HEDGING DOUBLE BARRIERS WITH SINGLES*

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Abstract

Double barrier options can be statically hedged by a portfolio of single barrier knockin options. The main part of the hedge automatically turns into the desired contract along the double barrier. Its residual part has small and comfortably stable value around the double barrier because its knockin price levels lie outside the double barrier. Residual value stability is robust to misspecification of market dynamics.

JEL Classification: G13.

Keywords: Double barrier options, single barrier options, static hedging.

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Double barrier options can be statically hedged by a portfolio of single barrier knockin options. The main part of the hedge automatically turns into the desired contract along the double barrier. Its residual part has small and comfortably stable value around the double barrier because its knockin price levels lie outside the double barrier. Residual value stability is robust to misspecification of market dynamics.

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1 Introduction

European, continuously-monitored barrier options are European options with an American feature. Option's existence at maturity depends on whether the underlying price breaches, before or at maturity, some pre-specified levels, called barriers. Given one barrier, *single knockin options* come to life if the barrier is hit. *Single knockouts* expire in that contingency. Given two barriers, one has a double barrier corridor which encompasses the initial underlying price. *Double knockins* come to life if either barrier is hit. *Double knockouts* expire in that contingency. These tables summarize the barrier option taxonomy.

barrier option type	The underlying price touches the upper barrier before/at maturity
up-in	the option comes to life
up-out	the option dies
down-in	
down-out	
double-in	the option comes to life if the upper barrier is hit before the lower barrier
double-out	the option dies if the upper barrier is hit before the lower barrier

barrier option type	The underlying price touches the lower barrier before/at maturity
up-in	
up-out	
down-in	the option comes to life
down-out	the option dies
double-in	the option comes to life if the lower barrier is hit before the upper barrier
double-out	the option dies if the lower barrier is hit before the upper barrier

Knockins and knockouts are complementary contracts: A portfolio of a knockin and a knockout written on the same barriers and strike is equivalent to a vanilla option with the same strike. Thus, one can focus on knockins only.

Barrier options are popular, especially in the foreign exchange markets, because they are cheaper than their non-barrier counterparts. Through double knockouts, traders enjoy even greater leverage potential in expressing a directional view because single knockouts typically have barriers too close for comfort. A double knockin may be used by an active fund manager who doubts about market direction and whose compensation is marked to market. It may be used by a trader who foresees a bigger volatility than the market consensus' one. Double knockins have low additional cost and improve upon the chance of option triggering with respect to single knockins.

The double barrier clause states: *if either barrier is hit*, that is, as soon as the first among the two barriers is hit. There is a *double barrier interdependence* which makes pricing and hedging difficult: *A double knockin is not simply the sum of two single knockins written on the corridor extrema.* This is a super-replicating hedge: if the upper (lower) barrier is hit first, the single barrier contract written on the lower (upper) barrier contributes positive unwanted value. The hedger needs to add extra layers to get exact replication.

This work shows that, under certain assumptions, *double barrier interdependence commands extra hedging layers all made of single knockins with the same maturity as the double barrier option.*

This numerical example shows the structure of those hedging layers. Take as underlying asset a commodity. Its price per unit follows a geometric Brownian motion and its initial value is equal to \$90. Consider a double knockin call with lower barrier \$80 and upper barrier \$100. Its strike is \$90. The double knockin call[1] is priced \$12.8079. The double knockin price is mainly made of the \$100-in price (\$12.7587) as logprice drift is positive. The following table shows barriers, strikes, portfolio amounts, portfolio amounts in \$s of the single knockin positions that constitute the hedging layers. The table illustrates how these single barrier options have barriers which take progressive distance from the original barrier corridor \$80 / \$100. Summing the portfolio amounts in \$s of all the first 14 single knockins (from barrier level \$21 to barrier level \$381) gives the double knockin exact price. The super-replicating portfolio, the sum of a \$80-in and a \$100-in only, is priced \$16.5163. The full replicating portfolio is made of a countable infinity of single knockin positions and I call it *Double Barrier Exact Hedge* (*DBEH*). The first few positions of the DBEH are sufficient to achieve good replication of the double knockin.

An example of Double Barrier Exact Hedge (DBEH)

Knockin barrier in \$s	Strike in \$s	Amount	Amount in \$s
381.47	343.32	0.2434	0.000015
305.18	343.32	-0.2434	-0.000023
244.14	219.73	0.3898	0.009642
195.31	219.73	-0.3898	-0.011746
156.25	140.63	0.6243	0.835973
125.00	140.63	-0.6243	-0.887747
(upper barrier) 100.00	90.00	1.0000	12.758694
(initial spot price) 90.00	(original strike) 90		
(lower barrier) 80.00	90.00	1.0000	3.757592
64.00	57.60	-1.6017	-3.66775
51.20	57.60	1.6017	0.113625
40.96	36.86	-2.5655	-0.100540
32.77	36.86	2.5655	0.000559
26.21	23.59	-4.1093	-0.000413
20.97	23.59	4.1093	0.000000

2 Contributions

The DBEH has a five-fold advantage.

