Multiperiod Asset-Liability Management in a Mean-Variance Framework with Exogenous and Endogenous Liabilities

presented by

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Outline

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Objective of the paper

General:

- Extend Markowitz's basic intuition to the multiperiod setting for AL portfolios by studying the optimal policies and minimum variance frontiers (MVF) implied by discrete-time multiperiod AL portfolio selection models.
- Give a geometric decomposition of these problems that drastically simplifies the analysis and numerical implementation of such model settings.

Specific: Investigate impact of

- taking liabilities into account
- investment horizon,
- rebalancing frequency,
- exogeneity/endogeneity of liabilities,
- determination of optimal funding ratios
- numerically efficient incorporation of optimization constraints.

Challenges

- Discrete-time multiperiod mean variance model only recently solved for the asset-only case (Li and Ng (2000)).
- AL surplus optimization requires working with two state variables: the aggregate value of (i) assets and (ii) liabilities.
- Treating liability as exogenous or endogenous.
- Economic interpretation of the structure of the implied solutions and MVF in a multi-period setting.
- Inclusion of intertemporal constraints of different forms (for instance, downside risk limitations).

Solution

- Discrete-time multiperiod mean variance model: we embed the AL-MV problem in an equivalent mean-second moments problem accessible to dynamics programming.
- State variables for assets and liabilities: we adopt a geometric approach that drastically simplifies the computations.
- Economic interpretation in exogenous case: an orthogonal geometric decomposition represents the implied policies (MVF) as linear combinations of some simple basis strategies (returns)
- Economic interpretation in endogenous case: a non-trivial geometric decomposition represents the implied policies (MVF) as linear combinations of some basis strategies (returns).

Contribution to Existing Literature

- Exogenous Liabilities:
  - General recursive analytical representation of the implied policies and MVF (extending results in Li and Ng (2000).
  - Orthogonal representation of k-period policies and MVF as a linear combination of $k+1$ basis policies and returns (extending results in Li and Ng (2000) and Hansen and Richard (1987)).
  - Inclusion of liabilites only affects the multi period minimum second moment (MSM) return (extending results in Keel and Müller (1995))
  - In the i.i.d. setting: closed form solutions
- Endogenous Liabilities:
  - ...to our knowledge, no related work!

Notation

- For given aggregate initial wealth x_0 and initial liabilities l_0 the investor is allowed to rebalance portfolios at dates $0, 1, ..., T-1$.
- There are two assets and two liabilities with gross returns $R_t = (R_t^0, \widetilde{R}_t, Q_t^0, \widetilde{Q}_t)'$ with aggregate dynamics given by

$$x_{t+1} = R_t^0 x_t + R_t^1 u_t \, , \, l_{t+1} = Q_t^0 l_t + Q_t^1 v_t \quad , \tag{1}$$

  where $R_t^1 = \tilde{R}_t - R_t^0$, $Q_t^1 = \tilde{Q}_t - Q_t^0$.
- The initial balance sheet to be optimized takes the form as given below.

Balance Sheet at time t

Benchmark Asset:	$x_t - u_t$	Benchmark Liability:	$l_t - v_t$
Other Assets:	u_t	Other Liabilities:	v_t
Total:	x_t	Surplus:	S_t

Problem Reformulation

- Define a set $\mathcal{A}(z_t)$, $z_t = (x_t, l_t)'$, of linear constraints. This set includes, e.g., short-selling restrictions, positivity constraints on the surplus etc.
- For a final surplus $S_T := x_T - l_T$ and risk aversion $w > 0$ the following AL problem has to be solved

$$(\mathrm{P}) \quad \begin{cases} \max_{u,v \in \mathcal{A}(z)} \left[\mathbb{E}(S_T) - w\mathrm{var}(S_T)\right] \\ \qquad\qquad s.t.\,(1) \end{cases}$$

- For a free parameter λ set $\gamma = \lambda/w$ and define the optimization problem

$$(\mathrm{P1}) \quad \begin{cases} \max_{u,v \in \mathcal{A}(z)} \left[\mathbb{E}(\gamma S_T - S_T^2)\right] \\ \qquad\qquad s.t.\,(1) \end{cases}$$

- If ϕ^* is a solution to $(\mathrm{P1})$ for given (λ^*, w), then it is also a solution to $(\mathrm{P1})$ for:

$$\lambda^* = 1 + 2w\mathbb{E}(S_T)|_{\phi^*} \ .$$

- The implied MVF are the same and problem $(\mathrm{P1})$ is accessible to dynamic programming

General Solution

Let us first state the main result in the most general form:

Theorem 1. *Given a mean-variance AL optimization problem in (P1) given the AL dynamics in (1) the optimal final surplus S_T^* can be decomposed into two returns,*

$$S_T^* = S_{T-k}^{0e} + \gamma(z) S_{T-k}^{1e}, \quad k = 0, ..., T-1\,,$$

with $S_{T-k}^{0e} = \boldsymbol{I}' D_{T-k}^{e}(z) z_{T-k}$ and $S_{T-k}^{1e} = \sum_{i=0}^{k-1} R_{T-k+i}^{1e}(z)$, D_{T-k}^{e} and S_{T-k}^{1e} are projections upon properly defined vector spaces.

Before clarifying the above theorem and presenting the explicit expressions for the projections, we remark:

- The surplus can be decomposed into two returns - their features to be explored!
- Under the above representation, the risk-aversion parameter γ is becoming state-dependent for $k < T-1$.

Definitions

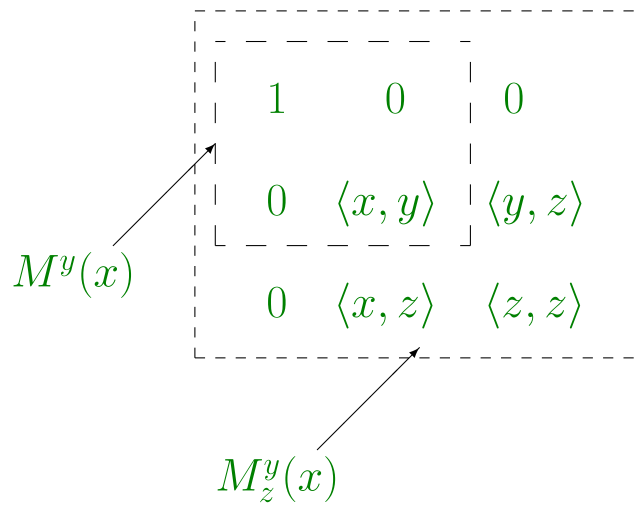
Before continuing, we have to introduce some definitions:

Definition 1. *Let $\mathbb{P} : L_2 \rightarrow L_2$ be an orthogonal projection, that is $\mathbb{P}$ is self-adjoint and $\mathbb{P}^2 = \mathbb{P}$. For any finite dimensional subspace $\mathcal{S} \subset L_2$, $\mathcal{S}^{\perp}$ is the orthogonal complement of $\mathcal{S}$. By $\langle \cdot, \cdot \rangle$ we mean the scalar product in the space L_2 of square integrable random variables.*

Definition 2. *A final surplus S_T^* belongs to the Minimum Variance Frontier (MVF) if the variance of the surplus is a minimum given a targeted expected return of the final surplus.*

Further, we denote the vector of assets and liabilities by $z_t = (x_t, l_t)'$, and by $d_t = (u_t, v_t)'$ the vector of amounts invested in the returns R_t^1 and Q_t^1.

