ASSET PRICING WITHOUT PROBABILITY

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Abstract. In this paper we propose a model of financial markets in which agents have limited ability to trade and no probability measure is given from the outset. In the absence of arbitrage opportunities, assets are priced according to a probability measure that lacks countable additivity. Pricing bubbles are shown to exist and a clear characterization is given in conditional terms. Despite finite additivity, we obtain an explicit representation of the expected value with respect to the pricing measure, based on some new results on finitely additive conditional expectation and finitely additive martingales. From this representation we derive a modified version of the *Capital Asset Pricing Model* according to which the expected value of augmented asset returns is explained by correlation with the market price of risk. In general this conclusion need not be true for original returns and this is shown to imply deviations from the *CAPM* that may potentially contribute to explain the equity premium puzzle. We also discuss special cases in which the above results can be improved.

1. Introduction.

Continuous time financial models adopt a wide definition of the trading activity of agents, no matter the degree of market imperfections considered. The ground for such definition is laid by two basic assumptions, that a probability measure is given from the outset (and known to agents) and that, with reference to this, gains from trade may be modeled as semimartingales. We shall henceforth refer to such modeling choice as the traditional setting. As clearly illustrated by Duffie and Huang [25], the continuous time framework permits to rewrite under full generality the model of financial markets originally proposed by Arrow [4] for a finite state space. Furthermore, it has fostered a large number of important results in the theory of asset pricing and portfolio selection which represent the backbone of modern financial theory. In all such developments it is therefore implicit the view of investors as agents of considerably refined ability, both in the assessment of uncertainty and in the trading of assets. The delta hedging strategy of Black and Scholes [10], a standard textbook case regardless of its overwhelming complexity, is a good case in point.

We present in this paper a theory of financial prices in continuous time and with a general state space but based on a more realistic picture of individual capabilities. We introduce in section 2 a model with two distinctive features: no probability measure is taken as given, as in the spirit of Arrow's model, and the trading of assets is considerably restricted. More precisely, we will only consider trading strategies which (i) extend over a finite time horizon, (ii) prescribe rebalancing positions a finite number of times and (iii) are contingent on a finite number of possible scenarios. On the other hand, given our focus on the constraints to financial activity coming from the subjective side, we assume that markets are free of imperfections and that investment (discounted) returns are bounded. In sections 3 to 8 we analyze the implications ensuing from the basic economic principle of absence of arbitrage opportunities. We obtain

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versions of the *Fundamental Theorem of Asset Pricing* and of the *Capital Asset Pricing Model* that, while comparing with the corresponding results developed in the traditional setting, exhibit at the same times differences of significant economic content.

An explicit motivation for the assumption of a given probability measure may hardly be found in contributions to the theory of finance. However, two main arguments underlying it refer to either the possibility of retrieving such a measure from historical data or to the circumstance that this be embodied in individual preferences over uncertain outcomes, i.e. expected utility. The first argument, helps explaining how is it that all agents have the same starting probability, given ease of accessing past data. Nevertheless, averaging data is not free of troubles whenever time series are not sufficiently stationary, as this may either lead to unstable estimates or to severe undervaluation of rarely occurring events. Non stationarity of financial time series is one of the of stylized facts in financial analysis and it contributes significantly to explaining well known puzzles, such as the equity premium ([7] and [17] provide alternative explanations of the equity premium puzzle based on non stationarity).

As for the second argument, a vast stream of literature, taking its moves from paradoxes of expected utility, has questioned the idea that a probabilistic assessment of uncertainty be implicit in preferences both on theoretical and empirical ground. Although in experimental psychology, subadditivity is a long-standing evidence (see [54] and [55] for pioneering contributions), more recent theoretical work has laid ground for models in which choice is not based on probability but rather on set functions with considerably poorer structure. Examples are Choquet expected utility [51], case-based decision making [29], prospect theory [42] and [57] and support theory [56]. On the other side, much experimental evidence has been obtained (see [18] and [19] for comprehensive reviews) showing how deeply individual choice is influenced by psychological elements such as the framing of decisions. These elements may lead investors to attach importance to events in a selective way and be responsible of market phenomena such as over- or under-reactions.