1. It is static;

2. It exhibits an automatically-in feature along the barriers.

[1] Other option parameters are: annualized riskfree rate equal to 5%, annualized volatility of logprice equal to 30%, 1-year maturity, and convenience yield equal to 0.

3. It has smoother value around the barriers than the value of other hedges made of non-barrier securities.
4. It takes account of the drift towards either barrier generated by a non-trivial cost of carrying the underlying asset.
5. It establishes an explicit link between single barrier pricing and double barrier pricing.

Portfolio amounts of the DBEH are static, that is, not time-varying except for the first passage time of the underlying price through either barrier. Static hedging has the advantage of suffering less from transaction costs and pricing model misspecification as you trade at most 2 times. The path-dependent options I examine often have high gammas and vegas, that is, their delta is highly time-varying and option prices are quite sensitive to volatility changes. In this case, static hedging is much likely to be easier and cheaper than dynamic hedging. The first analysis of static hedging of path-dependent options is due to Bowie and Carr (1994) and Derman, Ergener, and Kani (1994).

Static hedging of double barrier options by means of non-barrier options has been proposed by Carr, Ellis and Gupta (1998) and by Andersen, Andreasen, and Eliezer (2000). Along the barriers, the hedger should fully unwind the hedge because the double barrier contract automatically either comes to life or terminates. Trading along the barriers may be difficult so that hedging a double barrier with single knockins becomes important. With the DBEH, the hedger must only unwind its non-triggered legs. This is because, if \$100 is reached before \$80 in due time, the \$100-in leg automatically kicks in. The non-triggered legs have barriers which are fairly distant from the original barrier corridor. This gives greater comfort when unwinding as the value of the DBEH is going to be stable around the corridor extrema, even if the hedger has a misspecified perception of the underlying market dynamics.

If the underlying asset commands a positive (negative) cost of carry, then its risk-adjusted price exhibits a drift towards the upper barrier (lower barrier). Even in presence of such non-trivial risk-adjusted drift, the DBEH remains exact. I show that, with zero cost of carry, the DBEH specializes to the hedge proposed by Carr, Ellis and Gupta (1998). This is because, if you break down the DBEH legs into subportfolios of non barrier options, the two hedges correspond layer by layer. The DBEH needs a countable infinity of single knockins while the hedge proposed by Andersen, Andreasen, and Eliezer (2000) needs an along-all-strikes continuum of European options and an along-all-maturities continuum of calendar spreads.

Single barrier option prices are well known (see Merton (1973), Cox and Rubinstein (1985), Benson and Daniel (1991), Hudson (1991), Reiner and Rubinstein (1991), Heynen and Kat (1994), Rich (1994), and Trippi (1994)).

However, double barrier pricing is difficult because of the double barrier interdependence. The mathematics which unravels that interdependence is somewhat awkward, so that existing closed-form prices (Hui (1996), Kunitomo and Ikeda (1992), Lin (1997), Pelsser (1997)) hardly give financial intuition. The DBEH states that the double barrier option price is a weighted sum of single barrier option prices with weights which do not depend on the initial underlying price.

Geman and Yor (1996) and Jamshidian (1997) start from techniques based on time-horizon Laplace transforms and suggest numerical techniques for double option pricing. The analysis here develops the financial-engineering potential in those techniques by carving out explicit pricing and static-hedging results.

The rest of this work is organized as follows. Section 3 and Section 3 shows how the DBEH works. Sections 4 and 5 describe its advantages. Section 6 concludes. The appendix in Section 7 gives technical details and proofs of the propositions.

3 The Double Barrier Exact Hedge (DBEH)

Here I show that, under the Black-Scholes assumptions, the double barrier option price is a weighted sum of single barrier option prices. Such pricing results cast light on the financial nature of the contract. The key feature is that they project the risk of double instruments on to single barrier instruments.

Let $C^L_{knockin}(S_0, K, T)$ ($C^U_{knockin}(S_0, K, T)$) denote the price of a single knockin call with barrier L (U). The three arguments of the price function are the initial price S_0 of the underlying asset , the strike price K, and the option maturity T. The lower barrier L and upper barrier U straddle the initial underlying price S_0 and the strike K ($L \leq S_0 \leq U$ and $L \leq K \leq U$). The double knockin call, with price

$$C^{L,U}_{knockin}(S_0, K, T),$$

is a call option which is initiated whenever the upper barrier U or the lower barrier L is touched before or at option maturity. The instantaneous return rate of the riskfree asset is the constant r and the underlying asset offers a constant instantaneous payout rate d. $C(S_0, K, T)$ denotes the standard call price.