Definition 3. *We define a matrix* $M_z^y(x)$ *and its first leading principle submatrix* $M^y(x)$ *as:*



With the above definition the determiniant $|M_z^y(x)|$ equals the determinant of the second principle submatrix. Recall that the determinant is a multilinear skewsymmetric form. Therefore, for two scalars α and β we would have

$$|M_z^y(\alpha x_1 + \beta x_2)| = \alpha\, |M_z^y(x_1)| + \beta\, |M_z^y(x_2)|\,.$$

To abbreviate notation, we write $|M_z^y|$ for $|M_z^y(y)|$ and $|M_y^z(z)|$ as they are both the same. Finally, we will work with the following notation for $t = 0, .., T$:

$$D_t = \begin{pmatrix} R_t^0 & 0 \\ 0 & Q_t^0 \end{pmatrix} \ , \ G_t = \begin{pmatrix} R_t^1 & 0 \\ 0 & Q_t^1 \end{pmatrix} \ ,$$

$$\widetilde{D}_t = \begin{pmatrix} R_t^0 & 0 \\ 0 & \widetilde{Q}_t \end{pmatrix} \ \widehat{D}_t = \begin{pmatrix} \widehat{R}_t & 0 \\ 0 & Q_t^0 \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ , \ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ .$$

One-Period Optimization

Consider none of the constraints to be binding and an investor who...

a) ...totally neglects existence of liabilities

b) ...assumes liabilities to be exogenously determined

c) ...assumes endogenous liabilities.

In case a) the investor solves

$$(\text{P2-a}) \qquad \begin{cases} \max_u \mathbb{E}[\gamma x_T - \frac{1}{2}x_T^2] \\ s.t. \;\; x_{t+1} = R_t^0 x_t + R_t^1 u_t, \quad t = T-1. \end{cases}$$

The investor with exogenous liabilities, case b), solves the problem,

$$(\text{P2-b}) \qquad \begin{cases} \max_u \mathbb{E}[\gamma \mathbf{I}' z_T - \frac{1}{2} z_T' \mathbf{I}\mathbf{I}' z_T] \\ s.t. \;\; \mathbf{I}' z_{t+1} = \mathbf{I}' D_t z_t + \mathbf{I}' G_t e_1 u_t, \quad t = T-1, \end{cases}$$

whereas the investor with the endogenous liabilities, case c), solves

$$(\text{P2-c}) \qquad \begin{cases} \max_d \mathbb{E}[\gamma \mathbf{I}' z_T - \frac{1}{2} z_T' \mathbf{I}\mathbf{I}' z_T] \\ s.t. \;\; \mathbf{I}' z_{t+1} = \mathbf{I}' D_t z_t + \mathbf{I}' G_t d_t, \quad t = T-1. \end{cases}$$

A. The Solutions

According to Theorem 1, the MVF-surplus for all problems is given by

$$S_T^* = S_{T-1}^{0j} + \gamma S_{T-1}^{1j} = \mathbf{I}' D_{T-1}^{j} z_{T-1} + \gamma R_{T-1}^{1j}, \quad j \in \{a, b, c\}.$$

Depending on the investor's assumption about liabilities, we get for case c)

$$\begin{aligned} D^c &= \begin{pmatrix} \mathbb{P}_{R^{1\perp}Q^1}^{\mathcal{D}_{QR}\mathcal{D}_{RR}}(R^0) & 0 \\ 0 & \mathbb{P}_{R^1Q^{1\perp}}^{\mathcal{D}_{QQ}\mathcal{D}_{RQ}}(Q^0) \end{pmatrix}, \\ R^{1c} &= \mathbb{P}_{R^1Q^1}^{\mathcal{D}_{Q\mathbb{1}}\mathcal{D}_{R\mathbb{1}}}(\mathbb{1}). \end{aligned}$$

For case b)

$$\begin{aligned} D^b &= \begin{pmatrix} \mathbb{P}_{R^{1\perp}}(R^0) & 0 \\ 0 & \mathbb{P}_{R^{1\perp}}(Q^0) \end{pmatrix}, \\ R^{1b} &= \mathbb{P}_{R^1}(\mathbb{1}). \end{aligned}$$

Finally, for case a)

$$\begin{aligned} D^a &= \begin{pmatrix} \mathbb{P}_{R^{1\perp}}(R^0) & 0 \\ 0 & Q^0 \end{pmatrix}, \\ R^{1a} &= R^{1b} = \mathbb{P}_{R^1}(\mathbb{1}). \end{aligned}$$

B. Asset-Only vs. Exogenous Liabilities

The results for cases a) and b) seem to be interpretable in a straightforward manner. The formalism using projections was introduced in Leippold, Trojani, and Vanini (2002a). In particular,

- the projections are explicitly given as

$$\begin{aligned}
\mathbb{P}_{R^{1\perp}}(R^0) &= R^0_{T-1} - \frac{\mathbb{E}\left(R^1_{T-1} R^0_{T-1}\right)}{\mathbb{E}\left(\left(R^1_{T-1}\right)^2\right)} R^1_{T-1}, \\
\mathbb{P}_{R^{1\perp}}(Q^0) &= Q^0_{T-1} - \frac{\mathbb{E}\left(R^1_{T-1} Q^0_{T-1}\right)}{\mathbb{E}\left(\left(R^1_{T-1}\right)^2\right)} R^1_{T-1}, \\
\mathbb{P}_{R^1_{T-1}}(\mathbb{1}) &= \frac{\mathbb{E}\left(R^1_{T-1}\right)}{\mathbb{E}\left(\left(R^1_{T-1}\right)^2\right)} R^1_{T-1}.
\end{aligned}$$

- S^*_T is a linear combination of two returns:

$$S^*_T = \begin{cases} x_{T-1}\mathbb{P}_{R^{1,\perp}_{T-1}}\left(R^0\right) - l_{T-1}Q^0 & +\gamma \mathbb{P}_{R^1_{T-1}}(\mathbb{1}), \qquad \text{for case a)} \\ x_{T-1}\mathbb{P}_{R^{1,\perp}_{T-1}}\left(R^0\right) - l_{T-1}\mathbb{P}_{R^{1,\perp}_{T-1}}\left(Q^0\right) & +\gamma \mathbb{P}_{R^1_{T-1}}(\mathbb{1}), \qquad \text{for case b)} \end{cases}$$

- There is a subtle difference between the surplus for case a) and b):
  - S^{0b}_{T-1} is the minimum-second-moment (MSM) surplus, the difference of an asset-only and a liabilities-only MSM payoff.
  - S^{0a}_{T-1} is the minimum-second-moment (MSM) surplus only if $l_{T-1} = 0$ or $Q^0 = 0$.
  - $S^{1j}_{T-1} = \mathbb{P}_{R^1_{T-1}}(\mathbb{1})$, $j \in \{a, b\}$, is the asset excess return nearest to the "risk-free" pay-off $\mathbb{1}$.
  - The MSM surplus is the return of a portfolio with generally non zero initial position in AL while $\mathbb{P}_{R^1_{T-1}}(\mathbb{1})$ is an asset excess return
- With exogenous liability (case b)), the surplus is decomposed into two orthogonal returns, such that

$$\begin{aligned} \mathbb{E}\left[S^{0b}_{T-1} S^{1b}_{T-1}\right] &= 0, \\ \mathbb{E}\left[(S^{1b}_{T-1})^2\right] &= \mathbb{E}\left[S^{1b}_{T-1}\right]. \end{aligned}$$

- In the asset-only case, only the second equation above holds, i.e.,

$$\mathbb{E}\left[(S^{1a}_{T-1})^2\right] = \mathbb{E}\left[S^{1a}_{T-1}\right],$$

  but $\mathbb{E}\left[S^{0a}_{T-1} S^{1a}_{T-1}\right] \neq 0$. These differences have some impacts on the MVF.