This brief discussion motivates the choice to abandon the familiar assumption of a given probability measure and to treat as the primitive of our model the collection $\mathcal{N}$ of events that do not affect individual decisions. In the traditional setting such events, called *negligible*, would simply consist of the null sets generated by some probability prior Q. We want to stress, however, that in our approach this is only one important special case – another being $\mathcal{N} = \varnothing$. Our stance is that negligible events need not stem from a probabilistic assessment but, for example, from some sort of bounded rationality making individuals unable to make decisions contingent on some specific events. This part of the model is presented in section 2.2.

If the market is free of arbitrage opportunities we obtain a number of conclusions. First, there exists a *pricing measure*, m, (a risk neutral measure, in the traditional terminology) that does not charge negligible events and that will in general only be finitely additive. Second, associated to m is a countably additive probability measure P – termed the *representing measure*. The role of this compares to that of the "physical" or "objective" measure in traditional models but it is worth highlighting that in our approach P is endogenous (and typically non unique) and that it is generated by the pricing measure, rather than the other way round. The interplay between P and m is a distinguished feature of our model and drives most of our results. In some sense one should think of the representing measure as the one the investor would adopt were he to write a mathematical model of financial markets. In fact P permits an explicit and analytically tractable representation of the pricing rule arising from m, described in Proposition 2. As a consequence of this result, and upon stopping, asset returns turn out being are P semimartingales. These findings, the core mathematical results of the paper, allow to overcome some of the difficulties involved in finitely additive expectation and, in some sense, restore the traditional properties of financial models but in purely endogenous terms.

The existence of a representing measure P induced by m relies on a new decomposition result for finitely additive measures (proved in [12] but restated in Lemma 1 below) which, to some extent, translates the celebrated result of Yosida and Hewitt [59] in the framework of filtered probability spaces. Some additional results int he theory finitely additive probability also play an important role in our analysis. In particular, we prove in Proposition 1 the existence of a conditional expectation operator for finitely additive probabilities which possesses several important properties of ordinary conditional expectation and actually coincides with it whenever countable additivity obtains. This operator, of which we provide an explicit and familiar example in section 3.2, is employed to show that the pricing of assets, though forward looking, may be affected by the existence of pricing bubbles, of which we offer a conditional version.

Our model also contains strong financial implications. We construct an auxiliary return process, the *augmented* return, and show that the augmented equity premium is indeed explained by the correlation with the market price of risk. However, this explanation need not work relatively to the original return process, in contrast with the predictions of the *CAPM* and *CCAPM*. In fact we obtain a *CAPM* like formula from which it ensues that the traditional explanation of the equity premium overlooks the role of one additional risk factor associated to the discontinuous part of the return process. This further term, that disappears for predictable returns (as defined in section 7), will not in general be itself a jump process, suggesting that unpredictable discontinuities, such as market crashes, may have a long lasting influence on equity premia. Many a paper has extended the *CAPM* to the case of discontinuous asset returns (see [5], [39] and [52], among others) but in the traditional setting there cannot be but one risk factor unless *ad hoc* structure of individual preferences are invoked, e.g. [22] and [24]. On the other hand, it has long been recognized that the existence of more than one factor could be responsible for the poor performance of the *CAPM* in empirical terms.

Our preceding result may be reformulated by saying that (augmented) asset returns are turned into local martingales if discounted by a positive local martingale Z (a *martingale density* in the terminology of [52]): in other words, in a market free of arbitrage opportunities there exists a "state price density" process. This situation is typical of most financial models, often simply as the consequence of a convenient assumption. Indeed, the existence of a state price density process was shown to be equivalent to the absence of *free lunches* by Delbaen and Schachermayer [21] in a highly influential paper (see also [40]). But neither implication holds unless Z is strictly positive and of class D and the latter property, in particular, places substantial restrictions to the modelling of the volatility process. It may thus prove useful the conclusion that the existence of a martingale density is necessary for a consistent modelling of financial markets. In section 8 we prove that if the market does not allow for free lunches and if all pricing measures admit the same representing measure then the martingale density is strictly positive.