The DBEH unravels the pricing and hedging difficulty of double barrier options in a way which makes it easily comparable with the existing double barrier option literature, in particular with the double barrier option decomposition of Carr, Ellis and Gupta (1998).

Proposition 1 *Under the Black-Scholes assumptions, the double knockin call price has the following exact decomposition:*

$$C^{L,U}_{knockin}(S_0,K,T) = \qquad \text{(`Double Barrier Exact Hedge')}$$

$$C^{U}_{knockin}(S_0,K,T) + C^{L}_{knockin}(S_0,K,T)$$

$$+\sum_{n=1}^{\infty}\left(\frac{m_{BS}(0,\ln U,\ln L)}{m_{BS}(0,\ln L,\ln U)}\right)^{+n} \times \left(\frac{U}{L}\right)^{-2n} \times C^{U\left(\frac{U}{L}\right)^{+2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{+2n}, T\right)$$
$$-\sum_{n=1}^{\infty}\left(\frac{m_{BS}(0,\ln L,\ln U)}{m_{BS}(0,\ln U,\ln L)}\right)^{+n} \times \left(\frac{U}{L}\right)^{+2n} \times C^{U\left(\frac{U}{L}\right)^{-2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{-2n}, T\right)$$

$$+\sum_{n=1}^{\infty}\left(\frac{m_{BS}(0,\ln L,\ln U)}{m_{BS}(0,\ln U,\ln L)}\right)^{+n} \times \left(\frac{U}{L}\right)^{+2n} \times C^{L\left(\frac{U}{L}\right)^{-2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{-2n}, T\right)$$
$$-\sum_{n=1}^{\infty}\left(\frac{m_{BS}(0,\ln U,\ln L)}{m_{BS}(0,\ln L,\ln U)}\right)^{+n} \times \left(\frac{U}{L}\right)^{-2n} \times C^{L\left(\frac{U}{L}\right)^{+2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{+2n}, T\right),$$

where the constant σ is the local volatility of the underlying instantaneous percentage returns on the underlying asset, $\frac{dS_t}{S_t}$. The portfolio-weight factors are

$$m_{BS}(0,\ln L,\ln U) = e^{+(\ln L-\ln U)\left(-\frac{r-d-\frac{1}{2}\sigma^2}{\sigma^2}\right)} e^{+|\ln L-\ln U|\left(-\frac{\left|r-d-\frac{1}{2}\sigma^2\right|}{\sigma^2}\right)}$$

and

$$m_{BS}(0,\ln U,\ln L) = e^{+(\ln U-\ln L)\left(-\frac{r-d-\frac{1}{2}\sigma^2}{\sigma^2}\right)} e^{+|\ln U-\ln L|\left(-\frac{\left|r-d-\frac{1}{2}\sigma^2\right|}{\sigma^2}\right)}.$$

Proof. See the appendix. ■

Table 1 displays the structure of the DBEH. Portfolio amounts and single barriers are fully characterized in terms of the risk-adjusted probability of the price ever travelling the distance $[L,U]$ from L to U and in the opposite direction, $m_{BS}(0,\ln L,\ln U)$ and $m_{BS}(0,\ln U,\ln L)$. Indeed, these two excursion probabilities makes the portfolio weights. The factor $\left(\frac{U}{L}\right)^{-1}$ rescales the single knockin option prices, their strikes, and their barriers. $\left(\frac{U}{L}\right)^{-1}$ would be the risk-adjusted probability of the price ever travelling from L to U if the risk-adjusted price had zero local drift, that is, if either the underlying asset had zero cost of carry or it was a forward.

Proposition 2 *Under the Black-Scholes assumptions, the static hedge proposed by Carr, Ellis, and Gupta (1998) and the DBEH coincide in every respect.*

Proof. The proof is lengthy. Given the assumption of a zero cost of carrying the underlying asset in the black-Scholes setting, it is based on the break down of the DBEH legs into subportfolios of non barrier options. I can provide the proof on request. ■