C. Mean-Variance-Frontier I

Any MVF can be represented in the $(\mathbb{V}(S), \mathbb{E}(S))$-space by a curvature parameter A, by a horizontal shift parameter B and a vertical shift parameter C. In particular, the one-period MVF in the unrestricted case can be represented as

$$\mathbb{V}(S_T) = A_j \mathbb{E}(S_T)^2 + 2B_j \mathbb{E}(S_T) + C_j, \quad j \in \{a, b, c\}.$$

The Minimum Variance Portfolio (MVP) has expectation value and variance according to

$$\mathbb{E}(S^{MVP}) = -B_j/A_j, \quad \mathbb{V}(S^{MVP}) = -B_j^2/4A_j + C_j, \quad j \in \{a, b, c\}.$$

For the MVF's of a) and b) the following holds:

$$A_b = \frac{1}{\mathbb{E}\left[S^{1b}\right]} - 1, \qquad A_a = \frac{1}{\mathbb{E}\left[S^{1a}\right]} - 1,$$

$$B_b = -\frac{\mathbb{E}\left[S^{0b}\right]}{\mathbb{E}\left[S^{1b}\right]}, \qquad B_a = \frac{\mathbb{E}\left[S^{0a}S^{1a}\right]}{\mathbb{E}\left[S^{1a}\right]} - \frac{\mathbb{E}\left[S^{0a}\right]}{\mathbb{E}\left[S^{1a}\right]},$$

$$C_b = \mathbb{E}\left[\left(S^{0b}\right)^2\right] + \frac{\mathbb{E}\left[S^{0b}\right]^2}{\mathbb{E}\left[S^{1b}\right]}, \quad C_a = \mathbb{E}\left[\left(S^{0a}\right)^2\right] - \frac{2\mathbb{E}\left[S^{0a}\right]\mathbb{E}\left[S^{0a}S^{1a}\right]}{\mathbb{E}\left[S^{1a}\right]} + \frac{\mathbb{E}\left[S^{0a}\right]^2}{\mathbb{E}\left[S^{1a}\right]}.$$

From the expression above, the following implications are derived

- The AL MVF if affected by the introduction of liabilities in only two ways (cf. also Keel and Müller (1995)):
  - By a "vertical" shift caused by the parameter C and a "sidewise" shift caused by the parameter B.
  - Not by a change of curvature in the parameter A.
- These shifts are caused by the orthogonality property of the two surplus returns in the exogenous liability case.
- This induces a pure translation of the MVF in the mean-variance space, caused by a lower global MSM surplus S^{0j}_{T-1}, $j \in \{a, b\}$.
- The direction of the translation is north-west in the $(\mathbb{V}, \mathbb{E})$-plane and can be computed explicitly.

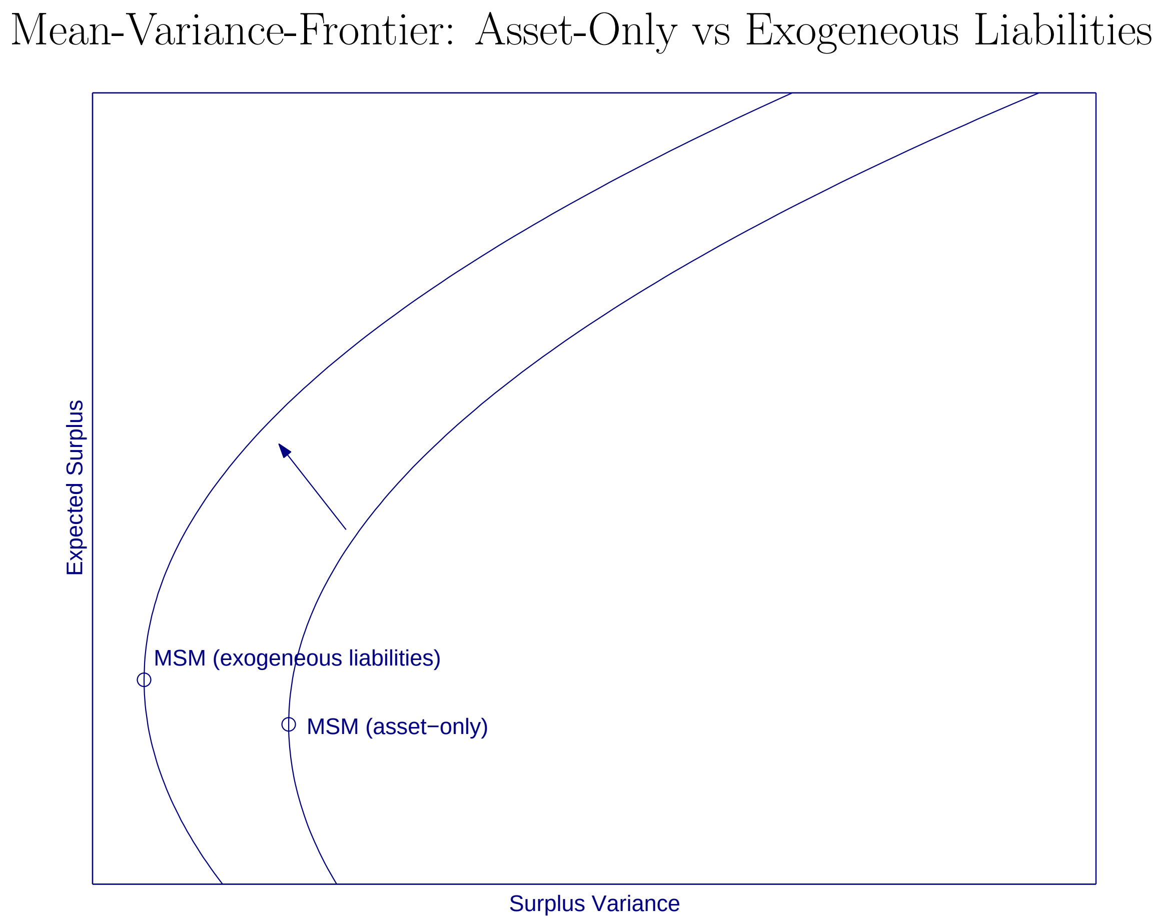


Figure 1: Mean-Variance-Frontiers. Moving from an asset-only optimization to a optimization where exogenous liabilities are taken into account, shifts the MVF in the upper-right corner of the $(\mathbb{V}(S), \mathbb{E}(S))$-plane.