The present paper is organized as follows. After describing the model, in section 2, we prove in section 3 the existence of the pricing measure m and discuss some of its properties. In particular, we obtain a characterization of asset bubbles in conditional terms, of which we provide an explicit example too. In section 4 we derive the existence of a representing measure P associated to m and, based on this, we obtain in the following section 5 an explicit characterization of the expected value of asset returns with respect to the pricing measure. This result, which heavily exploits the characterization of the structure of the separating measure over a filtered probability space studied in [12], allows to establish, in section 6, that asset returns, conveniently transformed, are P semimartingales and that a *CAPM*-like formula holds. In section 7 we restrict attention to predictable return processes, a class for which the preceding results may be significantly enhanced. In section 8 we replace the requirement that there be no arbitrage opportunities with the stronger notion of absence of free lunches, borrowed from [21]. Eventually, in section 9 we discuss the implication of our setting for empirical research and, in particular, we characterize the distribution function of asset returns with respect to the pricing measure m.

2. The Model.

2.1. **The Set-up.** The state space is described by an arbitrary set Ω. For each date $t \in \mathbb{R}_+$, $\mathcal{F}_t$ is a σ algebra of subsets of Ω representing the information available at time t. We posit that $(\mathcal{F}_t : t \in \mathbb{R}_+)$ is a right continuous filtration and, interpretating $t = 0$ as time present, that $\mathcal{F}_0 = \{\varnothing, \Omega\}$, an assumption not uncommon in the literature that will play some role. By $\mathcal{F}$ we denote the smallest algebra on Ω containing $\bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t$, and the collection $\mathcal{N}$, to be introduced below. Although, for the reasons addressed in the introduction, we will not refer to any probability measure, it will be important to know that a probability may be constructed on $\mathcal{F}$.

Assumption 1. *The set $\mathbb{P}(\mathcal{F})$ of probability measures on $\mathcal{F}$ is not empty.*

As for notation, $\mathcal{T}$ denotes the set of stopping times of the filtration; $\mathcal{T}_0 = \{\tau \in \mathcal{T} : \tau < \infty\}$. If $X = (X_t : t \in \mathbb{R}_+)$ and $\tau \in \mathcal{T}$, by X^τ we indicate the "stopped" process $(X_{t\wedge\tau} : t \in \mathbb{R}_+)$. $\tilde{\mathcal{F}}$ is the product σ algebra $\sigma\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ on $\tilde{\Omega} = \Omega \times \mathbb{R}_+$ – where $\mathcal{B}(\mathbb{R}_+)$ is the Borel σ algebra over $\mathbb{R}_+$ – and $\mathcal{P}$ is the predictable σ algebra of subsets of $\tilde{\Omega}$ (for standard terminology on stochastic processes we refer to [38] or [48]). The stochastic integral of θ with respect to X, whenever well defined, is indicated at will by $\int \theta dX$ or $\theta.X$. We do not distinguish between a set and its indicator (so that by FG we may denote the sets $F \cap G$ or $F \times G$ as well as their indicators); if $\mathcal{G}$ is a collection of subsets of Ω, by $\mathfrak{L}(\mathcal{G})$ we indicate the linear space spanned by the indicators of sets in $\mathcal{G}$; $\mathfrak{B}(\mathbb{X})$ denotes the space of all bounded, real valued functions on some set $\mathbb{X}$ as defined in [26]. By $ba(\mathcal{F})$ and $ca(\mathcal{F})$ we denote, as usual, the spaces of additive and countably additive set functions on $\mathcal{F}$ of bounded variation.

2.2. **Negligible Events.** Preferences are not the focus of this work and will therefore not be modelled explicitly. However, we introduce a weak notion of indifference, *negligibility.* This is defined with reference to a collection $\mathcal{N}$ of subsets of Ω, the class of *negligible events*, which is given *a priori.* Letting $\mathcal{N}$ take different forms, we can cover several situations of interest to financial modelling.

Assumption 2. *The collection $\mathcal{N}$ satisfies the following properties:*

(*i*) $\Omega \notin \mathcal{N}$*;*
(*ii*) $A \in \mathcal{N}$ *and* $B \subset A$ *imply* $B \in \mathcal{N}$*;*
(*iii*) $A, B \in \mathcal{N}$ *implies* $A \cup B \in \mathcal{N}$*.*

As suggested in the introduction, $\mathcal{N}$ should be interpreted from the point of view of a decision maker as describing the events that do not affect his choice. Several examples may be given. The most familiar one is the class $\mathcal{N}_Q$ of null sets generated by some prior $Q \in \mathbb{P}(\mathcal{F})$. Alternatively, the agent's attitude towards uncertainty may be associated with a capacity or a multiplicity of priors and $\mathcal{N}$ may thus amount to the collection of sets which are null with respect to the capacity or to all priors. We may however also consider situations in which agents are simply unable to carry out a proper assessment of the likelihood of events as they may feel, e.g., that the information available to them is too poor or that its processing is too costly. Being required to consider the likelihood of some scientific discovery without being in the field is perhaps a case in point. The source of negligibility may in other words lie in some form of bounded rationality.