Table 2 illustrates the equivalence between the two hedges in the case of zero carrying costs. Since they corrispond layer by layer, their hedging architecture is the same. Carr, Ellis and Gupta (1998), pp. 1174-1176, describe how this architecture works. Consider again the \$80 / \$100 double barrier knockin call example. You want to construct the replicating portfolio. If you knew in advance that the underlying price reaches \$80 before it reaches \$100, then buying a \$80 knockin call would be sufficient. Alternatively, if you knew in advance that the underlying price reaches the higher barrier \$100 first, buying a \$100 knockin call would do the job. Because you do not know in advance which barrier will be hit first, you shall try to combine the two portfolios. The problem with this combined portfolio is that each written-in call contributes (positive) value at the other barrier. You must zero out that contributed value along each barrier. For example, along \$100, the positive influence of the \$80 knockin is offset by selling an amount

$$\frac{m_{BS}\left(0, \ln 100, \ln 80\right)}{m_{BS}\left(0, \ln 80, \ln 100\right)} \times \left(\frac{100}{80}\right)^{-2} = 0.6243$$

of up-in calls with barrier $80\left(\frac{100}{80}\right)^{+2} = 125$ and strike $90\left(\frac{100}{80}\right)^{+2} = 140.63$.

Along \$80, the positive influence of the \$100 knockin is offset by selling an amount

$$\frac{m_{BS}\left(0, \ln 80, \ln 100\right)}{m_{BS}\left(0, \ln 100, \ln 80\right)} \times \left(\frac{100}{80}\right)^{+2} = 1.6017$$

of down-in calls with barrier $100\left(\frac{100}{80}\right)^{-2} = 64$ and strike $90\left(\frac{100}{80}\right)^{-2} = 57.60$. However, these short positions generate negative along the opposite barrier so that other knockin positions must be added. Each additional positions hedges at one barrier but creates an error at the other barrier. The size of that error decreases to zero with the number of hedging layers added.

4 The barrier effect

The hedger can project the risk of barrier instruments, and in particular double barriers, on to simple European options. This means that, as soon as either barrier is hit, 'manual' unwinding of the hedge must take place. Along the triggers, traders have a strategic edge on barrier option hedgers who must in principle unwind their positions. This exposes barrier option hedgers to underlying market price manipulation and spurious volatility.

Hence originates Taleb's (1998) risk management hint: avoid hedging discontinuous exposures (barrier instruments) with continuous ones (non-barrier instruments).

The DBEH projects the risk of double barriers on to single barriers. The key part of this projection is given by two single knockins with barriers equal to the corridor extrema. Thus, the hedge exhibits an automatically-in feature along the barriers. This automatically does most of the unwinding. The rest of the hedge has small and stable value around the corridor extrema because it is made of not-yet-triggered knockins. This gives comfort to hedger as it protects her from price gaps through the corridor extrema.

Montecarlo simulations are on their way to show the reaction of residual hedge value to underlying price movements in proximity of the barriers. In the simulations, the underlying price will exhibit stochastic volatility and jumps, whereas the DBEH is implemented as suggested in the Black-Scholes setting. The objective is to show that, even with a misspecified base for the DBEH, such an hedge still offers protection.

5 The cost-of-carry effect

How important is keeping track of a drift towards either barrier generated by a non-zero cost of carry of the underlying asset? An answer to this question is the evaluation, along both barriers, of the part of the hedge proposed by Carr, Ellis and Gupta (1998) which is meant to be zero over there if the carrying cost had been zero.

6 Conclusions

Barrier derivatives are becoming increasingly liquid, especially in the foreign exchange markets. Double barrier options provide risk managers with cheaper means to hedge their exposures without paying for the price ranges which they believe unlikely to occur. Double barrier options stipulate a double barrier price corridor which encompasses the initial level of the underlying asset price and the options are triggered or terminated whenever the underlying asset price breaches either barrier for the first time before or at maturity.

The mutual dependence of the two barriers makes these options difficult to price. This work represents their price like a weighted sum of single barrier knockin option prices, where the latter are well known.

The mutual dependence of the two barriers also makes these options difficult to hedge. The pricing representation implies a static hedging strategy (the DBEH) device of risk immunization

Double barrier hedges offer full protection only if unwound along the barriers. Along, the DBEH has automatic unwinding. Its residual part after automatic unwinding is likely to have small and stable value because it is made of knockins with barriers laying outside the double barrier corridor. These features help hedgers in overcoming possible trading issues along the barriers.

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7 Appendix

The log of the economically relevant state variable follows a diffusion process with dynamics:

$$dx_t = \mu(x_t)\,dt - \frac{1}{2}\sigma^2(x_t)\,dt + \sigma(x_t)\,dW_t,$$

where W_t is a Standard Brownian Motion and μ and σ are time-homogeneous and satisfy the conditions that allow for x_t's existence and uniqueness.

