A Note on Asset Bubbles in Continuous-Time

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Abstract

In this paper we propose a model of asset prices consistent with the no-arbitrage principle but allowing for the existence of "bubbles" which are explicitely characterized. From a mathematical point of view the main theorem may be read as a measure-theoretic interpretation of local martingales.

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1. Introduction

It is a common sense observation that financial prices may at times appear uncorrelated to all main economic indicators. In some equilibrium models the price of assets exceeds the discounted, expected value of intermediate returns (more briefly their fundamental value) i.e. there exist price bubbles (see, among others, [22], [24], [13]). The noteworthy property of equilibrium models with bubbles is that these emerge as a fully rational phenomenon, and particularly the fundamental principle is respected according to which, if agents prefer more to less, there cannot exist any arbitrage opportunity on the market. An arbitrage opportunity consists of a portfolio with no initial cost and non negative terminal payoff being strictly positive with positive probability. The Fundamental Theorem of Asset Pricing (FTAP) asserts that the absence of arbitrage profits is equivalent to the existence of a pricing rule: a strictly positive linear functional mapping assets (discounted) payoffs into prices. Further to that, it is often the case that the pricing rule admits the representation as an expected value. It is clear that in models in which this theorem holds true - like the ones with

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a finite state space - bubbles cannot arise since assets prices, emerging from the maximizing behaviour of agents concerned with investments over finite horizons, may in these economies be evaluated from the perspective of no future resale.

It is true and well known, at least since the seminal paper of Kreps [17] (but see also [3] and [5]), that in general spaces the FTAP may fail. In particular it is the implication from absence of arbitrage opportunities to the existence of a pricing rule that breaks down (the reverse one remains clearly true). The valuation operator that, in the absence of arbitrage opportunities, maps cumulative payoffs into prices may lack continuity or strict positivity, two properties that are deeply connected when the underlying space is such that the positive orthant has non empty interior. The proof of the theorem is usually accomplished by reformulating the no arbitrage condition as the statement that two convex sets - the set of final payoffs of portfolios with no initial cost and the strictly positive orthant - have empty intersection and remarking that the positive linear functional separating the two acts as the valuation operator. This line of proof suggests itself a way out: pricing discontinuities follow from the lack of topological properties of the sets to be separated. One may then reformulate the no arbitrage principle into a mathematically more convenient notion, the so called absence of free lunches. The choice of the topology is indeed the crucial step here.

In mathematical finance prices, inclusive of dividend yields, are treated as stochastic processes. It is therefore particularly convenient to establish the existence of a pricing rule and that this may be represented as an equivalent probability law with respect to which prices behave as martingales, more briefly an equivalent martingale measure. Although this version of the FTAP has been proved only rather recently (see [6]), the martingale machinery that it brings into play is so powerful that actually most models treat the existence of an equivalent martingale measure as a starting assumption. In these models pricing bubbles forcefully remain outside of the picture.

In the present paper we investigate the properties of a model of financial prices in continuous time which is free of arbitrage opportunities but still does not admit equivalent martingale measures: bubbles arise in a fairly natural way. As remarked in [19] (the first paper to adopt this setting) continuous time improves our comprehension by adding new features with respect to the discrete time case. The main point is that bubbles may arise over a finite horizon, so that their path need not grow unboundedly.

The building block of our model is the notion of martingale density (first introduced in [23]), which is considerably weaker than that of an equivalent martingale density. Prices are turned into local martingales by virtue of a state price process which is the product of the discount factor with a positive local martingale. Technically speaking, unlike the case of equivalent martingale measures, the density process lacks here uniform integrability. We start by describing a simple example of a model with two financial assets with bounded prices and show that this mathematical requirement immediately translates into an explicit mispricing. In fact we construct a sequence of contingent claims consisting of an exchange CALL option with different maturities and such that their price does not converge to 0 as the strike price diverges to infinity. This is shown to be due to the existence of a pricing bubble regardless of the fact that the underlying assets are bounded and the time horizon is finite. This example illustrates the discontinuity of the valuation operator to which we referred above.

This result can be put into a much more general setting. To do so we establish a fundamental relationship between (local) martingales and charges, or finitely additive measures. One of the main attractiveness of models admitting equivalent martingale measures is that prices can be viewed as expected values and pricing formulas are considerably simple. In our setting the first outcome still holds true safe that the intervening measure is not countably additive - so that pricing is not as clean. This fact is particularly interesting and its first important implication is caught as a model for bubbles since it allows a clear-cut decomposition of the asset price into its fundamental value and the bubble, based on the decomposition of Yosida and Hewitt. We establish that pricing bubbles may have two different sources, either unbounded asset prices or lack of uniform integrability of the density process: both components are identified explicitly. The connection between lack of countable additivity of the pricing measure and bubbles was first introduced in [10] and is indeed intuitive. In models with an infinite state space (for example an infinite set of dates) it is often useful to consider finite sub models by restricting the agents strategies (e.g. portfolios) to be supported by a given finite set of states. Consider approximating the unrestricted model by finite ones with increasing support. Since finite models admit no bubbles, these can only arise on the complement of such restricted sets, which decrease though to the empty set. A valuation operator that assigns a positive price to assets characteristics which emerge on sets of arbitrarily small dimension, cannot be represented by a countably additive set function.

Another aspect under which our construction proves to be useful is the FTAP itself. Although various examples have been given in which models that admit martingale densities are arbitrage free, a general prove that this implication is indeed direct is missing. We cannot prove that absence of arbitrage opportunities in financial models implies the existence of a martingale density. This result, much sought, has been established only for very special models in which prices have continuous sample paths (see [7] and [18]). Nevertheless we prove that martingale densities exclude arbitrage opportunities; we also show, rather easily, that whenever such processes remain strictly positive in the limit, then we are back to the case of an equivalent martingale measure.

The paper is organized as follows. After having introduced the set up, in section 3 we construct the example that was discussed already in which it is established a strong connection between martingale densities, bubbles and discontinuity of the pricing rule. In section 4 we construct the main tool of our analysis, consisting of the relationship between martingales and finitely additive measures. This result is obtained in some generality in order to recover a notion of conditional expectation with respect to the measure constructed which is not obvious for finitely additive set functions. This is important when we want to consider the model from time t onwards: a much simpler proof could be obtained if we confined our interest in time $t = 0$. In the following section 5 we apply the previous result to general pricing formulas and obtain, as main implication, an explicit decomposition of asset prices into the fundamental value and the bubble, explaining how bubbles may arise even for bounded prices over a finite interval. Eventually, in section 6 we examine the problem of arbitrage, proving the aforementioned results. Here some recent work has been done by Lowenstein and Willard, mainly in [20] (but see also [19]). In these papers, in which the main focus is the notion of viability of financial prices that is not addressed here, a different notion of arbitrage is proposed, based on a discussion of the role of credit constraints. Our approach differs from theirs exactly on this point, and most closely refer to [6]. Furthermore we consider general semimartingales as our model for prices rather than diffusion processes.

2. The model

In the following paragraphs we will work with a stochastic basis $\left(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\in\mathbf{T}}\right)$ that is with a probability space $(\Omega, \mathcal{F}, P)$ and a time set $\mathbf{T} \subseteq \mathbb{R}_+ \cup \{\infty\}$ on which the evolution of information is modeled via a right continuous, increasing family of sub σ algebras $\mathcal{F}_t$ of $\mathcal{F}$ - more concisely, a filtration. We set $\mathcal{F} = \vee_{t\in\mathbf{T}}\mathcal{F}_t \equiv \mathcal{F}_\infty$. A martingale is a process adapted to the filtration and such that (i) M_t is integrable for each $t \in \mathbf{T}$ and (ii) if $s < t$, $s, t \in \mathbf{T}$ then $E\left(M_t | \mathcal{F}_s\right) = M_s$. A *local* martingale is a process $M = (M_t)_{t\in\mathbf{T}}$ that admits a localizing sequence - i.e. a sequence $\langle\tau_k\rangle_{k\in\mathbb{N}}$ of stopping times which increases to ∞ a.s. - such that for each k the process $M^k = \left(M_{t\wedge\tau_k}\right)_{t\in\mathbf{T}}$ is a martingale. To stress the distinction with local martingales, martingales will sometimes be referred to as *true* martingales. A uniformly integrable martingale is a martingale M such that the family $\{M_t : t \in \mathbf{T}\}$ of random variables is uniformly integrable.

If X is a semimartingale we shall write $X = M + V$ for his decomposition (with M a local martingale and V of finite variation while predictable only if S is special): then X^d is the jump part of X while $\hat{X}^h$ is the process obtained by summing the jumps of absolute value greater than $h > 0$. $\hat{X}^h$ is known to be of finite variation and $X^h = X - \hat{X}^h = V^h + M^h$ is a special martingale together with its canonical decomposition.

The "cum dividend", discounted price process S will be modeled as a d dimensional semimartingale. No other assumption is needed right now but we will later discuss the implications of the existence of upper or lower bounds.

