a call with cash-flow $(S_{t+h}/S_t - k)^+ = (\exp(y_{t+1}+\ldots+y_{t+h}) - k)^+$. Due to the aggregation of geometric returns, the associated price has no simple expression and has to be computed by simulation.

exponential spline affine distribution, which depends on the past by means of the most recent regime :

$$p(y_{t+1}|\underline{y_t}) = p(y_{t+1}|z_t). \tag{3.4}$$

These conditional distributions differ by the value of parameter b which is not the same in each regime. b^j denotes the value of b, when $z = j$:

$$p(y_{t+1}|z_t) = p(y_{t+1}; c, b^j), \text{ if } z_t = j. \tag{3.5}$$

As in the previous subsection, the qualitative process (z_t) defines a Markov chain, with a transition matrix Π, with elements $\pi_{ij} = P[z_{t+1} = j|z_t = i]$ functions of the basic parameters $c, b^j, j = 0, \ldots, J$.

Then the conditional historical distribution h steps ahead is :

$$p(y_{t+h}|y_t) = \sum_{j=0}^{J} p(y_{t+h}; c, b^j)\pi_{i,j}^{(h-1)}, \text{ if } z_t = i, \tag{3.6}$$

whereas the conditional risk-neutral density is :

$$q(y_{t+h}|y_t) = \sum_{j=0}^{J} p(y_{t+h}; c, \tilde{b}^j)\tilde{\pi}_{ij}^{(h-1)} \text{ if } z_t = i, \tag{3.7}$$

where : $\tilde{b}_0^j = b_0 + \alpha^j, \tilde{b}_l^j = b_l^j$, if $l = 1, \ldots, J$, and $\tilde{\pi}$ is deduced from π after replacement of b^j by $\tilde{b}^j$.

This dynamic approach is the basis for dynamic nonparametric pricing under the assumption of a Markov process for geometric return. Indeed when the partition $c_j, j = 0, \ldots, J$ increases, the exponential-affine spline approximation with multiregime will tend to the conditional p.d.f. $p(y_{t+1}|y_t)$ itself. It provides numerical approximations for call-prices in the Markov framework, which mix a standard multinomial tree with a spline smoothing. The numerical advantage of the additional smoothing is to diminish the erratic evolutions of the approximated derivative prices, which are usually observed when the number of nodes in the tree increases.

4. Concluding remarks

The success of the Black-Scholes approach is due to a simple analytical formula for european call prices. However this formula is based on restrictive assumptions and may induce various mispricing. For instance the implied volatility has to be constant with the moneyness strike, whereas smile effects are often observed ; it has to be independent of the time to maturity, whereas an increasing dependence may be observed. Moreover it is varying with time and environment, since it neglects time dependency. The aim of this paper was to introduce alternative analytical formulas, which can be used to approximate the derivative prices for given date and residual maturity. We first derive a pricing formula for the skewed conditional Laplace distribution, before extending the analysis to exponential-affine splines. This leads to a nonparametric pricing approach. Finally, we introduce underlying Markov regimes in order to link the derivative prices for different dates and residual maturities.

Appendix 1 :
Pricing with Laplace distribution

i) The truncated Laplace transform

Let us assume $\gamma > c$ and $u < b_1$; we get :

$$\begin{aligned}
\psi(u,\gamma) &= E[\exp(uy)\mathbb{1}_{y>\gamma}] \\
&= \exp(uc)E\{\exp[u(y-c)]\mathbb{1}_{y>\gamma}\} \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_\gamma^\infty \exp[-(b_1-u)(y-c)]dy \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-u)(\gamma-c)]}{b_1-u}.
\end{aligned}$$

If $\gamma < c$, we get :

$$\begin{aligned}
\psi(u,\gamma) &= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_c^\infty \exp[-(b_1-u)(y-c)]dy \\
&+ \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_\gamma^c \exp[(b_0+u)(y-c)]dy \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_1-u} \\
&+ \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_0+u}\{1-\exp[(b_0+u)(\gamma-c)]\}.
\end{aligned}$$

Note that the truncated Laplace transform is defined for $u \in (-b_0, b_1)$.

ii) The arbitrage free conditions

If $-b_0 < u < b_1$ the Laplace transform is given by :

$$\begin{aligned}\psi(u,-\infty) &= exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_1-u}+\exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_0+u}\\ &= \exp(uc)\frac{b_0 b_1}{(b_0+u)(b_1-u)}.\end{aligned}$$

Thus the arbitrage free conditions become :

$$\begin{cases}\exp(\beta+r)\psi(\alpha,-\infty)=1,\\ \\ \exp(\beta)\psi(\alpha+1,-\infty)=1,\end{cases}$$

$$\Longleftrightarrow \quad \begin{cases}\exp(\beta+r+\alpha c)\dfrac{b_0 b_1}{(b_0+\alpha)(b_1-\alpha)}=1,\\ \\ \exp[\beta+(\alpha+1)c]\dfrac{b_0 b_1}{(b_0+\alpha+1)(b_1-\alpha-1)}=1.\end{cases}$$

In particular the risk correcting factor is the solution of the second degree equation, satisfying $-b_0<\alpha<b_1-1$:

$$\exp(c-r)(b_0+\alpha)(b_1-\alpha)=(b_0+\alpha+1)(b_1-\alpha-1).$$

It is easily checked that this equation has a unique solution in the interval $(-b_0,b_1-1)$, where the Laplace transforms $\psi(\alpha,-\infty)$ and $\psi(\alpha+1,-\infty)$ are both defined.

iii) **The price of the call**.