The process is Strong Markov. Thus, it starts afresh from any stopping time, that is, $x_{s-\tau} - x_\tau$ is stochastically independent from every function of any x_t's stopping time τ $(s > \tau)$. The initial level of the state variable is x_0 and its level at the finite time horizon T is x. The probability density of x_t's transition from x_0 to x during T, $p(x_0, x, T)$ has time-horizon Laplace transform given by:

$$L(\lambda, x_0, x) = \int_0^\infty \exp(-\lambda T)\, p(x_0, x, T)\, dT, \qquad \lambda \geq 0.$$

Taking time-horizon Laplace transforms simplifies the analysis. The Partial Differential Equation (PDE) dynamics of $p(x_0, x, T)$ turns into an Ordinary Differential Equation (ODE) dynamics.

Proposition 3 *The time-horizon Laplace transform $L(\lambda, x_0, x)$ satisfies the ODE*

$$\frac{1}{2}\sigma^2(x_0)\, L_{x_0 x_0} + \left(\mu(x_0) - \frac{1}{2}\sigma^2(x_0)\right) L_{x_0} - \lambda L = 0, \qquad \text{(Laplace ODE)}$$

where L_{x_0} and $L_{x_0 x_0}$ denote $L(\lambda, x_0, x)$'s first and second derivatives with respect to x_0. $L(\lambda, x_0, x)$ is positive and unique.

Proof. The probability density of x_t's transition is an Itô process and it can be conceived as a conditional expectation, that is, as a local martingale. Thus, its local drift must be zero, which means that the expectation, conditional on x_0, of p's infinitesimal changes is null, $E(dp \mid x_0) = 0$. This is p's backward equation and one gets the Laplace ODE by taking time-horizon Laplace transforms in it. L is positive because p is non-negative in all its arguments and it is unique because of Laplace transforms' uniqueness. ■

The moment generating function of x_t's first exit time through some barrier b,

$$m(\lambda, x_0, b),$$

has a natural link with the Laplace transform of the probability density of x_t's transition from x_0 to b as well as that of the probability density of x_t's transition from b to the same level b.

Proposition 4 *The moment generating function of x_t's first exit time through an upper barrier b^+ (lower barrier b^-), $m(\lambda, x_0, b^\pm)$, satisfies the Laplace ODE ($b^- \leq x_0 \leq b^+$) with these initial conditions:*

$$m(\lambda, b^\pm, b^\pm) = 1,$$

$$0 < m(\lambda, x_0, b^\pm) \leq 1. \qquad \text{(`Probability Bound I')}$$

The solution to the Laplace ODE is given by

$$m(\lambda, x_0, b^{\pm}) = \frac{L(\lambda, x_0, b^{\pm})}{L(\lambda, b^{\pm}, b^{\pm})}. \tag{Single Barrier M.G.F.}$$

The Single Barrier M.G.F.s enjoy the following properties. The Single Barrier M.G.F. $m\left(\lambda, x_0, b^-\right)$ is strictly decreasing in x_0 and the Single Barrier M.G.F. $m\left(\lambda, x_0, b^+\right)$ is strictly increasing in x_0. For finite $c \geq 0$,

$$m(\lambda, x_0, b^+ + c) = m(\lambda, x_0, b^+)m(\lambda, b^+, b^+ + c), \tag{'Strong Markov Up'}$$

$$m(\lambda, x_0, b^- - c) = m(\lambda, x_0, b^-)m(\lambda, b^-, b^- - c), \tag{'Strong Markov Down'}$$

For any $\lambda > 0$ and $x_0 \neq b^{\pm}$,

$$m(\lambda, x_0, b^{\pm}) < m(0, x_0, b^{\pm}) \leq 1. \tag{'Probability Bound II'}$$

Proof. Let $\tau_{b^{\pm}}$ be x_t's first exit time through $b^{\pm}$. $\tau_{b^{\pm}}$'s moment generating function satisfies the Laplace ODE as it is the Laplace transform of $\tau_{b^{\pm}}$'s probability density, which in turn satisfies the backward equation (its local drift is zero). If $x_0 = b^{\pm}$, the first exit time is zero for sure, that is, $\exp\left(-\lambda\tau_{b^{\pm}}\right)$ is constant and equal to 1. This gives the first initial condition. The second condition, 'Probability Bound I', comes from the fact that $\exp\left(-\lambda\tau_{b^{\pm}}\right)$ times $\tau_{b^{\pm}}$'s probability density is not greater than $\tau_{b^{\pm}}$'s probability density and that $m(0, x_0, b^{\pm})$ is the probability of ever reaching the barrier $b^{\pm}$. The result for Single Barrier M.G.F. follows from $L(\lambda, x_0, x)$'s structure and properties. The result also comes from Jamshidian (1997) who makes use of the Strong Markov Property and of the Convolution Property of Laplace transforms. When λ is positive, Single Barrier M.G.F.s' monotonicity comes from Breiman (1968), p. 380. The results for 'Strong Markov Up' and 'Strong Markov Down' follow from $L(\lambda, x_0, x)$'s structure and properties. The Strong Markov Property and of the Convolution Property of Laplace transforms prompt an alternative derivation of them. ■