If Q is a new probability measure equivalent to P, then the Radon-Nikodým derivative

$$Z_t = \frac{dQ_{\mathcal{F}_t}}{dP_{\mathcal{F}_t}}$$

- where $P_{\mathcal{F}_t}$ and $Q_{\mathcal{F}_t}$ denote the restrictions of P and Q respectively to the σ algebra $\mathcal{F}_t$ - can be taken to be a strictly positive uniformly integrable martingale such that $E(Z_t) = Z_0 = 1$. If, further to these properties, ZS is itself a (local) martingale, then Q is said to be an equivalent (local) martingale measure for S[1]. Whenever Z and ZS are simply *local* martingales, with Z positive, then Z is said to be a martingale density. More formally

Definition 1. *Let $\mathfrak{F}$ be a family of adapted stochastic processes. The positive local martingale Z such that $Z_0 = 1$ is a martingale density for $\mathfrak{F}$ if Zf is a local martingale for each $f \in \mathfrak{F}$*

This definition was initially given by Schweizer ([23]): it is though deprived of a clear interpretation in measure theoretic terms. In the following sections we will assume that financial prices admit a martingale density and treat the various implications of this assumption. The only issue we shall consider here is to understand what kind of restriction does this assumption impose on asset prices.

Theorem 2.1. *Let $S = M + V$, $h > 0$, $S = S^h + \hat{S}^h = M^h + V^h + \hat{S}^h$ be the above mentioned decompositions of S. Let furthermore $\mathcal{L}(Z)$ be the unique solution to the stochastic differential equation $\int \frac{1_{\{Z_- > 0\}}}{Z_-} dZ$. Then the following conditions are equivalent to Z being a martingale density for S.*

1. $(V + [\mathcal{L}(Z), M])^p = 0$;
2. $\begin{cases} \left(V^h + \left[\mathcal{L}(Z), M^h\right]\right)^p = 0 \\ \left(\hat{S}^h\right)^p = 0 \end{cases}$

[1] For a general reference on so-called martingale problems and equivalent martingale measures see Jacod and Shiryaev ([14]).

3. $\begin{cases} \left(V^c + \left[\mathcal{L}(Z), M^c\right]\right)^p = 0 \\ \left(V^d + \left[\mathcal{L}(Z), M^d\right]\right)^p = 0 \end{cases}$

Although completely standard this result generalizes the conclusions of [2] and helps clarifying some aspects of our theory. First if there exists an asset whose price is a predictable process of finite variation, then it has to be the numeraire: for this reason we will consider normalized prices as our strating point. Sometimes it has conjectured that special semimartingales are the most general model for viable asset prices in continuous time: in this case V can be taken to predictable and since $V = V^h + \hat{S}^h$, $\hat{S}^h$ itself must be predictable so that, by 2.b it must be true that $\hat{S}^h = 0$ and, h being arbitrary, S is therefore constrained to be continuous. As a consequence, if we want to consider the case of possibly discontinuous price paths, we must be prepared to consoder the case of general semimartingales, as we shall do in the following sections.

The connection between martingale densities and asset bubbles is rather straightforward and can be readily obtained as an application of Fatou's lemma. Let to this end $\mathfrak{F}$ in the above definition be the class of lower bounded processes f, describing the net, discounted revenue from a given set of financial investments, and a martingale density Z for $\mathfrak{F}$: For a given initial capital $W_0^f \in \mathbb{R}_+$ we have that the gross (discounted) revenue $W_0^f + f$ will be positive a.s. at each date. Let $\mathfrak{G} = \left\{W_0^f + f : f \in \mathfrak{F}\right\}$: clearly Z is a martingale density for $\mathfrak{G}$ as well. By definition there exists a sequence $\langle \tau_k \rangle_{k \in \mathbb{N}}$ of finite stopping times increasing a.s. to ∞ and such that $Z^k \equiv \left(Z_{t \wedge \tau_k}\right)_{t \in \mathbf{T}}$ is a *uniformly* integrable martingale; the process $Z^k g^k$, $g \in \mathfrak{G}$, is then itself a uniformly integrable martingale. A simple application of Fatou's lemma gives

$$\begin{aligned} Z_t g_t &= \lim_k Z_{t \wedge \tau_k} g_{t \wedge \tau_k} \\ &= \lim_k E\left(Z_{\tau_k} g_{\tau_k} \middle| \mathcal{F}_t\right) \\ &\geq E\left(Z_\infty g_\infty \middle| \mathcal{F}_t\right) \end{aligned} \tag{2.1}$$

for all $g \in \mathfrak{G}$. In all cases in which the preceding inequality holds strictly, then we have a "mispricing" of the portfolio g, given that time t price deviates from its "fundamental value" expressed by the expected value of its cumulative future returns $E\left(Z_\infty f_\infty \middle| \mathcal{F}_t\right)$. We may therefore write

$$g_t = \phi_t(g) + \beta_t(g) \tag{2.2}$$

where

$$\phi_t(g) = E\left(\frac{Z_\infty}{Z_t} g_\infty \middle| \mathcal{F}_t\right)$$

represents the fundamental value of the asset, while $\beta_t(g)$ represents the non-negative process describing the "bubble" that arises in the valuation of the asset. One of the main point of this paper is investigating the structure of β and to this end we will need some machinery, to be developed in section (4). The results of that section will also help achieving the subordinate goal of extending the treatment of bubbles outside the narrow range of Fatou's lemma, namely positive prices (actually (2.2) holding for $g = W_0^f + f$ need not hold for f). Before that we shall construct an interesting example of asset prices admitting martingale densities and bubbles.

3. An Example.

This example[2] is based on Ito's lemma and time changed properties of Brownian motion. The conclusions we will reach in Proposition 1 are though completely general and apply to any economy with complete financial markets. Let $\mathbf{T} = [0,1]$ and $\mathfrak{B} = \left(\Omega, \mathcal{F}, P; \left(\mathcal{F}_t^B\right)_{t\in[0,1]}\right)$ where $\left(\mathcal{F}_t^B\right)_{t\in[0,1]}$ is the natural filtration of the Brownian motion B. For any function $\theta : [0,1] \to \mathbb{R}$ define

$$\varphi_t(\theta) = \int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds$$

Consider now a deterministic, differentiable function of time k_s: then by ordinary rules of the stochastic integral one gets

$$\varphi_t(kB^n) = \frac{1}{1+n}\left\{k_t B_t^{1+n} - \int_0^t \left[\left(B_s^{1+n}k_s' + \frac{1+n}{2}B_s^{2n}k_s^2\right) + \frac{n^2}{2}B_s^{n-1}k_s\right]dt\right\}$$

Choose $n = 1$, $k_s = -2(1-s)^{-2}$ and define, for $a \le 1$,

$$\tau^a = \inf\left\{t \in [0,1] : B_t^2 = a(1-t)\right\} \tag{3.1}$$

Set $\varkappa(t) = (1-t)^{-\frac{1}{2}}$ and remark that $B_t \varkappa_t$ is isomorphic to the Brownian motion $\left(\hat{B}, \mathcal{F}_t^{\hat{B}}\right)_{t\in\mathbb{R}_+}$ where $\hat{B}_t = B_{\varkappa^{-1}(t)}$. As a consequence one may conclude that P a.s. any level a is reached by $\hat{B}$ in finite time, that is $P(\tau^a < 1) = 1$. Putting things together we then have

$$\begin{aligned}
\varphi_{\tau^a}(kB^n) &= -\frac{a}{(1-\tau^a)} - \frac{1}{2}\int_0^{\tau^a} B_s^2 s ds - \int_0^{\tau^a}\left(\frac{1}{1-s}\right)^2 ds \\
&\le -\frac{a}{(1-\tau^a)} - \int_0^{\tau^a}\left(\frac{1}{1-s}\right)^2 ds \\
&\le -a
\end{aligned}$$

[2] A few number of others may be found in [18], p. 232-238.

Define then

$$dS_{1,t} = S_{1,t}\alpha_t dt + S_{1,t}\sigma_t dB_t$$

with $\alpha_t = 2B_t 1_{\{t\leq\tau^a\}}$ and $\sigma_t = 1_{\{t\leq\tau^a\}}(1-t)^{-2}$ and $\theta = -\alpha\sigma^{-1}$. Let

$$dZ_t = Z_t\theta_t dB_t$$

By the above arguments we have

$$\begin{aligned} P\left(\int_0^1 \sigma_t^2 dt < \infty\right) &= P\left(\frac{1}{3}\left[\frac{1}{(1-\tau^a)^3} - 1\right] < \infty\right) \\ &= P(\tau^a < 1) \\ &= 1 \end{aligned}$$

and - since $B_t \leq a$ on $\{t \leq \tau^a\}$ -

$$\begin{aligned} P\left(\int_0^1 \theta_t^2 dt < \infty\right) &= P\left(4\int_0^{\tau^a} B_t^2 (1-t)^4 dt < \infty\right) \\ &\geq P\left(4a^2 < \infty\right) \\ &= 1 \end{aligned}$$

so that the above equations admit as unique solutions

$$S_{1,t} = \exp\left(\int_0^t \left(2B_s - \frac{1}{2(1-s)^4}\right) 1_{\{s\leq\tau^a\}} ds + \int_0^t \frac{1_{\{s\leq\tau^a\}}}{(1-s)^2} dB_s\right)$$

$$Z_t = \exp\left(-\int_0^t \frac{2B_s 1_{\{s\leq\tau^a\}}}{(1-s)^2} dB_s - \int_0^t \frac{2B_s^2 1_{\{s\leq\tau^a\}}}{(1-s)^4} ds\right)$$

The price process S_1 is bounded and

$$dZ_t S_{1,t} = Z_t S_{1,t} \frac{1-2B_s}{(1-s)^2} 1_{\{s\leq\tau^a\}} dB_s$$

i.e.

$$Z_t S_{1,t} = \exp\left(\int_0^t \frac{1-2B_s}{(1-s)^2} 1_{\{s\leq\tau^a\}} dB_s - \frac{1}{2}\int_0^t \left(\frac{1-2B_s}{(1-s)^2}\right)^2 ds\right)$$

which is clearly a positive, continuous local martingale. Define now the price process

$$S_{2,t} = \frac{E\left(Z_\infty | \mathcal{F}_t\right)}{Z_t}$$

with the convention $\frac{0}{0} = 0$: then clearly S_2 is a bounded semimartingale and furthermore $ZS_2 = E(Z_\infty | \mathcal{F}_t)$ is a P strictly positive, uniformly integrable martingale so that Z is a martingale density for $S = (S_1, S_2)$.