D. Endogenous Liabilities

Writing out explicitly the terms for D^c and R^{1c}, we arrive at

$$D^c = \begin{pmatrix} R^0 - \frac{\langle R^0,Q^1\rangle\langle R^1,Q^1\rangle - \langle R^0,R^1\rangle\langle Q^1,Q^1\rangle}{\langle R^1,Q^1\rangle^2 - \langle R^1,R^1\rangle\langle Q^1,Q^1\rangle} R^1 & \frac{\langle Q^0,Q^1\rangle\langle R^1,Q^1\rangle - \langle Q^1,Q^1\rangle\langle R^1,Q^0\rangle}{\langle R^1,Q^1\rangle^2 - \langle R^1,R^1\rangle\langle Q^1,Q^1\rangle} R^1 \\ \frac{\langle R^0,R^1\rangle\langle R^1,Q^1\rangle - \langle R^1,R^1\rangle\langle R^0,Q^1\rangle}{\langle R^1,Q^1\rangle^2 - \langle R^1,R^1\rangle\langle Q^1,Q^1\rangle} Q^1 & Q^0 - \frac{\langle R^1,Q^0\rangle\langle R^1,Q^1\rangle - \langle R^1,R^1\rangle\langle Q^0,Q^1\rangle}{\langle R^1,Q^1\rangle^2 - \langle R^1,R^1\rangle\langle Q^1,Q^1\rangle} Q^1 \end{pmatrix},$$

and

$$R^{1c} = \frac{\left(\langle Q^1,\mathbb{1}\rangle\langle R^1,Q^1\rangle - \langle R^1,\mathbb{1}\rangle\langle Q^1,Q^1\rangle\right) R^1 + \left(\langle R^1,\mathbb{1}\rangle\langle R^1,Q^1\rangle - \langle Q^1,\mathbb{1}\rangle\langle R^1,R^1\rangle\right) Q^1}{\langle R^1,Q^1\rangle^2 - \langle R^1,R^1\rangle\langle Q^1,Q^1\rangle}.$$

The above expressions are rather involved and there is now way to detect the mathematical structure of the problem. Luckily, using the definition of M, we can simplify

$$D^c = \begin{pmatrix} R^0 - \frac{\left|M^{R^1}_{Q^1}(R^0)\right|}{\left|M^{R^1}_{Q^1}\right|} R^1 - \frac{\left|M^{Q^1}_{R^1}(R^0)\right|}{\left|M^{R^1}_{Q^1}\right|} Q^1 & 0 \\ 0 & Q^0 - \frac{\left|M^{R^1}_{Q^1}(Q^0)\right|}{\left|M^{R^1}_{Q^1}\right|} Q^1 - \frac{\left|M^{Q^1}_{R^1}(Q^0)\right|}{\left|M^{R^1}_{Q^1}\right|} R^1 \end{pmatrix},$$

and

$$R^{1c} = \frac{\left|M_{Q^1}^{R^1}(\mathbb{1})\right|}{\left|M_{Q^1}^{R^1}\right|} R^1 + \frac{\left|M_{R^1}^{Q^1}(\mathbb{1})\right|}{\left|M_{Q^1}^{R^1}\right|} Q^1.$$

To simplify the notation, we define[1]

$$\mathcal{D}_{QR} = \frac{\left|M_{Q^1}^{R^1}(R^0)\right|}{\left|M_{Q^1}^{R^1}\right|}, \quad \mathcal{D}_{RR} = \frac{\left|M_{R^1}^{Q^1}(R^0)\right|}{\left|M_{Q^1}^{R^1}\right|}, \quad \mathcal{D}_{QQ} = \frac{\left|M_{Q^1}^{R^1}(Q^0)\right|}{\left|M_{Q^1}^{R^1}\right|},$$
$$\mathcal{D}_{RQ} = \frac{\left|M_{R^1}^{Q^1}(Q^0)\right|}{\left|M_{Q^1}^{R^1}\right|}, \quad \mathcal{D}_{Q\mathbb{1}} = \frac{\left|M_{Q^1}^{R^1}(\mathbb{1})\right|}{\left|M_{Q^1}^{R^1}\right|}, \quad \mathcal{D}_{R\mathbb{1}} = \frac{\left|M_{R^1}^{Q^1}(\mathbb{1})\right|}{\left|M_{Q^1}^{R^1}\right|}.$$

From the positive definiteness of the expected return matrix, $\mathcal{D} \in (0, \infty)$.

[1]The first subscript of $\mathcal{D}$ determines the subscript of M in the nominator. The second subscript of $\mathcal{D}$ enters as the argument in the parenthesis of M in the nominator and superscript of M can just be deduced from the subscript of $\mathcal{D}$.

Lemma 1. *The expressions*

$$R^0 - \frac{\left|M_{Q^1}^{R^1}(R^0)\right|}{\left|M_{Q^1}^{R^1}\right|}R^1, \quad \frac{\left|M_{R^1}^{Q^1}(R^0)\right|}{\left|M_{Q^1}^{R^1}\right|}Q^1, \quad Q^0 - \frac{\left|M_{Q^1}^{R^1}(Q^0)\right|}{\left|M_{Q^1}^{R^1}\right|}Q^1,$$
$$\frac{\left|M_{R^1}^{Q^1}(Q^0)\right|}{\left|M_{Q^1}^{R^1}\right|}R^1, \quad \frac{\left|M_{Q^1}^{R^1}(\mathbb{1})\right|}{\left|M_{Q^1}^{R^1}\right|}R^1, \quad \frac{\left|M_{R^1}^{Q^1}(\mathbb{1})\right|}{\left|M_{Q^1}^{R^1}\right|}Q^1,$$

are all (orthogonal) projections.

Proof. The lemma follows from the multilinearity of the determinant. □

Hence,

$$D^c = \begin{pmatrix} \mathbb{P}_{R^{1\perp}}^{\mathcal{D}_{QR}}(R^0) - \mathbb{P}_{Q^1}^{\mathcal{D}_{RR}}(R^0) & 0 \\ 0 & \mathbb{P}_{Q^{1\perp}}^{\mathcal{D}_{QQ}}(Q^0) - \mathbb{P}_{R^1}^{\mathcal{D}_{RQ}}(Q^0) \end{pmatrix},$$

and

$$R^{1c} = \mathbb{P}_{R^1}^{\mathcal{D}_{Q\mathbb{1}}}(\mathbb{1}) + \mathbb{P}_{Q^1}^{\mathcal{D}_{R\mathbb{1}}}(\mathbb{1}).$$

Going one step further we can represent the diagonal elements of the matrix

D^c as projections on the span of $(R^{1\perp}, Q^1)$ and $(R^1, Q^{1\perp})$, i.e.,

$$D^c = \begin{pmatrix} \mathbb{P}^{\mathcal{D}_{QR}\mathcal{D}_{RR}}_{R^{1\perp}Q^1}(R^0) & 0 \\ 0 & \mathbb{P}^{\mathcal{D}_{QQ}\mathcal{D}_{RQ}}_{R^1 Q^{1\perp}}(Q^0) \end{pmatrix},$$