Let

$$m^+\left(\lambda, x_0, b^-, b^+\right) \qquad \left(\ m^-\left(\lambda, x_0, b^-, b^+\right)\ \right)$$

be the moment generating function of x_t's first exit time through the upper barrier b^+ (lower barrier b^-) whithout any passage through the lower barrier b^- (upper barrier b^+). The sum of m^+ and m^- gives the moment generating function of x_t's first exit time through either barrier. m^+ and m^- satisfy the Laplace ODE with these initial conditions:

$$\begin{aligned} m^+\left(\lambda, b^+, b^-, b^+\right) &= 1,\ m^+\left(\lambda, b^-, b^-, b^+\right) = 0, \\ m^-\left(\lambda, b^+, b^-, b^+\right) &= 0,\ m^-\left(\lambda, b^-, b^-, b^+\right) = 1. \end{aligned}$$

This is because, if $x_0 = b^+$, the upper barrier is reached for sure and from the very beginning, without touching the lower barrier b^-. This implies $m^+ = 1$ and $m^- = 0$. The reverse holds for $x_0 = b^-$.

Proposition 5 *If x_t is an Arithmetic Brownian Motion (μ and σ are constants), the moment generating functions $m^{\pm}\left(\lambda, x_0, b^-, b^+\right)$ can be decomposed as follows:*

$$m^+\left(\lambda, x_0, b^-, b^+\right) = \tag{'m^+'s form'}$$

$$+\sum_{n=0}^{\infty}\left(\frac{m\left(0, b^+, b^-\right)}{m\left(0, b^-, b^+\right)}\right)^n m\left(\lambda, x_0, b^+ + 2n\left(b^+ - b^-\right)\right)$$
$$-\sum_{n=0}^{\infty}\left(\frac{m\left(0, b^-, b^+\right)}{m\left(0, b^+, b^-\right)}\right)^{n+1} m\left(\lambda, x_0, b^+ - 2\left(n+1\right)\left(b^+ - b^-\right)\right),$$

and

$$m^{-}\left(\lambda, x_0, b^{-}, b^{+}\right) = \qquad (\text{‘}m^{-}\text{’s form’})$$

$$+\sum_{n=0}^{\infty}\left(\frac{m\left(0, b^{-}, b^{+}\right)}{m\left(0, b^{+}, b^{-}\right)}\right)^{n} m\left(\lambda, x_0, b^{-} - 2n\left(b^{+} - b^{-}\right)\right)$$
$$-\sum_{n=0}^{\infty}\left(\frac{m\left(0, b^{+}, b^{-}\right)}{m\left(0, b^{-}, b^{+}\right)}\right)^{n+1} m\left(\lambda, x_0, b^{-} + 2\left(n+1\right)\left(b^{+} - b^{-}\right)\right).$$

Proof. I focus on the ‘m^{+}’s form’. Similar arguments justify the ‘m^{-}’s form’. The operator that generates the Laplace ODE is linear so that an absolutely convergent series of Single Barrier M.G.F.s satisfies it. Absolute convergence is sufficient for a safe reversal of order in the infinite-sum and derivative operations. ‘Probability Bound I’ and ‘Probability Bound II’ imply that $m\left(\lambda, b^{-}, b^{+}\right)$ times $m\left(\lambda, b^{+}, b^{-}\right)$ is less than 1. Thus, the absolutely convergent series

$$+m\left(\lambda, x_0, b^{+}\right)\sum_{n=0}^{\infty}\left(m\left(\lambda, b^{+}, b^{-}\right) m\left(\lambda, b^{-}, b^{+}\right)\right)^{n}$$
$$-m\left(\lambda, x_0, b^{-}\right)\sum_{n=0}^{\infty}\left(m\left(\lambda, b^{-}, b^{+}\right) m\left(\lambda, b^{+}, b^{-}\right)\right)^{n} m\left(\lambda, b^{-}, b^{+}\right)$$

satisfies the Laplace ODE and meets m^{+}’s two initial conditions. The same preliminary decomposition can be obtained from Jamshidian’s (1997) analysis by expanding

$$\left(1 - \frac{L\left(\lambda, b^{-}, b^{+}\right)}{L\left(\lambda, b^{+}, b^{+}\right)}\frac{L\left(\lambda, b^{+}, b^{-}\right)}{L\left(\lambda, b^{-}, b^{-}\right)}\right)^{-1}$$

in power series. The Arithmetic Brownian Motion hypothesis yields

$$m\left(\lambda, b^{+}, b^{-}\right) = m\left(\lambda, b^{-}, b^{+}\right)\frac{m\left(0, b^{+}, b^{-}\right)}{m\left(0, b^{-}, b^{+}\right)}.$$