Consider now a CALL option to receive the first asset at the price of γ units of the second that may be exercised by the holder at any random time τ_n along the localizing sequence for Z: its time $t = 0$ price is then

$$\begin{aligned} p_{\gamma,n}(0) &= L\left((S_{1,\tau_n} - \gamma S_{2,\tau_n})^+\right) \\ &= E\left((Z_{\tau_n} S_{1,\tau_n} - \gamma E(Z_\infty))^+\right) \end{aligned}$$

Let $p^*_\gamma = \sup_n p_{\gamma,n} = \lim_n p_{\gamma,n}$. If γ increases to ∞ then necessarily $p^*_\infty(t) = 0$ for each t since the corresponding option pays nothing with probability 1.

Proposition 1. *Pricing bubbles do not exist if and only if* $\lim_{\gamma\to\infty} p^*_\gamma(0) \to 0$.

Indeed the above example shows how uniform integrability of the density process translates into a pricing anomaly. Observe that the proposition remains valid after replacing S_1 by the price of the riskless asset - equal to 1 upon normalization - : in this case we can clearly see that there is a bubble on the riskless asset. Clearly, asset 2 is of a very special nature, and it may seem that this makes the whole example rather non generic. Nevertheless, since its price is strictly bounded, it can easily be replicated by the remaining assets in an economy with complete financial markets.

It is clear that the pricing functional in this example lacks continuity when the topology is that of convergence in measure or L^p and the question if the described price system is viable (i.e. supports the choice of some agents with monotonic preferences) is rather naturally.

4. Martingale Charges

In this section we are going to introduce the main tool of analysis needed for what follows: the main reult of this section, which may be of interest per se, will provide us with much insight in the theory of bubbles.

The main question linked to martingale densities is whether they still provide a well defined pricing rule as equivalent martingale measures. It turns out that the answer to this question also clarifies the measure theoretic interpretation of martingales, which is straightforward only as far as uniformly integrable martingales are concerned.

A true martingale is clearly as good a pricing and mathematical model as long as attention is restricted to finite dates (or dates preceding the terminal one, if the time-set is bounded).

The same conclusion applies to the local martingale if we restrict attention to $\mathcal{F}_{\tau_n}$ measurable assets (events), on which $L(A) = E(Z_{\tau_n} 1_A)$ is a well defined (and countably additive) measure. In the preceding example in fact we used this as our pricing rule. $\mathcal{A} = \bigcup_n \mathcal{F}_{\tau_n}$ is an algebra of subsets of Ω and the problem here is how to extend it to $\sigma(\mathcal{A})$. In the uniformly integrable case it is equivalent to define, for $A \in \mathcal{F}$, either $Q(A) = E(Z_\infty 1_A)$ or $Q(A) = \lim_t E(Z_t 1_A)$: in the case of martingale densities the former is not an extension of $E(Z_{\tau_n} 1_A)$ while $E(Z_{\tau_n} 1_A)$ does not converge for each $A \in \mathcal{F}$. Since $\delta(A) = \limsup_n E(Z_{\tau_n} 1_A)$ exists for each $A \in \mathcal{F}$, majorizes (actually coincides with) Q on $\mathcal{A}$ and is sub additive, we can use Hahn-Banach theorem to extend L in such a way that $L(A) \leq \limsup_n E(Z_{\tau_n} 1_A)$: this also implies $L(A) \geq \liminf_n E(Z_{\tau_n} 1_A)$ so that L is positive. By construction L is associated to a positive and continuous linear functional on $L^\infty(\Omega, \mathcal{F}, P)$ and is therefore a finitely additive measure, i.e. a charge.

We may consider L as the main ingredient of the pricing rule associated to Z: in this case though we only have a notion of the price of an asset at time $t = 0$. At times $t > 0$ one would be tempted to construct the conditional expectation of L with respect to the σ algebra $\mathcal{F}_t$: it though follows from the failure of existence of exact Radon-Nykodim derivatives for finitely additive measures that such conditional expectation does not exist in general. We shall therefore construct a notion of conditional expectation $L_\mathcal{G}$ on the σ algebra $\mathcal{G}$ which is much weaker tha the usual one, describe its properties and show some special cases in which it is actually a true conditional expectation. In the special case $\mathcal{G} = \{\Omega, \emptyset\}$, $L_\mathcal{G} = L$ and the proof of the following result, that we give with more details and in several steps in the appendix for the general case, goes true fairly easily.

For convenience we write Z_n and $\mathcal{F}_n$ rather than Z_{τ_n} than $\mathcal{F}_{\tau_n}$ respectively.

Theorem 4.1. *Let Z be a nonnegative martingale on the stochastic basis $\mathfrak{B} = \left(\Omega, \mathcal{F}, P; (\mathcal{F}_n)_{n\in\mathbb{N}}\right)$, $\mathcal{G}$ a σ algebra on Ω and let $\mathbb{X} = L^1(\Omega, \mathcal{G}, P)$. Then Z is associated to a finitely additive set function $L_\mathcal{G}$ of bounded variation defined on $(\Omega, \mathcal{F})$ that takes values in the Banach space $\mathbb{X}^{**}$, vanishes on P null sets and satisfies the property that if $\mathcal{G}$ is such that $E((Z_{n+1} - Z_n) b | \mathcal{G}) = 0$ for $b \in L^\infty(\Omega, \mathcal{F}_n, P)$ then the restriction $L_{\mathcal{G},n}$ of $L_\mathcal{G}$ to $\mathcal{F}_n$ admits the representation*

$$L_{\mathcal{G},n}(A) = E(Z_n 1_A | \mathcal{G}) \tag{4.1}$$

1. *$L_\mathcal{G}$ is countably additive if and only if Z is a uniformly integrable martingale and this is necessarily the case if $L_\mathcal{G}$ is unique.*

2. *If $E(Z_n | \mathcal{G})$ is uniformly integrable with respect to n then $L_\mathcal{G}(A) = \lim_n E(Z_n 1_A | \mathcal{G})$ (a.s.) and $L(A | \mathcal{G}) = L_\mathcal{G}(A) L_\mathcal{G}(\Omega)^{-1}$.*

3. *There exists a unique decomposition*

$$L_{\mathcal{G}} = L_{\mathcal{G}}^{c} + L_{\mathcal{G}}^{\perp} \tag{4.2}$$

with $L_{\mathcal{G}}^{c}$ *and* $L_{\mathcal{G}}^{\perp}$ *strongly additive measures and such that* $|L_{\mathcal{G}}| = |L_{\mathcal{G}}^{c}| + |L_{\mathcal{G}}^{\perp}|$, *for any* $\varepsilon > 0$ *and any measure* m *on* $(\Omega, \mathcal{F})$ *then* $|m|(A^c) + |L_{\mathcal{G}}|(A) < \varepsilon$ *for some* $A \in \mathcal{F}$ *and* $L_{\mathcal{G}}^{c}(A) = E(Z_\infty 1_A | \mathcal{G}_\infty)$.

The exact pricing rule associated to Z and L will be described in the next section, in which we will show its role as a pricing measure. However, we can highlight some of the features of the measure L right away. First of all, L is not unique unless we are in the case of equivalent martingale measures. In mathematical finance the existence of a multiplicity of pricing measures is related to (but actually often interpreted as) incompleteness of financial markets[3], that is tp the case in which some contingent claims are not attainable by existing financial strategies. The above example clarifies that this need not be the case here: in fact replacing the filtration with $(\mathcal{G}_t)_{t\in\mathbf{T}}$ where $\mathcal{G}_t = \mathcal{F}_{t\wedge\tau^a}$ and $\mathcal{F}$ by $\mathcal{G}_1$ leaves the price process a semimartingale and $B_t^a = B_{t\wedge\tau^a}$ brownian motion generating $(\mathcal{G}_t)_{t\in\mathbf{T}}$. Standard techniques on martingale representation over the brownian filtration apply.

5. Continuous-Time Finance and Bubbles

We now move to the applications of the previous result to Mathematical Finance. The main object of interest is of course the vector valued semimartingale process S which describes the process of asset prices discounted by the money market account. The stochastic integral $\int \theta dS$ describes the process of gains from trade. At each time t the wealth W_t of an investor with initial capital w and choosing portfolio strategy θ will satisfy the inequality

$$W_t \leq w + \int_0^t \theta_u dS_u \tag{5.1}$$

P a.s. where the inequality becomes strict only at times when withdrawals (for example for consumption needs) are considered. Negative values of W_t represent therefore contingencies in which extra funding is needed further to the initial endowment w. This possibility is clearly going to be subject to some restriction and it turns out that such feature plays an important role. No credit institution is likely to be involved in providing credit to investors

[3]The equivalence between the two statements is often referred to as the Second Fundamental Theorem of Asset Pricing. The first proof was given in [12].

that pursue portfolio strategies whose cost may increase without bound (even if with probability that converges to 0). On the other side it is unreasonable that credit constraints are set independently of the strategy θ and its proceeds. In fact any θ such that the integral $\int \theta dS$ turns eventually positive a.s. guarantees full repayment of extra credit received. A reasonable restriction, that is customary to adopt, is to impose tat the process $\int \theta dS$ be bounded from below in L^∞. Of course other choices may be considered as well: as is well known these must be tailored to leave out of the set of admissible trading strategies the so called "doubling" strategies.