For $logk>c$, we get :

$$
\begin{aligned}
C(k) &= \exp\beta[\psi(\alpha+1, logk) - k\psi(\alpha, logk)] \\
&= \exp\beta\left\{\exp[(\alpha+1)c]\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-\alpha-1)(logk-c)]}{b_1-\alpha-1}\right. \\
&\quad \left. -k\exp(\alpha c)\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-\alpha)(logk-c)]}{b_1-\alpha}\right\} \\
&= \exp\beta\exp[(\alpha+1)c]\exp[-(b_1-\alpha-1)(logk-c)]\frac{b_0 b_1}{b_0+b_1}\frac{1}{(b_1-\alpha)(b_1-\alpha-1)} \\
&= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}\exp[-(b_1-\alpha-1)(\log k - c)],
\end{aligned}
$$

by the arbitrage free condition.

The computation is similar for $logk < c$ and provides :

$$
C(k) = 1 - k\exp(-r) + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_o+\alpha}\exp[(b_0+\alpha+1)(logk-c)].
$$

iv) **Continuity of the pricing function**.

The value of the call is a continuous function of k. Indeed we get :

$$
\begin{aligned}
C_1(\exp c) &= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}, \\
C_2(\exp c) &= 1 - \exp(c-r) + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_0+\alpha} \\
&= 1 - \frac{(b_0+\alpha+1)(b_1-\alpha-1)}{(b_0+\alpha)(b_1-\alpha)} + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_0+\alpha} \\
&= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}
\end{aligned}
$$

The continuity property is still satisfied for the derivative of the value of the call with respect to k. Indeed the first order derivative of the pricing function is :

$$
\begin{aligned}
\frac{dC_1(k)}{dk} &= -\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)}\exp(-c)\exp[-(b_1-\alpha)(\log k - c)], \\
\frac{dC_2(k)}{dk} &= -\exp(-r)+\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)}\exp(-c)\exp[(b_0+\alpha)(\log k - c)].
\end{aligned}
$$

At the limiting point $k = \exp c$, we get :

$$
\begin{aligned}
\frac{dC_1(\exp c)}{dk} &= -\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)}\exp(-c), \\
\frac{dC_2(\exp c)}{dk} &= -\exp(-r)+\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)}\exp(-c).
\end{aligned}
$$

We get :

$$
\begin{aligned}
\frac{dC_1(\exp c)}{dk} &= \frac{dC_2(\exp c)}{dk} \\
\Longleftrightarrow \quad \exp(c-r) &= \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)}+\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)} \\
&= \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_1-\alpha)(b_0+\alpha)},
\end{aligned}
$$

which is exactly the equation defining α

v) **Risk neutral distribution**

The p.d.f. of the risk neutral distribution is still a Laplace distribution. Indeed this p.d.f. is given by :

$$
\begin{aligned}
q(y) &= \exp(r)\frac{b_0 b_1}{b_0+b_1}\exp(\beta+\alpha c)\exp[(b_0+\alpha)(y-c)], \text{ if } y \leq c, \\
& \exp(r)\frac{b_0 b_1}{b_0+b_1}\exp(\beta+\alpha c)\exp[-(b_1-\alpha)(y-c)], \text{ if } y \geq c.
\end{aligned}
$$

By using the arbitrage free condition, we get :

$$q(y) = \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[(b_0+\alpha)(y-c)], \text{ if } y \leq c,$$

$$\frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[-(b_1-\alpha)(y-c)], \text{ if } y > c.$$

Finally it is easily checked that the risk neutral distribution depends on b_0, b_1 through b_0+b_1 and c only. This property is satisfied if both $\alpha_0 = b_0+\alpha$ and $\alpha_1 = b_1-\alpha$ depend on b_0+b_1 and c only. It is easily seen that α_0 and α_1 are solutions of the equations :

$$\exp(c-r)\alpha_0(b_0+b_1-\alpha_0) = (\alpha_0+1)(b_0+b_1-\alpha_0-1),$$

$$\exp(c-r)\alpha_1(b_0+b_1-\alpha_1) = (\alpha_1-1)(b_0+b_1-\alpha_1+1),$$

and the result follows.

Appendix 2 :
Pricing with exponential-affine splines

i) **The historical distribution**

The distribution is given by :

$$p(y) = \exp[a + b_0 y + \sum_{j=1}^{J} b_j (y - c_j)^+],$$

where the constant a is fixed by the constraint of unit mass. This p.d.f. can also be written as :

$$p(y) = \exp(a - A_j + B_j y), \text{ if } y \in (c_j, c_{j+1}),$$

where :

$$\begin{aligned} A_j &= \sum_{l=1}^{j} b_l c_l \text{ (with) } A_0 = 0), \\ B_j &= \sum_{l=1}^{j} b_l. \end{aligned}$$

Then the integral of the p.d.f. is :

$$\begin{aligned} \int_{-\infty}^{+\infty} p(y)dy &= \sum_{j=0}^{J} \int_{c_j}^{c_{j+1}} \exp(a - A_j + B_j y) dy \\ &= \exp(a) \sum_{j=0}^{J} (\exp(-A_j) \frac{\exp B_j y}{B_j}]_{c_j}^{c_{j+1}}) \\ &= \exp(a) \sum_{j=0}^{J} \frac{\exp -A_j}{B_j} [\exp B_j c_{j+1} - \exp B_j c_j]. \end{aligned}$$