Similarly, R^{1c} is a projection of $\mathbb{1}$ on the span of (R^1, Q^1),

$$R^{1c} = \mathbb{P}^{\mathcal{D}_{Q\mathbb{1}}\mathcal{D}_{R\mathbb{1}}}_{R^1 Q^1}(\mathbb{1}).$$

Note that the orthogonality properties will simplify the subsequent optimization procedure for the subsequent time steps. Indeed, tedious calculations show that

$$\mathbb{P}^{\mathcal{D}_{QR}\mathcal{D}_{RR}}_{R^{1\perp}Q^1}\left(\mathbb{P}^{\mathcal{D}_{Q\mathbb{1}}\mathcal{D}_{R\mathbb{1}}}_{R^1 Q^1}(\mathbb{1})\right) = \mathbb{P}^{\mathcal{D}_{QQ}\mathcal{D}_{RQ}}_{R^1 Q^{1\perp}}\left(\mathbb{P}^{\mathcal{D}_{Q\mathbb{1}}\mathcal{D}_{R\mathbb{1}}}_{R^1 Q^1}(\mathbb{1})\right) = 0,$$

and

$$\mathbb{P}^{\mathcal{D}_{QR}\mathcal{D}_{RR}}_{R^{1\perp}Q^1}\left(\mathbb{P}^{\mathcal{D}_{QQ}\mathcal{D}_{RQ}}_{R^1 Q^{1\perp}}(Q^0)\right) = \mathbb{P}^{\mathcal{D}_{QQ}\mathcal{D}_{RQ}}_{R^1 Q^{1\perp}}\left(\mathbb{P}^{\mathcal{D}_{QR}\mathcal{D}_{RR}}_{R^{1\perp}Q^1}(R^0)\right) = 0.$$

Whenever a restriction is binding, the projection changes, but the basic structure of the optimal surplus as the sum of two returns is preserved. Indeed, whenever a constraint in u and v is binding, the optimization problem is equivalent to an optimization where either the assets or the liabilities are "exogenized".

E. The Mean-Variance-Frontier with Endogenous Liabilities

With endogenous liabilities, the parameters of the MVF are given by

$$
\begin{aligned}
A_c &= \frac{1}{\mathbb{E}\left[S^{1c}\right]} - 1, \\
B_c &= \frac{\mathbb{E}\left[S^{0c}S^{1c}\right]}{\mathbb{E}\left[S^{1c}\right]} - \frac{\mathbb{E}\left[S^{0c}\right]}{\mathbb{E}\left[S^{1c}\right]}, \\
C_c &= \mathbb{E}\left[\left(S^{0c}\right)^2\right] - \frac{2\mathbb{E}\left[S^{0c}\right]\mathbb{E}\left[S^{0c}S^{1c}\right]}{\mathbb{E}\left[S^{1c}\right]} + \frac{\mathbb{E}\left[S^{0c}\right]^2}{\mathbb{E}\left[S^{1c}\right]}.
\end{aligned}
$$

Besides that S^{0c} is no longer the MSM portfolio, the following holds:

Proposition 1. *Endogenous liabilities influence both shift parameters as well as the convexity parameter of the MVF. Exogenizing or neglecting liabilities in the optimization program makes the convexity independent of liabilities.*

We can even go one step further:

Proposition 2. *The risk-return tradeoff*[2] *on the MVF with exogenous liabilities is always less or equal as the risk-return tradeoff on the MVF with endogenous liabilities. The MVFs for b) and c) collide as* $Q_1 \to 0$*.*

Proof. We have to show that the curvature of the MVF with endogenous liabilities is always less or equal than the curvature with exogenous liabilities, i.e.,

$$\frac{1}{\mathbb{E}\left[\mathbb{P}_{R^1}(\mathbb{1})\right]} \geq \frac{\mathbb{E}\left[\left(\mathbb{P}_{R^1 Q^1}^{\mathcal{D}_{Q\mathbb{1}}\mathcal{D}_{R\mathbb{1}}}(\mathbb{1})\right)^2\right]}{\mathbb{E}\left[\mathbb{P}_{R^1 Q^1}^{\mathcal{D}_{Q\mathbb{1}}\mathcal{D}_{R\mathbb{1}}}(\mathbb{1})\right]^2}.$$

Note, from the positive definiteness assumption of the return matrix, all determinants in the above expression are strictly positive. After some algebraic manipulation, the above inequality can be considerably simplified to

$$\frac{\langle R^1, R^1\rangle}{\langle \mathbb{1}, R^1\rangle^2} \geq \frac{\langle R^1, R^1\rangle\langle Q^1, Q^1\rangle - \langle R^1, Q^1\rangle^2}{\langle \mathbb{1}, R^1\rangle^2\langle Q^1, Q^1\rangle - 2\langle \mathbb{1}, R^1\rangle\langle \mathbb{1}, Q^1\rangle\langle R^1, Q^1\rangle + \langle R^1, R^1\rangle\langle \mathbb{1}, Q^1\rangle^2}.$$

[2] As measured by the Sharpe-Ratio

We now multiply the above equation with $\langle \mathbb{1}, R^1 \rangle$ and write the RHS as

$$\langle R^1, R^1 \rangle \frac{\langle Q^1, Q^1 \rangle - \frac{\langle R^1, Q^1 \rangle^2}{\langle R^1, R^1 \rangle}}{\langle Q^1, Q^1 \rangle - 2\frac{\langle \mathbb{1}, Q^1 \rangle \langle R^1, Q^1 \rangle}{\langle \mathbb{1}, R^1 \rangle} + \frac{\langle R^1, R^1 \rangle \langle \mathbb{1}, Q^1 \rangle^2}{\langle \mathbb{1}, R^1 \rangle^2}}.$$

Obviously, we have proven our result, when we can show that

$$\frac{\langle R^1, Q^1 \rangle^2}{\langle R^1, R^1 \rangle} \geq 2\frac{\langle \mathbb{1}, Q^1 \rangle \langle R^1, Q^1 \rangle}{\langle \mathbb{1}, R^1 \rangle} - \frac{\langle R^1, R^1 \rangle \langle \mathbb{1}, Q^1 \rangle^2}{\langle \mathbb{1}, R^1 \rangle^2}$$

We can write the latter inequality as

$$\langle R^1, Q^1 \rangle^2 \langle \mathbb{1}, R^1 \rangle^2 \geq 2\langle \mathbb{1}, Q^1 \rangle \langle R^1, Q^1 \rangle \langle \mathbb{1}, R^1 \rangle \langle R^1, R^1 \rangle - \langle \mathbb{1}, Q^1 \rangle \langle R^1, R^1 \rangle^2.$$