As remarked in [19], martingale densities are a model for asset bubbles in continuous time. In fact, imposing $S^i \geq 0$ we can apply Fatou lemma as in equation (2.1) so that

$$Z_t S_t \geq E\left(Z_\infty S_\infty \middle| \mathcal{F}_t\right) \tag{5.2}$$

This point can be made clear by using the result of the preceding section. In fact we have that

$$Z_t S_t = \lim_n Z_{\tau_n \wedge t} S_{\tau_n \wedge t} = \lim_n \lim_k E\left(Z_{t\wedge\tau_k} S_{t\wedge\tau_k} \middle| \mathcal{F}_{t\wedge\tau_n}\right)$$

Let $\mathcal{G}_n = \mathcal{F}_{t\wedge\tau_n}$ and $\mathcal{G} = \mathcal{F}_t$. Observe that, in terms of the notation of Lemma (??) $\mathcal{G}_n$is orthogonal to $\langle Z_{t\wedge\tau_k}\rangle_{k>m}$ and that $S_{t\wedge\tau_k}$ is $\mathcal{A}_m = \bigcup_{k>m} \mathcal{F}_{t\wedge\tau_k}$ measurable so that the hypotheses of the letter (2) of the Lemma are trivially satisfied. Then

$$\begin{aligned} Z_t S_t &= \lim_n \lim_k L\left(S_{\tau_k} \middle| \mathcal{G}_n\right) \\ &= \lim_r \lim_n \lim_k \left\{ L^c\left(S_{\tau_k} \wedge r \middle| \mathcal{G}_n\right) + L^c\left((S_{\tau_k} - r)^+ \middle| \mathcal{G}_n\right) + L^\perp\left(S_{\tau_k} \middle| \mathcal{G}_n\right)\right\} \\ &= L^c\left(S_\infty \middle| \mathcal{G}\right) + \lim_r \lim_n \lim_k L^c\left((S_{\tau_k} - r)^+ \middle| \mathcal{G}_n\right) + \lim_n \lim_k L^\perp\left(S_{\tau_k} \middle| \mathcal{G}_n\right) \end{aligned} \tag{5.3}$$

i.e.

$$S_t = \phi_t(S) + \beta_t^u(S) + \beta_t^\perp(S)$$

Equation (5.3) makes clear what are the sources of price bubbles. In our model bubbles may arise because prices are unbounded - this is a necessary but in general not a sufficient condition for $\beta^u(S) \neq 0$. This finding is common to most papers on this subject. If prices are bounded, though, necessarily $\beta^u(S) = 0$ but there could still exist price bubbles provided $\beta^\perp(S) \neq 0$: this is the case when $L_{\mathcal{F}_t}$ is not countably additive. Observe that indeed the decomposition (5.3) applies to more general contexts than non negative prices. For prices bounded from below, for example, we still have $\beta^u(S) \geq 0$ although $\beta^\perp(S)$ may take on either sign; the extension to other more general cases is also possible at the price of some further modification. As a result we also have that bounded prices may display negative bubbles.

Some further insight on the role of bubbles can be obtained from (5.3). Contrary to [19], in which this condition is necessary (at least as long as agents are allowed to have monotonic preferences over consumption at the terminal date), we have no reason to assume that Z_∞ is strictly positive a.s. or, equivalently, that L^c is equivalent to P - a condition that will be important in the next section. Suppose for example that $P(\inf_t Z_t = 0) = 1$: then by Doob submartingale inequality (applied to $X_t = 1 - Z_t$) we conclude that $P(\sup_t X_t \geq 1) \leq E\left(1_{\{\sup_t X_t \geq 1\}} X_\infty\right)$ from which $E\left(1_{\{\inf_t Z_t = 0\}} Z_\infty\right) = 0$ i.e. $Z_\infty = 0$ a.s.. In all cases in which $P(Z_\infty = 0) > 0$ there are contingent claims that have no fundamental value and in general the operator ϕ underestimates the value of the asset in a way that may be significant. Contrary to common intuition, in our model bubbles contribute to a correct evaluation of assets at least in the case of a non strictly positive martingale density. In the next section we will explore the connection between bubbles and the existence arbitrage opportunities: it will be shown that $P(Z_\infty = 0) > 0$ is the only case in which bubbles are compatible with economic equilibrium.

6. Arbitrage.

Absence of arbitrage opportunities is a key concept in mathematical finance and we want to investigate it in the case of martingale densities. This fundamental concept that has though been defined in different ways over the years. Since the seminal paper of Kreps [17] it has been recognized that the most natural and economic meaningful definition, an admissible trading strategy bearing no cost but positive profits, need not be sufficient to recover "well behaved" price functionals. Kreps examines abstract topological spaces endowed with an order relationship; see also [5]. The notion of free lunches is exactly the desired mathematical reinforcement. In continuous time models, the most significant and important notion is that of No Free Lunch with Vanishing Risk (introduced in [6]) that we now briefly describe.

Let $a > 0$ and $\mathfrak{F}_a = \left\{(\theta \cdot S)_{t \in \mathbf{T}} : \theta \in \Theta_a\right\}$: given our definition of Θ_a, if Z is a martingale density for the price process S then it is so for $\mathfrak{F}_a$ as well. Define then

$$\begin{aligned} \mathbb{K}_a &= \left\{F \in L^0(\Omega, \mathcal{F}, P) : F = \lim_t f_t,\ f \in \mathfrak{F}_a\right\} \\ \mathbb{K} &= \bigcup_{a>0} \mathbb{K}_a \end{aligned}$$

$\mathbb{C}_a = \left(\mathbb{K}_a - L^0_+\right) \cap L^\infty$, $\mathbb{C} = \bigcup_{a>0} \mathbb{C}_a$ and let $\overline{\mathbb{C}}$ be the strong closure of $\mathbb{C}$ in L^∞. Then *NFLVR* is defined by the condition

$$\overline{\mathbb{C}} \cap L_+^\infty = \{0\} \tag{6.1}$$

while absence of arbitrage opportunities (NA) may be defined as

$$\mathbb{K}_a \cap L_+^\infty = \{0\} \tag{6.2}$$

It turns out that (6.1) is necessary and sufficient for the existence of an equivalent martingale measure (at least when S is bounded, or of an equivalent local martingale measure when S is locally bounded, see [6]). This conclusion is obtained thanks to the following, difficult result

Lemma 1 (Delbaen and Schachermayer 94, Kabanov 97). *Under* NFLVR, $\overline{\mathbb{C}} = \overline{\mathbb{C}}^*$ *(where* $\overline{\mathbb{C}}^*$ *denotes the closure of* $\mathbb{C}$ *in the weak** *topology).*

It has to be remarked that, contrary to what might seem at a first glance, the definition that NA does not depend on the choice of the real number a, provided $a > 0$ (an equivalent definition would in fact be $\mathbb{C} \cap L_+^\infty = \{0\}$, since for $b = \lambda a$, $\mathbb{K}_b = \lambda \mathbb{K}_a$). This is indeed a crucial parameter and synthetizes the role of credit constraints: we only consider investments that can be financed by borrowing a limited amount of capital. L^p bounds have also been considered in the literature. In tow important papers (see [20] and [19]) the definition of an arbitrage opportunity is confined to the case where $a = 0$. This corresponds to the view of exogenous credit constraints: if investors have to meet a fixed borrowing ceiling over the whole of their life then they have a limited opportunity to exploit arbitrage unless this is perfectly costless. Adopting this extreme version it is easy to show that the existence of a martingale density Z is a sufficient condition for absence of arbitrage opportunities, provided $Z_\infty > 0$ a.s.: simply $E(Z_\infty F) \leq \lim_k E(Z_{\tau_k} f_{\tau_k}) = 0$ i.e. $F = 0$ a.s. for any $F \geq 0$. In more general cases the relationship between martingale densities and absence of arbitrage has not been clarified yet, and some aspects of it will be treated in the next proposition. The motivation for a higher degree of generality comes from the intuition that credit constraints are typically endogenous (even if we have no model of equilibrium here) and would then depend on the expected level of income available to agents, including gains from trade. Existence of true arbitrage opportunities would then discard any bounded constraint.

Weak absolute continuity between two finitely additive measures m and μ on an algebra $\mathcal{A}$ of subsets of the space Ω is defined as the condition $m(A) = 0$ when $\mu(A) = 0$ - and denoted by $\mu \gg^w m$ - for any $A \in \mathcal{A}$. In the case of true measures this is equivalent to the usual definition of absolute continuity.

Proposition 2. *Let Z be a martingale density and L the finitely additive measure associated to it.*

1. *If $L \gg^w P$, then (6.2) is satisfied;*
2. *If $L^c \gg^w P$, then (6.1) is satisfied.*

The case in which $Z_\infty > 0$ a.s. is somehow in contrast with the existence of price bubbles, as one deduces from the preceding proposition and lemma, at least for bounded asset prices (a different proof of this result is in [7]). If this is the case, in fact, martingale densities can be replaced by uniformly integrable martingales (although the two need not coincide, at least in the case of incomplete financial markets). Remark also that the existence of a positive (in the sense of the measure L) martingale density does not imply at all $L(F) \leq 0$: in fact if this inequality were true we would still be back in the NFLVR. Eventually observe that a positive martingale density implies a bit more than absence of arbitrage. Let to this end $\mathbb{C}_n$ be the set of those elements g in $\mathbb{C}$ such that $g \leq nH$ and set $\mathbb{C}_0 = \bigcup_n \mathbb{C}_n$: then $\mathbb{C}_0 \subset \mathbb{C} = \overline{\mathbb{C}_0}^P$ (where $\overline{\cdot}^0$ denotes the $L^0(P)$ closure) and $L\left[\overline{\mathbb{C}_0}^L\right] \leq 0$.