We deduce the expression of the p.d.f :

$$
\begin{aligned}
p(y) &= \sum_{j=0}^{J} \{\exp[-A_j + B_j y]\mathbf{1}_{(c_j,c_{j+1})}(y)\} \left\{\sum_{j=0}^{J} \frac{\exp -A_j}{B_j}(\exp B_j c_{j+1} - \exp B_j c_j)\right\}^{-1} \\
&= \left\{\sum_{j=0}^{J} \frac{\exp -A_j}{B_j}(\exp B_j c_{j+1} - \exp B_j c_j)\frac{B_j \exp B_j y}{\exp B_j c_{j+1} - \exp B_j c_j}\mathbf{1}_{(c_j,c_{j+1})}(y)\right\} \\
&\quad \left\{\sum_{j=0}^{J} \frac{\exp -A_j}{B_j}(\exp B_j c_{j+1} - \exp B_j c_j)\right\}^{-1}.
\end{aligned}
$$

ii) **The truncated Laplace transform :**

Let us assume $\gamma \in (c_j, c_{j+1})$; we get :

$$
\begin{aligned}
\psi(u,\gamma) &= E[\exp(uy)\mathbb{1}_{y>\gamma}] \\
&= \int_{\gamma}^{c_{j+1}} \exp(a - A_j + B_j y + uy)dy \\
&+ \sum_{l=j+1}^{J} \int_{c_l}^{c_{l+1}} \exp(a - A_l + B_l y + uy)dy \\
&= \frac{\exp(a - A_j)}{B_j + u}\{\exp[(B_j + u)c_{j+1}] - \exp[(B_j + u)\gamma]\} \\
&+ \sum_{l=j+1}^{J} \frac{\exp(a - A_l)}{B_l + u}\{\exp[(B_l + u)c_{l+1}] - \exp[(B_l + u)c_l]\}.
\end{aligned}
$$

iii) **The arbitrage free conditions**

The (untruncated) Laplace transform is given by :

$$\psi(u,-\infty) = \sum_{l=0}^{J} \frac{\exp(a - A_l)}{B_l + u}\{\exp[(B_l + u)c_{l+1}] - \exp[(B_l + u)c_l]\},$$

and the correcting factor α is solution of the equation :

$$\exp(r)\psi(\alpha, -\infty) = \psi(\alpha + 1, -\infty)$$

or equivalently :

$$\exp(r)\sum_{l=0}^{J}\left\{\frac{\exp(-A_l)}{B_l+\alpha}(\exp[(B_l+\alpha)c_{l+1}]-\exp[(B_l+\alpha)c_l])\right\}$$

$$=\sum_{l=0}^{J}\left\{\frac{\exp(-A_l)}{B_l+\alpha+1}(\exp[(B_l+\alpha+1)c_{l+1}]-\exp[(B_l+\alpha+1)c_l])\right\}.$$

iv) **The risk-neutral distribution**

By multiplying the historical p.d.f by the exponential stochastic discount factor, we get a risk-neutral density with an exponential-affine spline representation. The limiting points of the partition $c_j, j=1,\ldots,J$ are unchanged, whereas the parameters of the truncated exponential distributions become : $B_j^q = B_j+\alpha$. Since : $B_j^q=\sum_{l=0}^{j} b_l^q$, we immediately deduce that :