While property (i) only helps avoiding trivial cases, the brief discussion that precedes supports property (ii). As for (iii), although $\mathcal{N}$ need not be closed with respect to *countable* unions (as assumed in [6]), it is essential for what follows that it is so for finite unions. [1].

Definition 1. *$X : \Omega \to \mathbb{R}$ is negligible if $\{|X| > \eta\} \in \mathcal{N}$ for any $\eta > 0$.*

It is fairly clear that, due to property (ii), X is negligible if and only if $1 \wedge |X|$ belongs to the closure $\overline{\mathfrak{L}(\mathcal{N})}$ in $\mathfrak{B}(2^\Omega)$ of the linear space $\mathfrak{L}(\mathcal{N})$. This definition induces an equivalence relationship defined by saying that $X \sim_{\mathcal{N}} Y$ whenever $X - Y$ is negligible (we also say $X = Y$ up to a negligible set, shorted as $X = Y$ u.n.) as well as the quotient spaces $\mathfrak{B}(\mathcal{F},\mathcal{N}) = \mathfrak{B}(\mathcal{F}) \Big\backslash \overline{\mathfrak{L}(\mathcal{N})}$ and $\mathfrak{B}\left(\tilde{\mathcal{F}},\mathcal{N}\right) = \mathfrak{B}\left(\tilde{\mathcal{F}}\right) \Big\backslash \overline{\mathfrak{L}(\mathcal{N})}$. If $\mathcal{N} = \mathcal{N}_Q$ for some $Q \in \mathbb{P}(\mathcal{F})$ then $\mathfrak{B}(\mathcal{F},\mathcal{N}_Q) = L^\infty(\mathcal{F},Q)$ while $\mathfrak{B}(\mathcal{F},\mathcal{N}) = \mathfrak{B}(\mathcal{F})$ whenever $\mathcal{N} = \{\varnothing\}$. In Lemma 4 in the Appendix we prove, not surprisingly, that bounded linear functionals on $\mathfrak{B}(\mathcal{F},\mathcal{N})$ may be identified with finitely additive measures on $\mathcal{F}$ vanishing on $\mathcal{N}$ – i.e. elements of $ba(\mathcal{F},\mathcal{N})$. Borrowing from the theory of L^p spaces, we shall speak of the elements of $\mathfrak{B}(\mathcal{F},\mathcal{N})$ as if they were $\mathcal{F}$ measurable functions rather than sets of equivalent functions (with the only exception of Lemma 4, in the Appendix). We write $X \in \mathfrak{B}(\mathcal{F},\mathcal{N})_+$ whenever $\{X < -\eta\} \in \mathcal{N}$ for any $\eta > 0$; $X \in \mathfrak{B}(\mathcal{F},\mathcal{N})_{++}$ whenever $X \in \mathfrak{B}(\mathcal{F},\mathcal{N})_+$ and there is some $\eta > 0$ such that $\{X > \eta\} \notin \mathcal{N}$.

Our definition of an arbitrage opportunity and, more generally, the content of the next sections is compatible with any system of preferences which are monotonic in the following sense: if $X - Y \in \mathfrak{B}(\mathcal{F},\mathcal{N})_{++}$ then $X \succ Y$; if, in addition, $X \succsim Y$ when $X - Y \in \mathfrak{B}(\mathcal{F},\mathcal{N})_+$ then negligibility implies indifference. Preferences of this sort may indeed be considered as an exemplification of over confidence: the circumstance $X - Y \in \mathfrak{B}(\mathcal{F},\mathcal{N})_{++}$ does not exclude that events such as $\{X < Y - \eta\}$ may occur but simply implies that these will not be considered by the decision maker.

A natural question is whether negligibility, whatever its source, may be reconciled with probability. An answer is provided by the following result, to which we shall refer later on.