7. Mathematical Appendix

In this Appendix we shall give the detailed proof of all the results presented in the preceding sections.

7.1. Section3

Proposition 3. *Pricing bubbles do not exist if and only if* $\lim_{\gamma\to\infty} p^*_\gamma(0) \to 0$.

Proof. Choosing $\gamma_c = c\left(e^{-a}+1\right)^{-1}$ we have $E\left(Z_{\tau_n} - c\right)^+ \leq p_{\gamma_c,n}$. If $E\left(Z_{\tau_n} - c\right)^+$ converges to 0 uniformly with respect to n, then for each $A \in \mathcal{F}$, $\lim_n E\left(Z_{\tau_n} 1_A\right) = E\left(Z_\infty 1_A\right)$. By Vitali Hahn Saks theorm it follows that $E\left(Z_{\tau_n} 1_{A_r}\right)$ converges to 0 as $P(A_r) \to 0$, uniformly in n: by Doob convergence theorem we know that $\sup_n Z_{\tau_n}$ is finite almost surely so that we can choose $A_r = \{\sup_n Z_{\tau_n} > r\}$. Then

$$E\left(Z_{\tau_n} 1_{\{Z_{\tau_n} > c\}}\right) \leq E\left(Z_{\tau_n} 1_{\{\sup_n Z_{\tau_n} > r\}}\right)$$

and the result follows. On the other side, if Z is uniformly integrable then for $\gamma > \sup_n \|S_{1,\tau_n}\|_\infty$ (S_1 is bounded)

$$\begin{aligned} p_{\gamma,n} &\leq E\left(\left(Z_{\tau_n}S_{1,\tau_n} - \gamma E\left(Z_\infty \middle| \mathcal{F}_{\tau_n}\right)\right)^+ \middle| \mathcal{F}_t\right) \\ &= E\left(Z_{\tau_n}\left(S_{1,\tau_n} - \gamma\right)^+ \middle| \mathcal{F}_t\right) \\ &= 0 \end{aligned}$$

■

7.2. Section 2

Theorem 7.1. *Let $S = M + V$, $h > 0$, $S = S^h + \hat{S}^h = M^h + V^h + \hat{S}^h$ be the decompositions of S into its bounded and unbounded jumps parts. Let furthermore $\mathcal{L}(Z)$ be the unique solution to the stochastic differential equation $\int \frac{1_{\{Z_- > 0\}}}{Z_-} dZ$. Then the following conditions are equivalent to Z being a martingale density for S.*

1. $(V + [\mathcal{L}(Z), M])^p = 0$;
2. $\begin{cases} V^h + \left(\left[\mathcal{L}(Z), M^h\right]\right)^p = 0 \\ \left(\hat{S}^h\right)^p = 0 \end{cases}$
3. $\begin{cases} V^c + [\mathcal{L}(Z), M^c]^p = 0 \\ \left(V^d + \left[\mathcal{L}(Z), M^d\right]\right)^p = 0 \end{cases}$

Proof. As a consequence of the integration by parts formula we have that

$$V + [\mathcal{L}(Z), M] = \int \frac{1}{Z_-} d(SZ) - M - \int S_- d\mathcal{L}(Z)$$

where the left hand part is of finite variation by definition. Z is then a martingale density only if $V + [\mathcal{L}(Z), M]$ is a local martingale of finite hence of integrable variation, i.e. only if (1) is the case. The converse implication follows directly from the definition of dual predictable projection.

(1)⇒(3) Since by assumption $V + [\mathcal{L}(Z), M]$ is a local martingale of finite variation, its continuous part, $V^c + [\mathcal{L}(Z)^c, M^c]^p$, must vanish so $0 = (V + [\mathcal{L}(Z), M])^p = \left(V^d + \left[\mathcal{L}(Z), M^d\right]\right)^p$, i.e. 3 follows.

(3)⇒(2) If $\left(V^d + \left[\mathcal{L}(Z), M^d\right]\right)^p = 0$, then by decomposing V^d as $\Delta V^h + \hat{S}^h$, and noting that where ΔV^h is predictable while $\left[\mathcal{L}(Z), M^d\right]$ is of locally integrable variation, we obtain $\Delta V^h + \Delta\left(\left[\mathcal{L}(Z), M^h\right]\right)^p = 0$ from which $V^h + \left(\left[\mathcal{L}(Z), M^h\right]\right)^p = 0$ follows by sum.

(2)⇒(1) Obvious.

■

7.3. Section 4

In this section we will deal with a martingale Z on a discrete stochastic basis $\mathfrak{B} = \left(\Omega, \mathcal{F}, P; (\mathcal{F}_n)_{n\in\mathbb{N}}\right)$ satisfying the usual assumptions and with $\mathcal{F} = \bigvee_{n\in\mathbb{N}} \mathcal{F}_n$; let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Furthermore, let $\mathcal{G}$ be an arbitrary σ algebra on Ω: we say that the martingale Y is orthogonal to $\mathcal{G}$ if $E\left(\left(Y_{n+1} - Y_n\right) b \middle| \mathcal{G}\right) = 0$ whenever $b \in L^\infty\left(\Omega, \mathcal{F}_n, P\right)$.

If F is a vector valued measure on $(\Omega, \mathcal{F})$, i.e. $F : \mathcal{F} \to \mathbb{X}$ for some Banach space $\mathbb{X}$ then its variation over the set $A \in \mathcal{F}$ is defined to be

$$|F|(A) = \sup\left\{\sum_{k=1}^{K} \|F(A_k)\| : A_k \in \mathcal{F}, A = \bigcup_{k=1}^{K} A_k,\ A_k \cap A_{k'} = \emptyset\right\}$$

In general saying that F satisfies a given property weakly amounts to x^*F satisfying that same property for each $x^* \in \mathbb{X}^*$. In a few occasions we will identify $x \in \mathbb{X}$ and $y = f(x) \in \mathbb{Y}$ whenever f is an isometry on the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$.

Theorem 7.2. *Let Z be a nonnegative martingale on the stochastic basis $\mathfrak{B} = \left(\Omega, \mathcal{F}, P; (\mathcal{F}_n)_{n\in\mathbb{N}}\right)$, $\mathcal{G}$ a σ algebra on Ω and let $\mathbb{X} = L^1(\Omega, \mathcal{G}, P)$. Then Z is associated to a finitely additive set function $L_\mathcal{G}$ of bounded variation defined on $(\Omega, \mathcal{F})$ that takes values in the Banach space $\mathbb{X}^{**}$, vanishes on P null sets and satisfies the following property: if $\mathcal{G}$ is orthogonal to Z then the restriction $L_{\mathcal{G},n}$ of $L_\mathcal{G}$ to $\mathcal{F}_n$ admits the representation*

$$L_{\mathcal{G},n}(A) = E\left(Z_n 1_A \middle| \mathcal{G}\right) \tag{7.1}$$

Furthermore, the conditions (1)-(5) are equivalent one to another and are implied by (6).

1. *$L_\mathcal{G}$ is weakly countably additive;*
2. *$L_\mathcal{G}$ is countably additive;*
3. *$L_\mathcal{G}$ is absolutely continuous with respect to P;*
4. *$L_{\mathcal{G},\mathcal{A}}$ is absolutely continuous with respect to $P_\mathcal{A}$;*
5. *Z is a uniformly integrable martingale;*
6. *$L_\mathcal{G}$ is unique.*

Proof. Let us first show that (1)⇒(2)⇒(3)⇒(4).

(1)⇒(2) This is a well known result due to Pettis (see [9], Theorem IV.10.1, p. 318).

(2)⇒(3) If $L_{\mathcal{G}}$ is countably additive then it is absolutely continuous with respect to P if (and only if) it vanishes on P null sets. (3)⇒(4) is obvious.