$$b_0^q = b_0+\alpha, b_j^q = b_j, j=1,\ldots,J,$$

$$A_j^q = A_j, j=0,\ldots,J.$$

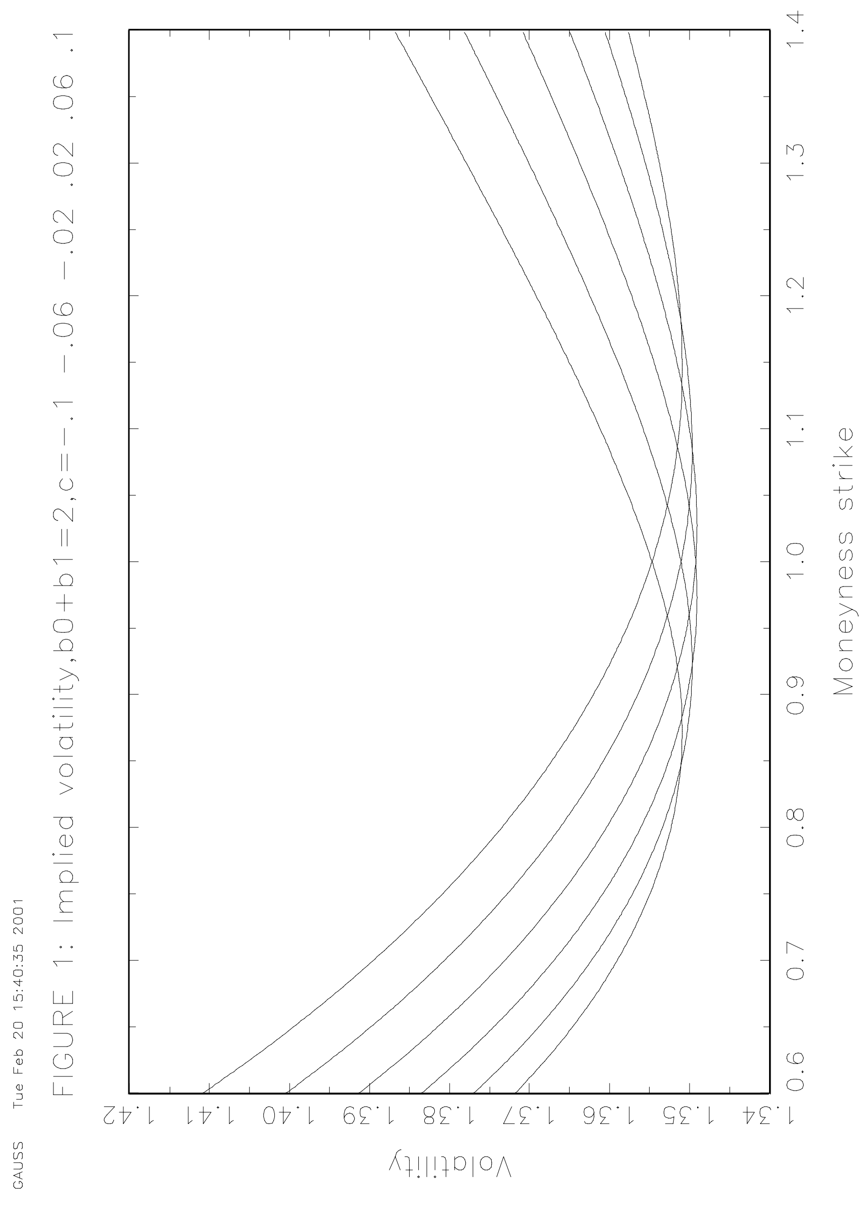
Thus the risk-neutral p.d.f. is :

$$\begin{aligned} q(y) &= \exp[a^q + y(b_0+\alpha)+\sum_{j=1}^{J} b_j(y-c_j)^+] \\ &= \sum_{j=0}^{J}[\exp[a^q - A_j+(B_j+\alpha)y]\mathbb{1}_{(c_j,c_{j+1})}(y)]. \end{aligned}$$

v) **The price of a call**

Let us assume $\gamma\in(c_j,c_{j+1})$; the price of a call is given by :

$$
\begin{aligned}
C(k) &= \frac{1}{\psi(\alpha+1,-\infty)}[\psi(\alpha+1,\log k) - k\psi(\alpha,\log k)] \\
&= [\sum_{l=0}^{J} \frac{\exp(-A_l)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\}]^{-1} \\
& \quad [\frac{\exp(-A_j)}{B_j+\alpha+1}\{\exp[(B_j+\alpha+1)c_{j+1}] - \exp[(B_j+\alpha+1)\log k]\} \\
&- k\frac{\exp(-A_j)}{B_j+\alpha}\{\exp[(B_j+\alpha)c_{j+1}] - \exp[(B_j+\alpha)\log k]\} \\
&+ \sum_{l=j+1}^{J} \frac{\exp(-A_l)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\} \\
&- k\sum_{l=j+1}^{J} \frac{\exp(-A_l)}{B_l+\alpha}\{\exp[(B_l+\alpha)c_{l+1}] - \exp[(B_l+\alpha)c_l]\}]
\end{aligned}
$$


GAUSS
Tue Feb 20 15:40:35 2001
FIGURE 1: Implied volatility,b0+b1=2,c=-.1 -.06 -.02 .02 .06 .1
Volatility
1.34 1.35 1.36 1.37 1.38 1.39 1.40 1.41 1.42
Moneyness strike
0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4