This, in turn, is just the same as

$$\big(\langle R^1, Q^1 \rangle \langle \mathbb{1}, R^1 \rangle - \langle \mathbb{1}, Q^1 \rangle \langle R^1, R^1 \rangle \big)^2 \geq 0,$$

and we are done. $\square$

Mean-Variance-Frontiers: Exogenous vs. Endogenous Liabilities

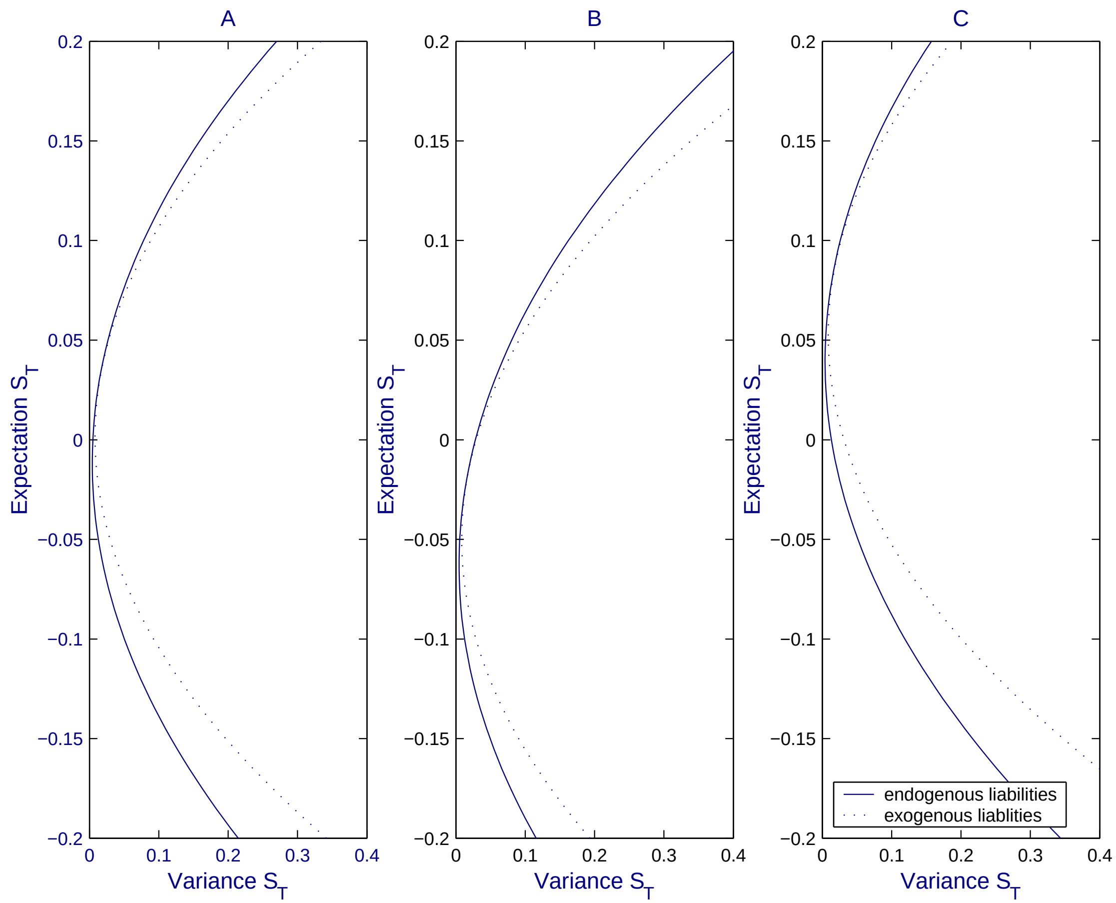


Figure 2: **One-Period MVFs. The curvature of the MVF with endogenous liabilities is smaller than the curvature of the MVF with endogenous liabilities.**

Multiperiod Model - Exogenous Liabilites

The following results are obtained:

- MSM returns

$$
\begin{aligned}
R^{0e}_{T-k} &= \mathbb{P}_{\left(R^1_{T-k} R^{0e}_{T-k+1}\right)^{\perp}} \left(R^0_{T-k} R^{0e}_{T-k+1}\right) \\
Q^{0e}_{T-k} &= \mathbb{P}_{\left(R^1_{T-k} R^{0e}_{T-k+1}\right)^{\perp}} \left(Q^0_{T-k} Q^{0e}_{T-k+1}\right).
\end{aligned}
$$

- Optimal surplus

$$
S_T = x_{T-k} R^{0e}_{T-k} - l_{T-k} Q^{0e}_{T-k} + \gamma \sum_{i=0}^{k-1} R^{1e}_{T-k+i},
$$

where $R^{1e}_{T-k+i} = \mathbb{P}_{\left(R^1_{T-k+i} R^{0e}_{T-k+i+1}\right)} (\mathbb{1})$.

A. Interpretation

- The same orthogonal structure prevails as in the one-period model.

- The set
$$\left\{x_{T-k}R^{0e}_{T-k} - l_{T-k}Q^{0e}_{T-k}, R^{1e}_{T-k}, ..., R^{1e}_{T-2}, R^{1e}_{T-1}\right\}$$
is an orthogonal system that spans the MVF.

- The return R^{1e}_{T-k+i} can be interpreted for any $i = 0, .., k-1$ as a "local", $(k-i)$-period, excess asset return.

- S_T is the orthogonal sum of a k-period MSM surplus $x_{T-k}R^{0e}_{T-k} - l_{T-k}Q^{0e}_{T-k}$ and k "local" excess asset returns $R^{1e}_{T-k}, .., R^{1e}_{T-1}$.

- Without any further assumptions about return dynamics, the optimal surplus has to be solved recursively.

B. Minimum Variance Frontier

- Defining $S^0_{T-k} = x_{T-k}R^{0e}_{T-k} - l_{T-k}Q^{0e}_{T-k}, S^1_{T-k} = \sum_{i=0}^{k-1} R^{1e}_{T-k+i}$ the k-period MVF is given by:

$$\begin{aligned}
\mathbb{V}\left(S_T\right) &= A_{T-k} \cdot \left[\mathbb{E}\left(S_T\right)\right]^2 + 2B_{T-k} \cdot \mathbb{E}\left(S_T\right) + C_{T-k} \\
A_{T-k} &= \frac{1}{\mathbb{E}\left(S^1_{T-k}\right)} - 1 \ , \ B_{T-k} = \frac{\mathbb{E}\left(S^0_{T-k}\right)}{\mathbb{E}\left(S^1_{T-k}\right)} \ , \\
C_{T-k} &= \frac{\left[\mathbb{E}\left(S^0_{T-k}\right)\right]^2}{\mathbb{E}\left(S^1_{T-k}\right)} + \mathbb{E}\left((S^0_{T-k})^2\right)
\end{aligned}$$

- While B_{T-k} and C_{T-k} depend on $\left(Q^0_{T-k}\right)_{k=1,..,T}$, A_{T-k} does not! (as in the one period model).

- This induces a pure translation of the MVF in the mean-variance space, caused by a lower global MSM surplus S^0_{T-k}

C. Results for the iid Case

- The basis returns are given in closed form, for instance:
  - Assets only MSM returns

$$R^{0e}_{T-k} = \prod_{i=0}^{k-1} \mathbb{P}_{R^{1,\perp}_{T-k+i}} \left(R^{0}\right),$$

  - Local asset excess returns:

$$R^{1e}_{T-k+i} = \mathbb{P}_{R^{1}_{T-k+i}} (\mathbb{1}) \prod_{j=i+1}^{k-1} \mathbb{P}_{\mathbb{P}_{R^{1,\perp}_{T-k+j}}} \left(R^{0}\right) (\mathbb{1})$$

- The closed form expression for Q^{0e}_{T-k} is given by:

$$Q^{0e}_{T-k} = \prod_{i=0}^{k-1} Q^{0}_{T-k+i} - \sum_{i=0}^{k-1} \left(\mathbb{P}_{R^{1}_{T-k+i}} \left(Q^{0}\right) \prod_{j=i+1}^{k-1} \mathbb{P}_{\mathbb{P}_{R^{1,\perp}_{T-k+j}}} (R^{0}) \left(Q^{0}\right) \prod_{j=0}^{i-1} Q^{0}_{T-k+j} \right) .$$

D. Myopic vs Dynamic with Exogenous Liabilites

Question 1: Given a fixed time horizon, what is the impact of the rebalancing frequency of the portfolio?