As for the existence of $L_{\mathcal{G}}$, let $A \in \mathcal{F}$ and $\phi_n(A) = E(Z_n 1_A | \mathcal{G})$ and observe that

$$\|\phi_n(A)\| \leq E(\|Z_n\| 1_A | \mathcal{G}) \leq Z_0$$

The sequence $\langle \phi_n(A) \rangle_{n \in \mathbb{N}}$ is therefore bounded and we may consider its Banach limit $\Phi(x^*, A) = \mathrm{LIM}_n\, x^* \phi_n(A)$. The Banach limit is defined to be any linear functional on l^∞ that separates the compact set $\{e\}$ (where e is the unitary sequence in l^∞) from the closure Y of the set of all sequences of the form $(x_1, x_2 - x_1, x_3 - x_2, \dots)$ when $x = \langle x_n \rangle_{n \in \mathbb{N}}$ belongs to l^∞. By Banach-Mazur it is continuous and may be normalized to have norm 1 and $\mathrm{LIM}_n\, e_n = 1$: clearly $\mathrm{LIM}_n\, y_n = 0$ for $y \in Y$. It also follows that it is a positive linear functional. By these properties, $|\Phi(x^*, A)| \leq Z_0 \|x^*\|$ and there will therefore exist an element $x^{**}_{\Phi,A} \in \mathbb{X}^{**}$, $\left\| x^{**}_{\Phi,A} \right\| = Z_0$ such that $x^{**}_{\Phi,A} x^* = \Phi(x^*, A)$: define

$$L_{\mathcal{G}}(A) = x^{**}_{\Phi,A}{}^{4} \tag{7.2}$$

It is clear that if $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$ then

$$\begin{aligned}
L_{\mathcal{G}}(A \cup B)\, x^* &= \Phi(x^*, A \cup B) \\
&= \mathrm{LIM}_n\, x^* \phi_n(A \cup B) \qquad (7.3)\\
&= \mathrm{LIM}_n\, x^* \phi_n(A) + x^* \phi_n(B) \\
&= \mathrm{LIM}_n\, x^* \phi_n(A) + \mathrm{LIM}_n\, x^* \phi_n(B) \qquad (7.4)\\
&= \Phi(x^*, A) + \Phi(x^*, B) \\
&= x^{**}_{\Phi,A} x^* + x^{**}_{\Phi,B} x^* \\
&= (L_{\mathcal{G}}(A) + L_{\mathcal{G}}(B))\, x^*
\end{aligned}$$

for any $x^* \in \mathbb{X}^*$, so that $L_{\mathcal{G}}(A \cup B)$ is weak* additive. Suppose that $L_{\mathcal{G}}(A \cup B) \neq L_{\mathcal{G}}(A) + L_{\mathcal{G}}(B)$ then there will be $x^{***} \in \mathbb{X}^{***}$ and positive scalars $\alpha > \beta$ such that $x^{***} L_{\mathcal{G}}(A \cup B) \geq \alpha$ while $x^{***}(L_{\mathcal{G}}(A) + L_{\mathcal{G}}(B)) \leq \beta$. Since $\mathbb{X}^{***} = \overline{\varkappa[\mathbb{X}^*]}^*$ - where $\varkappa$ is the natural embedding of $\mathbb{X}^*$ into $\mathbb{X}^{***}$ and $\bar{\cdot}^*$ denotes the weak* closure in $\mathbb{X}^{***}$ - by Goldstine theorem (see [9], Corollary V.4.6, p. 425) there will exist an element x^* such that $L_{\mathcal{G}}(A \cup B)\, x^* > (L_{\mathcal{G}}(A) + L_{\mathcal{G}}(B))\, x^*$ contradicting (7.4). Suppose now that $\mathcal{G}$ is orthogonal to Z and let $A \in \mathcal{F}_m$ and $n > m$: then

$$\begin{aligned}
\mathrm{LIM}_n\, x^* E(1_A Z_n | \mathcal{G}) &= \mathrm{LIM}_{n:\, n>m}\, x^* E(1_A Z_n | \mathcal{G}) \\
&= \mathrm{LIM}_{n:\, n>m}\, x^* E(E(1_A Z_n | \mathcal{F}_m) | \mathcal{G}) \\
&= x^* E(1_A Z_m | \mathcal{G})
\end{aligned}$$

so that $L_{\mathcal{G},n} = \varkappa\left(E\left(1_A Z_n\right|\mathcal{G}\right))$ and (7.1) is satisfied. $L_{\mathcal{G}}$ is clearly of bounded variation since for any finite, disjoint collection $\{A_k : k = 1, \ldots, K\}$

$$\begin{aligned}
\sum_{k=1}^{K} \|L_{\mathcal{G}}(A_k)\| &\leq \sum_{k=1}^{K} \operatorname*{LIM}_n \|E(Z_n 1_{A_k}|\mathcal{G})\| \\
&\leq \operatorname*{LIM}_n E\left(\sum_{k=1}^{K} \|Z_n 1_{A_k}\| \middle| \mathcal{G}\right) \\
&= \operatorname*{LIM}_n E\left(\|Z_n\| \sum_{k=1}^{K} 1_{A_k} \middle| \mathcal{G}\right) \\
&= \operatorname*{LIM}_n E\left(\|Z_n\| 1_{\cup_{k=1}^{K} A_k} \middle| \mathcal{G}\right) \\
&\leq Z_0
\end{aligned}$$

(4)⇒(5) Then the restriction $L_{\mathcal{G},\mathcal{A}}$ to the algebra $\mathcal{A}$ of $L_{\mathcal{G}}$ is countably additive: let in fact $\langle A_k\rangle_{k\in\mathbb{N}}$ be a decreasing sequence of sets in $\mathcal{A}$ with empty intersection so that $P(A)_k \to 0$ and for each δ we can find an ε such that being $P(A)_k < \varepsilon$ then also $|L_{\mathcal{G}}|(A_k) < \delta$, i.e. $\lim_k |L_{\mathcal{G}}|(A_k) = 0$. Then for each $x^{***} \in \mathbb{X}^{***}$ the scalar valued measure $x^{***}L_{\mathcal{G},\mathcal{A}}$ is also countably additive. It may be then be uniquely extended to the whole of $\mathcal{F}$ by preserving countably additivity and absolute continuity with respect to P. Let f be the Radon Nikodým derivative of $x^{***}L_{\mathcal{G}}$ with respect to P and choose x^{***} to be that element of $\mathbb{X}^{***}$ associated to $1_\Omega \in L^\infty$ via the natural embedding of $\mathbb{X}^*$ into $\mathbb{X}^{***}$. Then for each measurable set $A \in \mathcal{F}_m$ we have

$$\begin{aligned}
x^{***}L_{\mathcal{G}}(A) &= x^{***}L_{\mathcal{G},n}(A) \\
&= \operatorname*{LIM}_{n:\, n>m} E(Z_n 1_A) \\
&= E(Z_m 1_A) \\
&= E(f 1_A)
\end{aligned}$$

i.e. $Z_m = E(f|\mathcal{F}_m)$ and $Z_\infty = f$ P a.s.: Z is therefore uniformly integrable.

We prove the implication (5)⇒(1). Let $x_b^* \in \mathbb{X}^*$ be associated to $b \in L^\infty(\Omega, \mathcal{G}, P)$. Then the sequence $\langle Z_n\rangle_{n\in\mathbb{N}}$ is weakly sequentially compact and all of its subsequences admit themselves a susbsequence converging weakly to Z_∞: so the sequence $\langle Z_n\rangle_{n\in\mathbb{N}}$ itself converges weakly to Z_∞ and

$$\begin{aligned}
L_{\mathcal{G}}(A)\, x_b^* &= \operatorname*{LIM}_n E(Z_n 1_A b) \\
&= \lim_n E(Z_n 1_A b) \qquad (7.5) \\
&= E(Z_\infty 1_A b)
\end{aligned}$$

i.e. $L_{\mathcal{G}}(A)$ is weak* countably additive. Again by Goldstine theorem - and by the fact that $\varkappa$ is an isometry - we can find a sequence $\langle b_r \rangle_{r\in\mathbb{N}}$ in L^∞ such that $\left\| x^*_{b_r} \right\| \leq \|x^{***}\|$ and

$$\begin{aligned} \left| x^{***} L_{\mathcal{G}}(A) \right| &\leq \lim_r \left| L_{\mathcal{G}}(A)\, x^*_{b_r} \right| \\ &\leq E\left(Z_\infty 1_A\right) \|x^{***}\| \end{aligned}$$

so that $x^{***} L_{\mathcal{G}}(A)$ is dominated by a countably additive measure and is therefore itself countably additive.

Eventually we prove that (6) implies (5). It is clear that by selecting a subsequence $\langle Z_{n_k} \rangle_{k\in\mathbb{N}}$ we may define $L^k_{\mathcal{G}}$ as the measure associated to it and for any x^*_b we have $L^k_{\mathcal{G}}(A)\, x^*_b = \mathrm{LIM}_k\, E\left(Z_{n_k} 1_A b\right)$. Of course we may always select a subsequence such that $\mathrm{LIM}_k\, E\left(Z_{n_k} 1_A b\right) = \liminf_n E\left(Z_n 1_A b\right)$ and another one such that $\mathrm{LIM}_r\, E\left(Z_{n_r} 1_A b\right) = \liminf_n E\left(Z_n 1_A b\right)$. By uniqueness, and choosing $b = 1_\Omega$, we conclude that $\lim_n E\left(Z_n 1_A\right)$ exists which is equivalent to uniform integrability of the martingale Z (see [9]). ■

We will indicate by $L(A)$ the measure $L_{\mathcal{G}}(A)$ in the case $\mathcal{G} = \{\Omega, \emptyset\}$. We have $L(A) = \mathrm{LIM}_n\, E\left(Z_n 1_A\right)$.