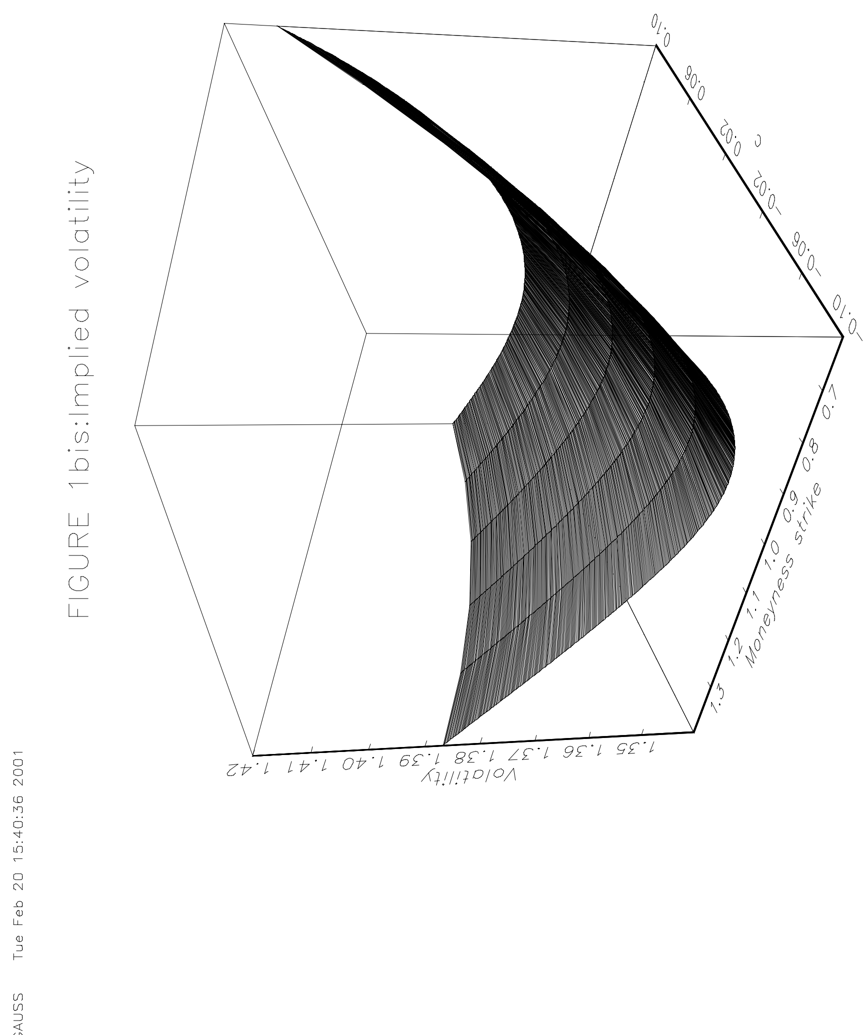


FIGURE 1bis:Implied volatility

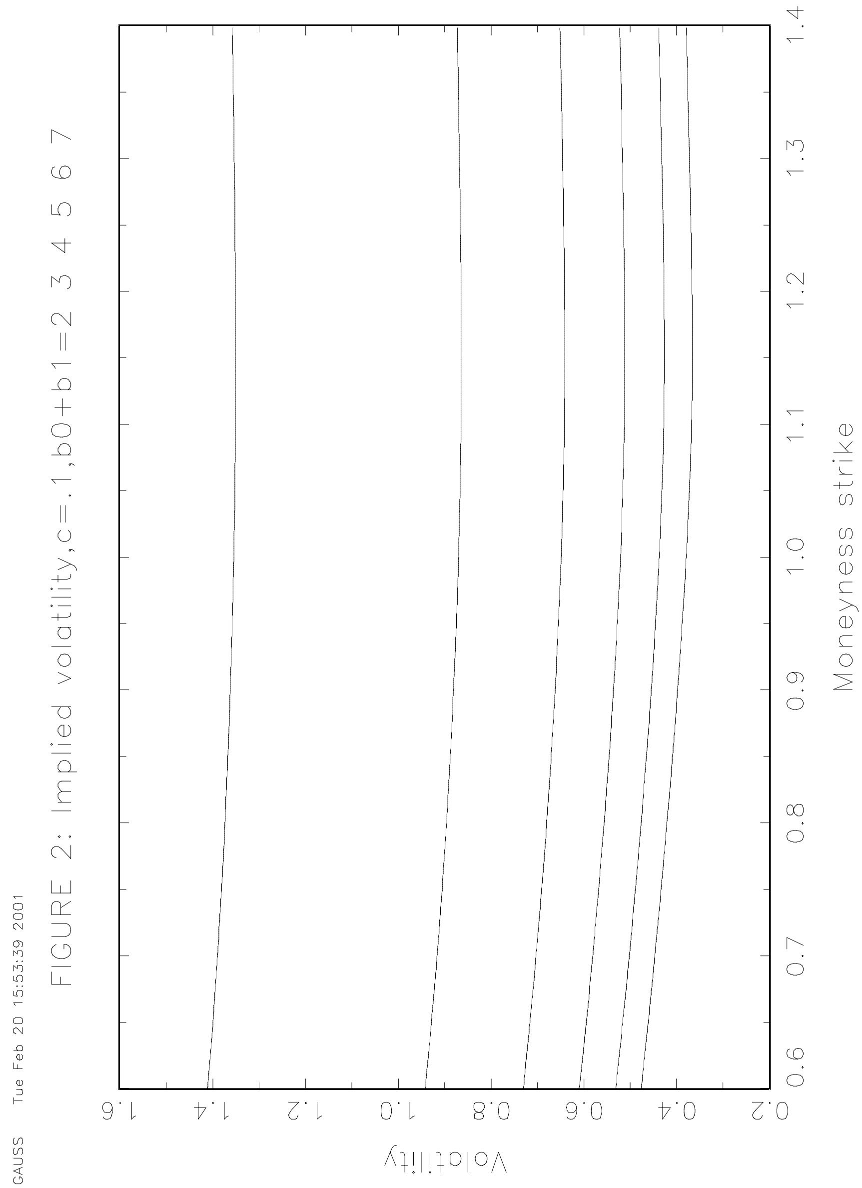