Question 2: How far off can a myopic investor be, compared to an investor who rebalances his portfolio?

Answer 1: Comparing the myopic strategy with the dynamic strategy, we can prove the following:

- The curvature of the MVF is always smaller in the dynamic case. This can be proven by using Jensen's inequality for the expectation operator.
- The vertical shift parameter of the dynamic MVF is always bigger in absolute terms.
- The horizontal shift parameter of the dynamic curve is always smaller.
- The parameters of the frontier converge to a finite value as $h \rightarrow \infty$.

This leads to the conclusion, that the MVF of the dynamic investor can only intersect the myopic MVF in the lower, inefficient part of the dynamic MVF.

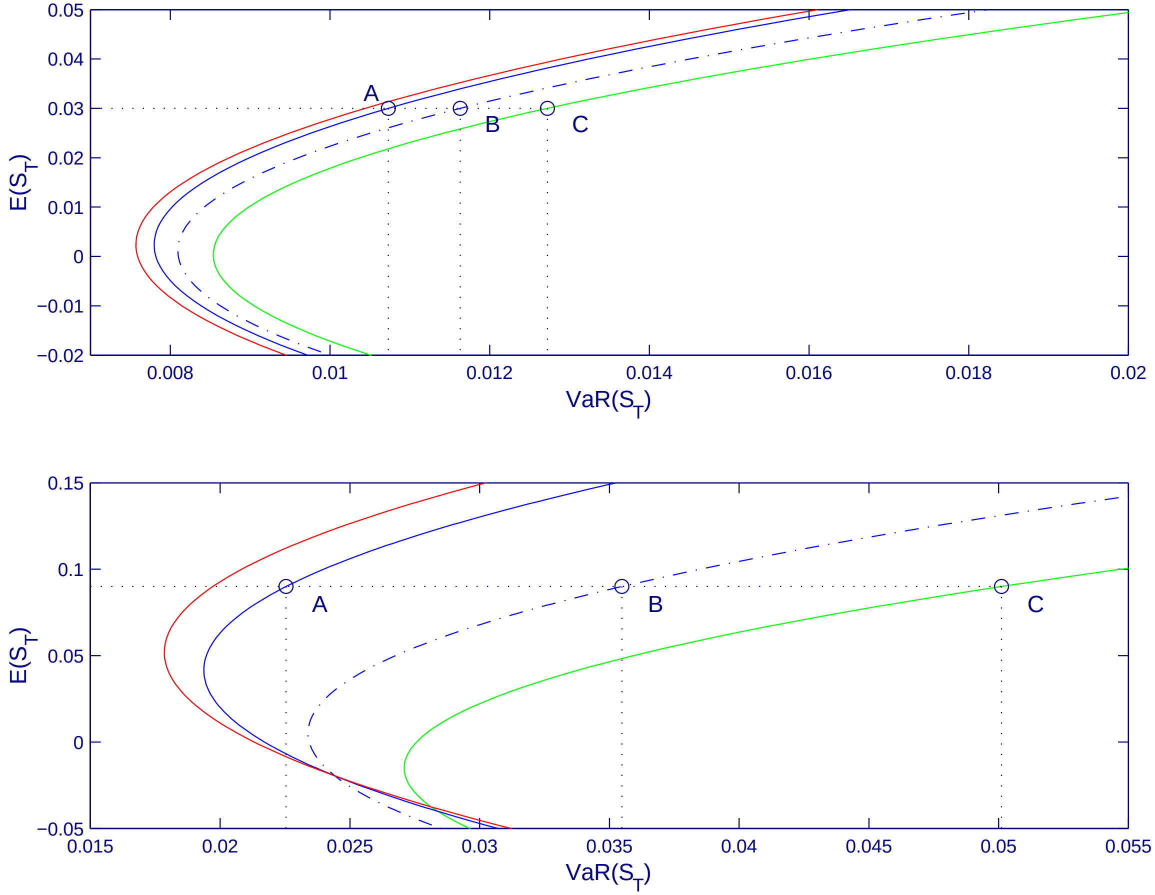


Figure 3: Effects of Time Diversification.

Answer 2: We compare to strategies with a fixed time-horizon of 1 year.

- The first strategy is a myopic one, where the investor does not rebalance his AL portfolio during the whole period.
- The second strategy rebalances the portfolio every month.
- We compare the initial MVFs of both strategies.
- We make a snapshot after 2 month and assume that

  a) The value of assets to liabilities is 1.1.

  b) The value of assets to liabilities is 0.95.

The results are plotted in the next figure...