Theorem 1. *Let $\mathcal{N}$ satisfy Assumption 2. Then:*

(i). *there exists $m \in ba(\mathcal{F},\mathcal{N})_+$ with $m(\Omega) = 1$;*

(ii). *the following two statements are equivalent:*

(a) *there exists $P \in \mathbb{P}(\mathcal{F})$ vanishing on $\mathcal{N}$, i.e. $P \in \mathbb{P}(\mathcal{F},\mathcal{N})$;*

(b) *there exists $Q \in \mathbb{P}(\mathcal{F})$ such that for any increasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of sets in $\mathcal{N}$*

$$\lim_n Q(F_n^c) > 0 \tag{2.1}$$

It is always possible to find some *finitely additive* probability which is compatible with $\mathcal{N}$ in the above sense – given Assumption 2 – but this may no longer be the case if countable additivity is required. Consider the case of a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{N}$ such that $\Omega = \bigcup_n F_n$: any $m \in ba(\mathcal{F},\mathcal{N})$ is then purely finitely additive. To see that this situation is not a pure mathematical *curiosum*, imagine an individual confronted

[1]In fact (ii) may fail in special cases such as that of prior beliefs represented by a superadditive capacity, an expression of the propensity to uncertainty. Assumption 2 may be reconciled with more general situations if we interpret $\mathcal{N}$ as the subcollection of negligible sets possessing the listed properties.

with a real valued experiment (the example is taken from [32, p. 74]) and with no informative prior. The uniform distribution would perhaps be a reasonable choice: two events of equal "size" are equally likely, thus necessarily falling into the above case whenever the values of the experiment are rational numbers between 0 and 1[2]. More generally, the uniform distribution over the natural numbers is a useful model both in probability (see [9, pp. 38-41] and [41]) and in economics (see [58] where measures of this family are employed to represent perfectly diversified portfolios in an APT framework) although it does not reconcile with countable additivity. Condition (2.1) clearly rules these cases out and all it requires is essentially that there exists a measure which is not in blatant contrast with the interpretation of $\mathcal{N}$ as a collection of null sets. Failure of (2.1) will bring several complications to the analysis that follows.

Assumption 3. *$\mathcal{N}$ satisfies Assumption 2 and*

(iv). $\mathbb{P}(\mathcal{F},\mathcal{N}) \neq \varnothing$. [3].

We let $\hat{\mathcal{F}}_t = \bigcap_{u>t} \sigma(\mathcal{F}_t \cup \mathcal{N})$.

2.3. **Asset Returns and Trading Strategies.** Throughout the paper we shall make the following assumption concerning the set $\mathbb{K}$ of admissible, discounted returns:

Assumption 4. *$\mathbb{K}$ is a linear subspace of $\mathfrak{B}\left(\tilde{\mathcal{F}}\right)$ such that if $K \in \mathbb{K}$*

(i). K_t is $\mathcal{F}_t$ measurable for each $t \in \mathbb{R}_+$ and $K_0 = 0$;

(ii). there exists $T \in \mathbb{R}_+$ such that $K = K^T$;

(iii). $\theta.K \in \mathbb{K}$ whenever θ belongs to the set Θ of all processes of the form

$$\theta(\omega, t) = \sum_{m=1}^{M} \theta_m \,]]\tau_m, \tau_{m+1}]] \tag{2.2}$$

where $\tau_m \in \mathcal{T}$ and $\theta_m \in \mathfrak{L}(\mathcal{F}_{\tau_m})$, $m = 1, \ldots, M$.

We also define

$$\mathcal{K} = \{k \in \mathfrak{B}(\mathcal{F},\mathcal{N}) : k = K_\infty \text{ u.n. for some } K \in \mathbb{K}\} \tag{2.3}$$

and

$$\mathcal{C} = \mathcal{K} - \mathfrak{B}(\mathcal{F},\mathcal{N})_+ \tag{2.4}$$

By assumption, then, $0 \in \mathbb{K}$ and the corresponding asset is to be interpreted as money or, more generally, the *numéraire* asset with respect to which the processes in $\mathbb{K}$ have been normalized. Observe that, given *(ii)* above, we could equivalently require that each $\theta \in \Theta$ vanishes on $\Omega \times]T, \infty[$ for some $T \in \mathbb{R}_+$.