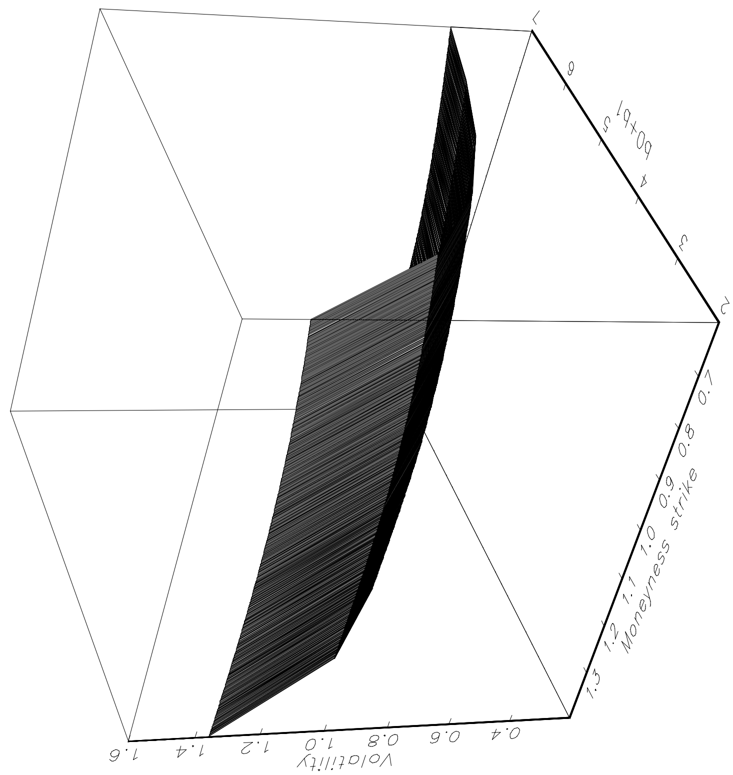
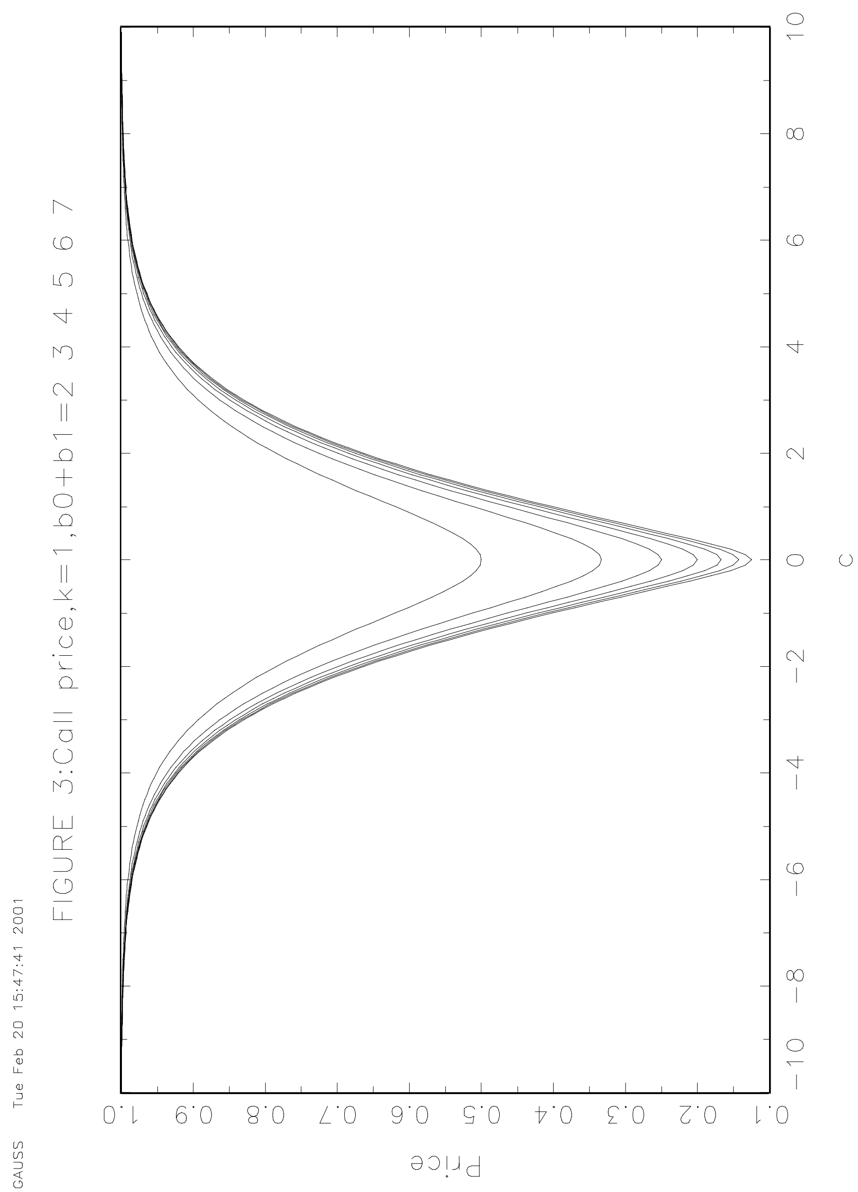