Corollary 1. *Suppose that for some $A \in \mathcal{F}$ the sequence $\langle E\left(Z_n 1_A \middle| \mathcal{G}\right) \rangle_{n\in\mathbb{N}}$ converges weakly in $L^1(\Omega, \mathcal{G}, P)$. Then $L_{\mathcal{G}}(A) = \lim_n E\left(Z_n 1_A \middle| \mathcal{G}\right)$.*

Proof. Let $\psi(A, \mathcal{G}) = \lim_n E\left(Z_n 1_A \middle| \mathcal{G}\right)$ where the limit is in the weak topology. For any $b \in L^\infty$,

$$\begin{aligned} L_{\mathcal{G}}(A)\, x^*_b &= \operatorname*{LIM}_n E\left(E\left(Z_n 1_A \middle| \mathcal{G}\right) b\right) \\ &= E\left(\psi(A, \mathcal{G})\, b\right) \end{aligned}$$

■

The construction achieved may at first sight appear rather unnatural and some comments are then in order. In fact to see things more explicitly, we have that for each $A \in \mathcal{F}$ $L_{\mathcal{G}}(A)$ belonging to $\mathbb{X}^{**}$ is in fact (in a one to one correspondence with) a (positive) boundedly additive set function μ_A on $(\Omega, \mathcal{G})$. This correspondence can be made clear by writing

$$\begin{aligned} \mu_A(B) &= L_{\mathcal{G}}(A)\, x^*_{1_B} \\ &= \operatorname*{LIM}_n E\left(Z_n 1_{A\cap B}\right) \\ &= L(A \cap B) \end{aligned} \tag{7.6}$$

when $B \in \mathcal{G}$. Observe that in the setting of countably additive, bounded measures, given (7.6) one may define the conditional expectation $L\left(A \middle| \mathcal{G}\right)$ as the Radon-Nikodym derivative of $\mu_A(B)$ with respect to the restriction $L|_{\mathcal{G}}$ of L to $\mathcal{G}$. It is though a well known fact that exact Radon Nikodým derivatives need not exist for finitely additive measures. It is also easy to see from (7.6) that the restriction of $\mu_{A,\mathcal{G}}$ to $\mathcal{H} \subset \mathcal{G}$ coincides with $\mu_{A,\mathcal{H}}$ and that $\mu_{A,\mathcal{G}}$ vanishes on P null sets it would then be important to establish under what conditions is $\mu_{A,\mathcal{G}}$ countably additive - and admits so a Radon Nikodým derivative. In the following lemma we give partial answers to these questions.

Lemma 2. *Let $\mathcal{G}$ be arbitrary and $Z_{\mathcal{G},n} = E(Z_n|\,\mathcal{G})$ be uniformly integrable, then*

1. *The family $\{\mu_{A,\mathcal{G}} : A \in \mathcal{F}\}$ is uniformly countably additive;*
2. *$L_{\mathcal{G}}(A) = \lim_n E(Z_n 1_A|\,\mathcal{G})$ and $L(A|\,\mathcal{G}) = L_{\mathcal{G}}(A)\, L_{\mathcal{G}}(\Omega)^{-1}$.*

Let $\mathcal{G}$ be orthogonal to Z and $A \in \mathcal{A}$, then $\mu_{A,\mathcal{G}}$ is countably additive and $L(A|\,\mathcal{G}) = L_{\mathcal{G}}(A)\, L_{\mathcal{G}}(\Omega)^{-1}$.

Proof. Let $\langle B_r \rangle_{r\in\mathbb{N}}$ to be a decreasing sequence of $\mathcal{G}$ measurable sets with empty intersection. If $\mathcal{G}$ is orthogonal to Z and $A \in \mathcal{F}_m$ then

$$\begin{aligned}
\mu_A\left(\bigcap_{r=1}^R B_r\right) &= L\left(A \cap \bigcap_{r=1}^R B_r\right) \\
&= \operatorname*{LIM}_{n>m} E\left(Z_n 1_{A\cap\bigcap_{r=1}^R B_r}\right) \\
&= E\left(Z_m 1_{A\cap\bigcap_{r=1}^R B_r}\right)
\end{aligned}$$

We now prove (1). To this end consider the inequality

$$\begin{aligned}
\mu_A\left(\bigcap_{r=1}^R B_r\right) &= \operatorname*{LIM}_{n} E\left(Z_n 1_{A\cap\bigcap_{r=1}^R B_r}\right) \\
&= \operatorname*{LIM}_{n} E\left(E(Z_n 1_A|\,\mathcal{G})\, 1_{\bigcap_{r=1}^R B_r}\right) \\
&\le \operatorname*{LIM}_{n} E\left(Z_{\mathcal{G},n} 1_{\bigcap_{r=1}^R B_r}\right) \\
&\le \sup_n E\left(Z_{\mathcal{G},n} 1_{\bigcap_{r=1}^R B_r}\right)
\end{aligned}$$

Whenever $E\left(Z_{\mathcal{G},n} 1_{\bigcap_1^R B_r}\right)$ converges to 0 uniformly in n then $\mu_{A,\mathcal{G}}\left(\bigcap_{r=1}^R B_r\right) \to 0$ uniformly in $A \in \mathcal{F}$.

To prove (2), let $B \in \mathcal{G}$: by Egoroff theorem we have a sequence $\langle F_r \rangle_{r\in\mathbb{N}}$, $F_r \in \mathcal{G}$ such that $Z_{\mathcal{G},n}$ converges uniformly on each F_r and $P(F_r^c) \downarrow 0$. By uniform integrability we have that $\sup_n E\left(Z_n 1_{A\cap B\cap F_r^c}\right)$ converges to 0 with r for any $A \in \mathcal{F}$ so that

$$\begin{aligned}
L_{\mathcal{G}}(A)\, x_{1_B}^* &= \lim_r \operatorname*{LIM}_n E\left(Z_n 1_{A\cap B\cap F_r}\right) + \lim_r \operatorname*{LIM}_n E\left(Z_n 1_{A\cap B\cap F_r^c}\right) \\
&= \lim_r \operatorname*{LIM}_n E\left(Z_n 1_{A\cap B\cap F_r}\right) \\
&= \lim_r \lim_n E\left(Z_n 1_{A\cap B\cap F_r}\right) \\
&= \lim_r E\left(\lim_n E\left(Z_n 1_A \mid \mathcal{G}\right) 1_{B\cap F_r}\right) \\
&= E\left(\lim_n E\left(Z_n 1_A \mid \mathcal{G}\right) 1_B\right)
\end{aligned}$$

Let $\Phi(A) = L_{\mathcal{G}}(A)\, L_{\mathcal{G}}(\Omega)^{-1}$: under the stated conditions by the preceding Corollary (1) $\Phi(A) = \lim_n E\left(Z_n 1_A \mid \mathcal{G}\right)\left(\lim_n E\left(Z_n \mid \mathcal{G}\right)\right)^{-1}$ and is therefore a $\mathcal{G}$ measurable function for each $A \in \mathcal{F}$ bounded by 1. Furthermore, for each $g \in L^\infty(\Omega, \mathcal{G}, P)$ we have by uniform integrability and a.s. convergence of $Z_{\mathcal{G},n}$

$$\begin{aligned}
L\left(g\Phi(A)\right) &= \operatorname*{LIM}_n E\left(Z_n \frac{\lim_n E\left(Z_n 1_A \mid \mathcal{G}\right)}{\lim_n E\left(Z_n \mid \mathcal{G}\right)} g\right) \\
&= \operatorname*{LIM}_n E\left(Z_{\mathcal{G},n} \frac{\lim_n E\left(Z_n 1_A \mid \mathcal{G}\right)}{\lim_n Z_{\mathcal{G},n}} g\right) \\
&= E\left(\lim_n Z_{\mathcal{G},n} \frac{\lim_n E\left(Z_n 1_A \mid \mathcal{G}\right)}{\lim_n Z_{\mathcal{G},n}} g\right) \\
&= E\left(\lim_n E\left(Z_n 1_A \mid \mathcal{G}\right) g\right) \qquad (7.7) \\
&= \lim_n E\left(E\left(Z_n 1_A \mid \mathcal{G}\right) g\right) \\
&= \lim_n E\left(Z_n 1_A g\right) \\
&= L\left(1_A g\right)
\end{aligned}$$

so that $L_{\mathcal{G}}(A)\left(L_{\mathcal{G}}(\Omega)\right)^{-1}$ is (a version of) the conditional measure $L\left(A \mid \mathcal{G}\right)$. If $\mathcal{G}$ is orthogonal to Z and $A \in \mathcal{F}_m$ then (7.7) still holds because $E\left(Z_n 1_A \mid \mathcal{G}\right) = E\left(Z_m 1_A \mid \mathcal{G}\right)$: it therefore holds for $A \in \mathcal{A}$. ■

It is clear that the uniform integrability of Z implies that of $Z_{\mathcal{G},n}$; if $\mathcal{G} = \mathcal{F}_k$ for some k, then $Z_{\mathcal{G},n} = Z_k$ so that uniform integrability trivially obtains.

The structure of the vector measure $L_{\mathcal{G}}$ can be further investigated. To our purposes the next result will be important.

Theorem 7.3. *Let $L_{\mathcal{G}}$ be as in the preceding Theorem (7.2). Then there exists a unique decomposition*

$$L_{\mathcal{G}} = L_{\mathcal{G}}^c + L_{\mathcal{G}}^\perp \qquad (7.8)$$

with $L_{\mathcal{G}}^c$ and $L_{\mathcal{G}}^\perp$ strongly additive measures and such that

1. $|L_{\mathcal{G}}| = \left|L^c_{\mathcal{G}}\right| + \left|L^{\perp}_{\mathcal{G}}\right|$;

2. $L^c_{\mathcal{G}} \ll P$;

3. *For any $\varepsilon > 0$ and any countably additive measure m on $(\Omega, \mathcal{F})$ the exists a set $A_{m,\varepsilon} \in \mathcal{F}$ such that $|m|(A^c) + |L_{\mathcal{G}}|(A) < \varepsilon$;*

4. *For each $x^{***} \in \mathbb{X}^{***}$, $x^{***}L_{\mathcal{G}} = x^{***}L^c_{\mathcal{G}} + x^{***}L^{\perp}_{\mathcal{G}}$ is the ordinary Yosida and Hewitt decomposition;*

5. $L^c_{\mathcal{G}}(A) = E(Z_\infty 1_A | \mathcal{G}_\infty)$.