The Arithmetic Brownian Motion hypothesis implies that the travel distance $\left[b^{-}, b^{+}\right]$ can be shifted by any shifting factor $\pm c$. Set c equal to either $n\left(b^{+} - b^{-}\right)$ or $\frac{1}{2}n\left(b^{+} - b^{-}\right)$). Then, ‘Strong Markov Up’ and ‘Strong Markov Down’ lead to ‘m^{+}’s form’’s actual form. ■

The mapping between this analysis and the options market setting is straightforward. One has $\ln U = b^{+}$, $\ln S_0 = x_0$, $\ln L = b^{-}$, $S_t = \exp\left(x_t\right)$, and $var\left(\frac{dS_t}{S_t} \mid S_t\right) = \sigma^2\left(x_t\right)$, where $var\left(\frac{dS_t}{S_t} \mid S_t\right)$ is the variance of the asset percentage returns conditional on the current asset price. After risk adjustment of the actual probabilites, one has:

$$E\left(\frac{dS_t}{S_t} \mid S_t\right) = \mu\left(x_t\right) = r - d\left(x_t\right).$$

The probability density which prices the double knockin contracts has the following option-maturity Laplace transform:

$$m^{+}L(\lambda, \ln U, \ln S_T) + m^{-}L(\lambda, \ln L, \ln S_T).$$

PROOF OF PROPOSITION 1

Proposition 5 as well as option prices’ homogeneity of degree 1 in the initial price, the strike, and the possible barriers, can be used. This gives the DBEH result and completes the proof. ■

Table 1: **The Double Barrier Exact Hedge (DBEH): any cost of carrying the underlying asset**

The arguments of the option price functions are the current underlying asset spot price, S_0, the strike price K, and the time to maturity, T. C denotes the price of a standard call. $C^{L,U}_{knock-in}$ denotes the price of a double knockin call with upper barrier U and lower barrier L. $C^{U}_{knock-in}$ denotes the price of a single knockin call with barrier U. $C^{U\left(\frac{U}{L}\right)^{-2(n+1)}}_{knock-in}$ denotes the price of a single knockin call with barrier $U\left(\frac{U}{L}\right)^{-2(n+1)}$ (single barrier down knockin call). Notice that the second price series is made up of prices of knockin call options which are standard call options. This is because their barrier level, $L\left(\frac{U}{L}\right)^{+2(n+1)}$, is always below their strike level, $K\left(\frac{U}{L}\right)^{+2(n+1)}$. r is the risk-free rate and d is the asset's payout ratio. σ is the local volatility of the returns on the underlying. r, d, and σ are constant.

$$
\begin{aligned}
& C^{L,U}_{knockin}\left(S_0,K,T\right) \\
&= C^{U}_{knockin}\left(S_0,K,T\right) + C^{L}_{knockin}\left(S_0,K,T\right) \\
&\underbrace{+\sum_{n=1}^{\infty} e^{-2\left(\frac{\left(r-d-\frac{1}{2}\sigma^2\right)}{\sigma^2}+1\right)n(\ln U-\ln L)} \times C^{U\left(\frac{U}{L}\right)^{+2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{+2n}, T\right)}_{\text{(single barrier up-and-in calls with barrier above the strike)}} \\
&\underbrace{-\sum_{n=1}^{\infty} e^{+2\left(\frac{\left(r-d-\frac{1}{2}\sigma^2\right)}{\sigma^2}+1\right)n(\ln U-\ln L)} \times C^{U\left(\frac{U}{L}\right)^{-2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{-2n}, T\right)}_{\text{(single barrier down-and-in calls with barrier above the strike)}} \\
&\underbrace{+\sum_{n=1}^{\infty} e^{+2\left(\frac{\left(r-d-\frac{1}{2}\sigma^2\right)}{\sigma^2}+1\right)n(\ln U-\ln L)} \times C^{L\left(\frac{U}{L}\right)^{-2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{-2n}, T\right)}_{\text{(single barrier down-and-in calls with barrier below the strike)}} \\
&\underbrace{-\sum_{n=1}^{\infty} e^{-2\left(\frac{\left(r-d-\frac{1}{2}\sigma^2\right)}{\sigma^2}+1\right)n(\ln U-\ln L)} \times C^{L\left(\frac{U}{L}\right)^{+2n}}_{knockin}\left(S_0, K\left(\frac{U}{L}\right)^{+2n}, T\right)}_{\text{(single barrier up-and-in calls with barrier below the strike = standard calls)}}
\end{aligned}
$$