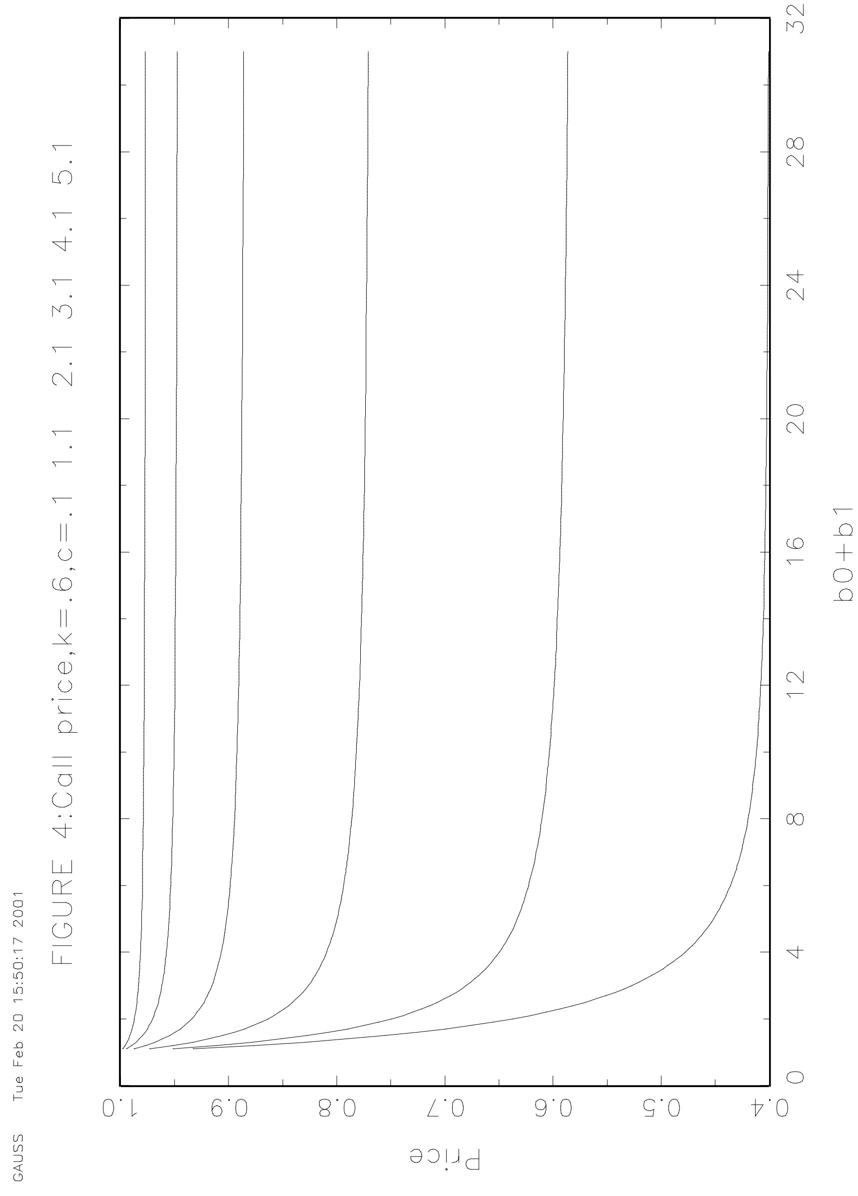
FIGURE 2: Implied volatility,c=.1,b0+b1=2 3 4 5 6 7

FIGURE 2bis :Implied volatility



GAUSS Tue Feb 20 15:58:51 2001


GAUSS Tue Feb 20 15:47:41 2001
FIGURE 3:Call price,k=1,b0+b1=2 3 4 5 6 7
Price
c
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
−10
−8
−6
−4
−2
0
2
4
6
8
10


GAUSS
Tue Feb 20 15:50:17 2001
FIGURE 4:Call price,k=.6,c=.1 1.1 2.1 3.1 4.1 5.1
Price
1.0
0.9
0.8
0.7
0.6
0.5
0.4
b0+b1
0
4
8
12
16
20
24
28
32

REFERENCES

Abken, P., Madan, D., and S., Ramamurtie (1996) : "Estimation of Risk Neutral and Statistical Densities by Hermite Polynomial Approximation : With an Application to Eurodollar Futures Options", Federal Reserve Bank of Atlanta.

Ait-Sahalia, Y. (1996) : "Nonparametric Pricing of Interest Rate Derivative Securities", Econometrica, 64, 527-560.

Ait-Sahalia, Y., and A., Lo (1995) : "Nonparametric Estimation of State Price Densities Implicit in Financial Asset Prices",

Bahra, B. (1996) : "Probability Distribution of Future Asset Prices Implied by Option Prices", Bank of England Quartely Bulletin, August, 299-311.

Ball, C., and A., Roma (1994) : "Stochastic Volatility Option Pricing", Journal of Financial and Quantitative Analysis, 29, 589-607.

Ball, C., and W., Torous (1985) : "On Jumps in Common Stock Prices and their Impact on Call Option Pricing", Journal of Finance, 50, 155-173.

Banz, R., and H., Miller (1978) : "Prices of State Contingent Claims : Some Estimates and Applications", Journal of Business, 51, 653-671.

Bates, D. (1996) : "Jumps and Stochastic Volatility : Exchange Rate Processes Implicit in Deutsche Mark Options", Journal of Financial Studies, 9, 69-107.

Black, F., and M., Scholes (1973) : "The Pricing of Options and Corporate Liabilities", Journal of Political Economy, 81, 637-659.

Breeden, D., and R., Litzenberger (1978) : "Prices of State Contingent Claims Implicit in Option Prices", Journal of Business, 51, 621-651.

Brennan, M. (1979) : "The Pricing of Contingent Claims in Discrete Time Models", Journal of Finance, 34, 53-68.

Buhlman, H., Delbaen, F., Embrechts, P. and A., Shiryaev (1996) : "No Arbitrage, Change of Measure and Conditional Esscher Transforms in a Semi-Martingale Model of Stock Processes", CWI Quarterly, 9, 291-317.

Campa, J., Chang, P., and R., Reider (1997) : "ERM Bandwidths for EMU and After : Evidence from Foreign Exchange Options", Economic Policy, 55-87.

Campbell, J., Lo, A., and C., McKinlay (1997) : "The Econometrics of Financial Markets", Princeton Univ. Press.