Proof. Since $L_{\mathcal{G}}$ is of bounded variation it is also strongly additive (see [8], Proposition I.1.15, p. 7) and as such it admits the vector valued measures version of the Yosida and Hewitt decomposition (see again [8], Theorem I.5.8, p. 30). Decomposition (7.8) follows at once as well as uniqueness and properties (1) through (3). As for (4) observe that for each $x^{***} \in \mathbb{X}^{***}$, $\left|x^{***}L^{\perp}_{\mathcal{G}}\right|(A) \le \|x^{***}\| \left|L^{\perp}_{\mathcal{G}}\right|(A)$ so that by (3) $x^{***}L^{\perp}_{\mathcal{G}}$ is orthogonal to any countably additive measure i.e. coincides with the unique purely finitely additive part of $x^{***}L_{\mathcal{G}}$.

As for (5) observe that for each $x^*_b \in \mathbb{X}^*$

$$\begin{aligned} \varkappa(x^*_b) L_{\mathcal{G}}(A) &= L_{\mathcal{G}}(A) x^*_b \\ &= \underset{n}{\mathrm{LIM}}\, E(Z_n \wedge k 1_A b) + E\left((Z_n - k)^+ 1_A b\right) \\ &= \underset{n}{\mathrm{LIM}}\, E(Z_n \wedge k 1_A b) + \underset{n}{\mathrm{LIM}}\, E\left((Z_n - k)^+ 1_A b\right) \\ &= \lim_k \underset{n}{\mathrm{LIM}}\, E(Z_n \wedge k 1_A b) + \lim_k \underset{n}{\mathrm{LIM}}\, E\left((Z_n - k)^+ 1_A b\right) \\ &= E(Z_\infty 1_A b) + \lim_k \underset{n}{\mathrm{LIM}}\, E\left((Z_n - k)^+ 1_A b\right) \end{aligned} \tag{7.9}$$

- the fourth line following from an application of bounded as well as monotone convergence. It is clear that $\xi(A) = \lim_k \mathrm{LIM}_n E\left((Z_n - k)^+ 1_A b\right)$ is null on the set $A_k = \bigcup_n \bigcap_{m>n} \{Z_m \le k\}$ for each k. On the other side any countably additive set function ϕ dominated by ξ is absolutely continuous with respect to P and therefore vanishes on $\bigcap_k A^c_k = \{\sup_n Z_n = \infty\}$ since this is a P null set given that Z_n converges a.s. to the a.s. finite valued random variable Z_∞ - due to Doob convergence theorem (see [25], Theorem 11.5, p. 109). Then for any $B \in \mathcal{F}$

$$\begin{aligned} \phi(B) &= \phi\left(B \cap \bigcap_k A^c_k\right) + \phi\left(B \cap \bigcup_k A_k\right) \\ &\le \xi\left(B \cap \bigcup_k A_k\right) \\ &= 0 \end{aligned}$$

so that $\phi = 0$ and ξ is purely finitely additive. Then the decomposition (7.9) coincides with the one of Yosida and Hewitt and this being unique we have for $b \in L^{\infty}(\Omega, \mathcal{G}_{\infty})$ that

$$\varkappa(x_b^*) L_{\mathcal{G}}^c(A) = E(Z_{\infty} 1_A b) = \varkappa(x_b^*) E(Z_{\infty} 1_A | \mathcal{G}_{\infty})$$

so that we may again conclude that $L_{\mathcal{G}}^c(A) = E(Z_{\infty} 1_A | \mathcal{G}_{\infty})$ by denseness. ■

Remark 1. *In (7.9) we actually showed that*

$$\lim_k \operatorname*{LIM}_n E\left((Z_n - k)^+ 1_A b\right) \tag{7.10}$$

is the purely finitely additive component of the scalar, finitely additive measures $L_{\mathcal{G}}(A) x_b^*$. Although the decomposition itself is unique, its representation is not. In particular the sequence of reals k could be replaced by any sequence $\langle h_k \rangle_{k\in\mathbb{N}}$ of elements in L^1 that converge to ∞ P a.s.. This remark will be of use later on.

The typical use of the above results will be as follows. Let $\left(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\in\mathbf{T}}\right)$ be the stochastic basis and Z a non negative local martingale defined on it. Let furthermore $\mathbf{A} = \mathbb{N} \times \mathbf{T}$ ordered by product order: then $(Z_\alpha)_{\alpha\in\mathbf{A}}$ is a martingale on $\left(\Omega, \mathcal{F}, P; (\mathcal{F}_\alpha)_{\alpha\in\mathbf{A}}\right)$ where for $\alpha = (n, t)$, $Z_\alpha = Z_{t\wedge\tau_n}$ and $\mathcal{F}_\alpha = \mathcal{F}_{t\wedge\tau_n}$ and so the preceding Theorem (7.2) applies whenever an increasing sequence $\langle \alpha_k \rangle_{k\in\mathbb{N}}$ has been selected. In particular we will choose $\mathcal{G} = \mathcal{F}_t$. As a last application, for fixed $t \in \mathbf{T}$ let $\mathbf{A}^t = \mathbb{N} \times \{t\}$, choose an increasing sequence $\langle \alpha_k^t \rangle_{k\in\mathbb{N}}$ in $\mathbf{A}^t$ and let $\mathcal{G} = \{\emptyset, \Omega\}$: define $L_t = L_{\mathcal{G}}$ as in the proof of Theorem (7.2) and let $L_*^\perp = \bigvee_{t\in\mathbf{T}} L_t^\perp$.

Corollary 2. *Let Z be a local martingale on the stochastic basis $\left(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\in\mathbf{T}}\right)$. Then it is a martingale if and only if $L_*^\perp = 0$.*

Proof. For each $s, t \in \mathbf{T}$ with $s < t$ and any $A \in \mathcal{F}_s$,

$$L_t(A) = \operatorname*{LIM}_n E(Z_{\tau_n\wedge t} 1_A) = \operatorname*{LIM}_n E(Z_{\tau_n\wedge s} 1_A) = L_s(A) \tag{7.11}$$

Since $L_t^\perp = L_s^\perp = 0$, (7.11) implies $L_t^c(A) = L_s^c(A)$ and by Theorem (7.3 (5)), $E(Z_t 1_A) = E(Z_s 1_A)$ i.e. $Z_s = E(Z_t | \mathcal{F}_s)$. For the converse, if Z is a martingale, then $L_t(A) = \operatorname{LIM}_n E(Z_{\tau_n\wedge t} 1_A) = E(Z_t 1_A)$ i.e. $L_t^\perp = 0$ and this, being true for any t, implies $L_*^\perp = 0$. ■

7.4. Section 6

Proposition 4. *Let Z be a martingale density and L the finitely additive measure associated to it.*

1. *If $L \gg^w P$, then (6.2) is satisfied;*

2. *If $L^c \gg^w P$, then (6.1) is satisfied.*

Proof. (1) Let $H = h\exp(-Z^*)$ where h is a strictly positive (i.e. $P(h > 0) = 1$) element of L^1 and $Z^* = \bigvee_t Z_t$. Observe that, by Doob theorem, Z^* is finite a.s., so that $P(H > 0) = 1$. Then

$$\begin{aligned} L(F \wedge H) &= \operatorname*{LIM}_k E(Z_{\tau_k} F \wedge H) \\ &= \operatorname*{LIM}_k E(Z_{\tau_k} f_k \wedge H) + \operatorname*{LIM}_k E(Z_{\tau_k}(F - f_k) \wedge H) \\ &\leq \operatorname*{LIM}_k E(Z_{\tau_k} f_k) + \limsup_k E(Z_{\tau_k}(F - f_k) \wedge H) \\ &\leq E\left(\limsup_k Z_{\tau_k}(F - f_k) \wedge H\right) \\ &= 0 \end{aligned} \tag{7.12}$$

by the reverse Fatou's lemma. Since L and P are weakly equivalent this implies that $P(F \wedge H \geq 0) = 1$ cannot be true unless $P(F \wedge H = 0) = 1$ and since H is arbitrary (but positive) $P(F \geq 0) = 1$ with $P(F > 0) > 0$ if and only if $P(F = \infty) > 0$, a situation that does not meet the requirement $F \in L^0$.

(2) Clearly (6.2) is satisfied by the preceding paragraph and, in particular,

$$\begin{aligned} 0 &\geq \lim_n L(F \wedge nH) \\ &= \lim_n L\left(F^+ \wedge nH\right) - L\left(F^-\right) \\ &= \lim_n \operatorname*{LIM}_k E\left(Z_{\tau_k}\left(F^+ \wedge nH\right)\right) - L\left(F^-\right) \\ &\geq E\left(Z_\infty F^+\right) - L\left(F^-\right) \end{aligned}$$

i.e. $L^c(F^+) \leq L(F^-)$. Take now a sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ uniformly bounded from below: then it is easy to conclude that $L^c(g^+) \leq \|L\| \liminf_n \|F_n^-\|_\infty$ for any $g \leq \liminf_n F_n$. Take a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ in $\mathbb{C}$ - i.e. dominated by some sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathbb{K}$ - and converging to some $g \geq 0$ with $L^c(g) > 0$ in the uniform topology. For n sufficiently large $\langle g_n \rangle_{n \in \mathbb{N}}$ (and therefore $\langle F_n \rangle_{n \in \mathbb{N}}$) has a uniform lower bound and $\liminf_n \|F_n^-\|_\infty \leq \lim_n \|g_n^-\|_\infty = 0$. It must then be true that $g = 0$, L^c a.s. and therefore P a.s.. ■

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