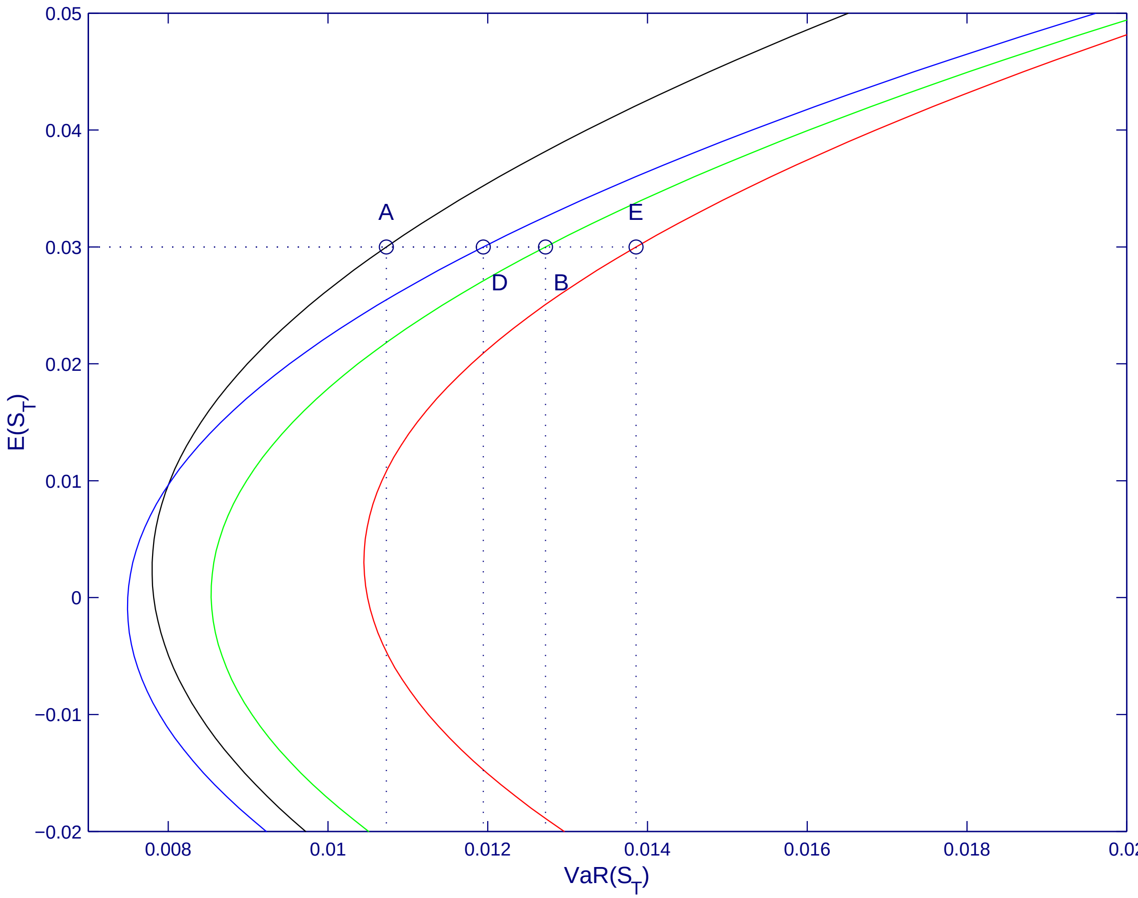


Figure 4: Movement of Dynamic MFVs in Time.

Multiperiod Model - Endogenous Liabilities

A. Time $T-2$

Dropping the term $\mathcal{C}(z)$, we can write for $T-2$ the unconstrained value function as

$$J(z) \equiv \max_{d \in A(z)} \mathbb{E}\left[\widehat{\gamma}(z_{T-1})\mathbf{I}'(\widehat{D}(z)z + \widehat{G}(z)d) - \frac{1}{2}(\widehat{D}(z)z + \widehat{G}(z)d)'\mathbf{II}'(\widehat{D}(z)z + \widehat{G}(z)d)\right],$$

where $\alpha_i \leq 0$ are the Kuhn-Tucker multipliers at $T-2$ and

$$\widehat{\gamma}(z_{T-1}) = \gamma\left(1 - \gamma R^{1e}(z_{T-1})\right).$$

Whenever at time $T-1$ a constraint is binding, we can set $\widehat{\gamma}(z_{T-1}) = \gamma$!

The J-function has the same form as the J-Function at time $T-1$, but with the matrices D and G replaced by the state-dependent matrices $\widehat{D}(z)$ and $\widehat{G}(z)$. More particularly,

$$\widehat{D}(z) \equiv \widehat{D}_{T-2}(z) = D^e_{T-1}(z)D_{T-2}, \quad \widehat{G}(z) \equiv \widehat{G}_{T-2}(z) = D^e_{T-1}(z)G_{T-2}.$$

B. Time $T - k$

For time $T - k$, $k = 3, ..., T$, we can continue recursively, substituting at each time step k,

$$\widehat{D}_{T-k}(z) = D^e_{T-k+1}(z) D_{T-k}, \quad \widehat{G}_{T-k}(z) = D^e_{T-k+1}(z) G_{T-k}.$$

Finally, we can write the optimal surplus belonging to the MVF as

$$\begin{aligned} S^*_T &= \mathbf{I}' D^e_{T-1}(z) z_{T-1} + \gamma R^{1e}_{T-1}, \\ &= \mathbf{I}' D^e_{T-k}(z) z_{T-k} + \gamma \sum_{i=0}^{k-1} R^{1e}_{T-k+i}(z), \end{aligned}$$

where D^e_{T-k} and R^{0e}_{T-k} have to be determined recursively. As an example, when no constraints are binding, we have

$$D^e_{T-k}(z) = D^e_{T-k+1}(z) \widetilde{D}_{T-k},$$

where $\widetilde{D}_{T-k}$ is the matrix given by

$$\begin{pmatrix} R^0_{T-k} - \begin{pmatrix} \dfrac{\left|\widetilde{M}^{R^1}_{Q^1}(R^0)\right|}{\left|\widetilde{M}^{R^1}_{Q^1}\right|} \\ \dfrac{\left|\widetilde{M}^{Q^1}_{R^1}(R^0)\right|}{\left|\widetilde{M}^{R^1}_{Q^1}\right|} \end{pmatrix}' \begin{pmatrix} R^1_{T-k} \\ Q^1_{T-k} \end{pmatrix} & 0 \\ 0 & Q^0_{T-k} - \begin{pmatrix} \dfrac{\left|\widetilde{M}^{R^1}_{Q^1}(Q^0)\right|}{\left|\widetilde{M}^{R^1}_{Q^1}\right|} \\ \dfrac{\left|\widetilde{M}^{Q^1}_{R^1}(Q^0)\right|}{\left|\widetilde{M}^{R^1}_{Q^1}\right|} \end{pmatrix}' \begin{pmatrix} Q^1_{T-k} \\ R^1_{T-k} \end{pmatrix} \end{pmatrix}.$$

Here, e.g. the matrix $\widetilde{M}^{R^1}_{Q^1}(R^0)$ is given by

$$\begin{pmatrix} \langle [D^e_{T-k+1}]_{11} R^0_{T-k}, [D^e_{T-k+1}]_{11} R^1_{T-k} \rangle & \langle [D^e_{T-k+1}]_{11} R^1_{T-k}, [D^e_{T-k+1}]_{22} Q^1_{T-k} \rangle \\ \langle [D^e_{T-k+1}]_{11} R^0_{T-k} [D^e_{T-k+1}]_{22} Q^1_{T-k} \rangle & \langle [D^e_{T-k+1}]_{22} Q^1_{T-k}, [D^e_{T-k+1}]_{22} Q^1_{T-k} \rangle \end{pmatrix},$$

and $[D^e_{T-k+1}]_{ii}$ is the ith diagonal element of D^e_{T-k+1}. The expression for R^{1e}_{T-k}

is given as

$$R^{1e}_{T-k} = \frac{\left|\widetilde{M}^{Q^1}_{R^1}(\mathbb{1})\right|}{\left|\widetilde{M}^{R^1}_{Q^1}\right|}[D^e_{T-k+1}]_{11}R^1_{T-k} + \frac{\left|\widetilde{M}^{R^1}_{Q^1}(\mathbb{1})\right|}{\left|\widetilde{M}^{R^1}_{Q^1}\right|}[D^e_{T-k+1}]_{22}Q^1_{T-k}.$$

Setting $S^{0e}_{T-k} \equiv \mathbf{I}' D^e_{T-k}(z) z_{T-k}$ and $S^{1e}_{T-k} \equiv \sum_{i=0}^{k-1} R^{1e}_{T-k+i}(z)$ finishes the proof of Theorem 1. More details can be found in Leippold, Trojani, and Vanini (2002b)

Conclusion

- We derived analytical solutions for the optimal policies and the MVF implied by a multiperiod mean-variance optimization of AL portfolios under some additional assumptions.
- We discussed some specific issues related to the impact of the rebalancing frequency and the characteristic of the liabilities, whether they are exogenously or endogenously determined.
- Work in progress:
  - Intertemporal constraints on, for instance, downside risk.
  - Efficient numerical implementation

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