Canina, L., and S., Figlewski (1973) : "The Informational Content of Implied Volatility", Review of Financial Studies, 6, 659-681.

Chesney, M., and L., Scott (1989) : "Pricing European Currency Options : A Comparison of the Modified Black-Scholes and a Random Variance Model", Journal of Financial and Quantitative Analysis, 24, 267-284.

Cochrane, J. (2001) : "Asset Pricing", Princeton University Press.

Cox, J., and M., Rubinstein (1985) : "Options Markets", Prentice Hall.

Darolles, S., Gourieroux, C., and J., Jasiak (2001) : "Compound Autoregressive Models", CREST DP.

Day, T., and C., Lewis (1988) : "The Behaviour of the Volatility Implicit in the Prices of Stock Index Options", Journal of Financial Economics, 22, 103-122.

Day, T., and C. Lewis (1992) : "Stock Market Volatility and the Information Content of Stock Index Options", Journal of Econometrics, 52, 267-288.

Derman, E., and I., Kani (1994) : "Riding on a Smile", Risk, 7, 32-39.

Dewald, L., and Lewis, P. (1985) : "A New Laplace Second Order Autoregressive Time Series Model NLAR(2)", IEEE Trans. Inf. Theory, 31, 645-651.

Duan, J.C., and J.G., Simonato (1999) : "American Option Pricing under GARCH by a Markov Chain Approximation", forthcoming Journal of Economic Dynamics and Control.

Dumas, B.,Fleming, J. and R. Whaley (1998) : "Implied Volatility Functions : Empirical Tests", Journal of Finance, 53, 2059-2106.

Dupire, B. (1994) : "Pricing with a Smile", Risk, 7, 18-20.

Esscher, F. (1932) : "On the Probability Function in the Collective Theory of Risk", Skandinavisk Aktuarietidskrift, 15, 175-195.

Garcia, R., Luger, R., and E., Renault (2000) : "Asymmetric Smiles, Leverage Effects and Structural Parameters", D.P. Montreal Univ.

Garcia, R., and E., Renault (1998) : "Risk Aversion, Intertemporal Substitution and Option Pricing", DP Montreal Univ.

Gerber, H., and E., Shiu (1994) : "Option Pricing by Esscher Transforms", Transactions of the Society of Actuaries, XLVI, 99-191.

Gouriéroux, C., and J., Jasiak (2001) : "Financial Econometrics", Princeton Univ.Press.

Gouriéroux, C., and A., Monfort (2001) : "Econometric Specifications of Stochastic Discount Factor Models", CREST DP.

Hansen, L., and S., Richard (1987) :"The Role of Conditioning Information in Deducing Testable Restriction Implied by Dynamic Asset Pricing Models", Econometrica, 55, 587-613.

Heston, S. (1993) : "A Closed Form Solution for Options with Stochastic Volatility with Application to Bond and Currency Options", Review of Financial Studies, 6, 327-343.

Huang, C., and R., Litzenberger (1988) : "Foundations for Financial Economics", North-Holland.

Hull, J. (1989) : "Options, Futures and Other Derivatives", Third Edition, Prentice Hall.

Hull, J., and A., White (1987) : "The Pricing of Options on Assets with Stochastic Volatilities", Journal of Finance, 42, 281-300.

Jarrow, R., and A., Rudd (1989) : "Approximate Valuation for Arbitrary Stochastic Processes", Journal of Financial Economics, 347-369.

Longstaff, F. (1992) : "An Empirical Examination of the Risk Neutral Valuation Model", Working Paper Anderson Graduate School of Management, UCLA.

Madan, D., and F., Milne (1994) : "Contingent Claims Valued and Hedged by Pricing and Investing in a Basis", Mathematical Finance, 4, 223-245.

Malz, A. (1996) : "Using Option Price to Estimate Realignment Probabilities in the European Monetary System : The Case of Sterling Mark", Journal of International Money and Finance, 717-748.

Melick, W., and C. Thomas (1997) : "Recovering on Asset's Implied PDF from Option Prices : An Application to Crude Oil During The Gulf Crisis", Journal of Financial and Quantitative Analysis, 32, 91-116.

Melino, A., and S., Turnbull (1990) : "Pricing Foreign Currency Options with Stochastic Volatility", Journal of Econometrics, 45, 239-265.

Merton, R. (1973) : "The Theory of Rational Option Pricing", Bell Journal of Economics and Management Science, 4, 141-183.

Merton, R. (1976) : "Option Pricing when Underlying Stock Returns are Discontinuous", Journal of Financial Economics, 125-144.

Ross, S. (1976) : "Options and Efficiency", Quarterly Journal of Economics, 90, 75-89.

Rubinstein, M. (1976) : "The Valuation of Uncertain Income Streams and the Pricing of Options", Bell Journal of Economics, 7, 407-425.

Shiryaev, A. (1999) : "Essentials of Stochastic Finance : Facts, Models, Theory", World Scientific Publishing, London.

Stein, E., and J., Stein (1991) : "Stock Price Distribution with Stochastic Volatility : An Analytical Approach", Review of Financial Studies, 4, 727-752.

Stutzer, M. (1996) : "A Simple Nonparametric Approach to Derive Security Valuation", Journal of Finance, 1633-1652.

Wiggins, J. (1987) : "Option Values under Stochastic Volatility : Theory and Empirical Estimates", Journal of Financial Economics, 19, 351-372.