Table 2: **The static hedge of Carr, Ellis, and Gupta (1998): zero cost of carrying the underlying asset**

The arguments of the option price functions are the current underlying asset spot price, S_0, the strike price K, and the time to maturity, T. $C^{L,U}_{knock-out}$ denotes the price of a double knockout call with upper barrier U and lower barrier L. C denotes the price of a standard call. P denotes the price of a standard put. BP (GP) is the price of a European bynary (gap) put option, BC (GC) is the price of a European bynary gap) call option. The risk-free rate and the asset's payout ratio are equal so that the risk-neutral drift of the underlying asset price is zero. The local volatility of the returns on the underlying asset can be time-dependent and price-dependent but must satisfy a logprice-symmetric condition: The volatility of the underlying asset price is a known function $\sigma(S_t, t)$ of the underlying price S_t at time t and it satisfies the symmetry condition $\sigma(S_t, t) = \sigma\left(\frac{S_0^2}{S_t}, t\right)$ for all $S_t \geq 0$ and $t \in [0, T]$, where S_0 is the current underlying price.

$$C^{L,U}_{knockout}(S_0, K, T)$$

$$= C(S_0, K, T)$$

$$-\underbrace{\left(KU^{-1}C\left(S_0, K^{-1}U^2, T\right) + (U-K)\left(2BC(S_0, U, T) + U^{-1}C(S_0, U, T)\right)\right)}_{\text{(single barrier } (U) \text{ up-and-in call with barrier below the strike } (K))}$$

$$-\underbrace{KL^{-1}P\left(S_0, \left(\frac{L}{K}\right)L\left(\frac{U}{L}\right)^{-2n}, T\right)}_{\text{(single barrier } (L) \text{ down-and-in calls with barrier below the strike } (K))}$$

$$-\underbrace{\sum_{n=1}^{\infty}\left(\left(\frac{U}{L}\right)^{-n}\left(\begin{array}{c} KU^{-1}C\left(S_0, \left(\frac{U}{K}\right)U\left(\frac{U}{L}\right)^{+2n}, T\right) \\ +(U-K)\left(\begin{array}{c} 2e^{+2n(\ln U - \ln L)}\times \\ BC\left(S_0, U\left(\frac{U}{L}\right)^{+2n}, T\right) \\ +U^{-1}C\left(S_0, U\left(\frac{U}{L}\right)^{+2n}, T\right) \end{array}\right) \end{array}\right)\right)}_{\text{(single barrier } (U\left(\frac{U}{L}\right)^{+2n}) \text{ up-and-in calls with barrier above the strike } (K\left(\frac{U}{L}\right)^{+2n}))}$$

$$+\underbrace{\sum_{n=1}^{\infty}\left(\left(\frac{U}{L}\right)^{+n}\left(\begin{array}{c} P\left(S_0, K\left(\frac{U}{L}\right)^{-2n}, T\right) \\ +(U-K)\left(\begin{array}{c} U^{-1}2GP\left(S_0, U\left(\frac{U}{L}\right)^{-2n}, T\right) \\ +U^{-1}C\left(S_0, U\left(\frac{U}{L}\right)^{-2n}, T\right) \end{array}\right) \end{array}\right)\right)}_{\text{(single barrier } (U\left(\frac{U}{L}\right)^{-2n}) \text{ down-and-in calls with barrier above the strike } (K\left(\frac{U}{L}\right)^{-2n}))}$$

$$-\underbrace{\sum_{n=1}^{\infty}\left(\left(\frac{U}{L}\right)^{+n}KL^{-1}P\left(S_0, \left(\frac{L}{K}\right)L\left(\frac{U}{L}\right)^{-2n}, T\right)\right)}_{\text{(single barrier } (L\left(\frac{U}{L}\right)^{-2n}) \text{ down-and-in calls with barrier below the strike } (K\left(\frac{U}{L}\right)^{-2n}))}$$

$$+\underbrace{\sum_{n=1}^{\infty}\left(\left(\frac{U}{L}\right)^{-n}C\left(S_0, K\left(\frac{U}{L}\right)^{2n}, T\right)\right)}_{\text{(single barrier } (L\left(\frac{U}{L}\right)^{+2n}) \text{ up-and-in calls with barrier below the strike } (K\left(\frac{U}{L}\right)^{+2n}) = \text{ standard calls })}$$