Assumption 4 seems to us a reasonable approximation to the way real markets actually work on three grounds. First, the strategies considered do not imply a life commitment on the side of investors. Second, trading only involves a finite number of transactions: the cost of trading – which may either consist of explicit transaction fees or be simply implicit in information processing – is then certain and reasonable. Eventually, each transaction is contingent on a finite number of scenarios, a feature making the actual implementation of the investment strategy realistically simple; it also captures the increasing importance of scenario analysis in the investment industry (see [45]). Observe that pathological situations which are of concern in the traditional approach – like so called "doubling strategies" – do not arise here.

[2]Bewley [8, p. 517] remarks that preferences supporting countable additivity of equilibrium prices over l^∞ imply "an asymptotic form of impatience".

[3]Let us just mention that omitting to impose condition (2.1) makes the whole construction in [6] potentially vacuous.

The boundedness property, although important in the sequel, may raise discussion. The existence of a lower bound on returns can be seen as the result of some form of financial regulation aiming at preventing the possibility of Ponzi schemes. The restriction of an upper bound is far less obvious and will therefore be relaxed in section 9, when dealing with applications.

A market in which the diversification of risk is to no extent restricted would allow investors to diversify their portfolios at will across admissible investment projects, provided their resulting position does not imply the possibility of unbounded losses. Portfolio returns would be described for such a market by the set

$$\mathbb{K}_\sigma = \left\{ \sum_n K^n : K^n \in \mathbb{K},\ n \geq 1,\ \sum_n \left\| K^{n-} \right\|_{\mathcal{B}(\bar{\mathcal{F}},\mathcal{N})} < \infty \right\} \tag{2.5}$$

from which the definition of $\mathcal{K}_\sigma$ and $\mathcal{C}_\sigma$ are obtained in analogy to (2.3) and (2.4).

A time honored issue in the theory of finance is that of completeness of markets, introduced in [33], [34] and [35] (see also [37] for a more recent treatment and [6] for a different approach). In our model completeness is defined as follows:

Definition 2. *Financial markets are complete if for every $f \in \bigcup_{t\in\mathbb{R}_+} \mathfrak{B}(\mathcal{F}_t, \mathcal{N})$ there exists $\gamma(f) \in \mathbb{R}$ such that $f - \gamma(f) \in \mathcal{K}$.*

Remark that our definition requires that claims have finite maturity.

2.4. **A Preliminary Result.** Starting from section 3, we shall be concerned with bounded, finitely additive measures over $\mathcal{F}$. Useful results on finitely additive measures are decomposition theorems, among which the one of Yosida and Hewitt [59, theorem 1.24, p. 52] is probably the best known. We shall use the following [12, theorem 1]:

Lemma 1. *Let $\mathcal{G}$ be a sub algebra of $\mathcal{F}$ and $\xi \in ba(\mathcal{G})$. There exists a unique way of writing*

$$\xi = \xi^e + \xi^p \tag{2.6}$$

with ξ^e, $\xi^p \in ba(\mathcal{G})$, where ξ^e admits a countably additive extension to $\mathcal{F}$ and any norm preserving extension of ξ^p to $\mathcal{F}$ is purely finitely additive. Furthermore,

(*i*). *if $\xi \geq 0$ then $\xi^e, \xi^p \geq 0$;*

(*ii*). *if $\mathcal{G}$ is a σ algebra, $\epsilon > 0$ and $P \in ca(\mathcal{F})_+$ there exists $G \in \mathcal{G}$ such that $|\xi^p|(G) = 0$ and $P(G^c) < \epsilon$;*

(*iii*). *if $\mathcal{H}$ is a sub σ algebra of $\mathcal{G}$, $\xi|\mathcal{H}$ the restriction of ξ to $\mathcal{H}$, and $\xi^e_{\mathcal{H}} + \xi^p_{\mathcal{H}}$ the decomposition of $\xi|\mathcal{H}$ in accordance to (2.6), then $\xi^e_{\mathcal{H}} \geq \xi^e|\mathcal{H}$ and $\xi^p_{\mathcal{H}} \leq \xi^p|\mathcal{H}$.*

In the case $\mathcal{G} = \mathcal{F}$ this decomposition coincides with that of Yosida and Hewitt. This result shows that a probability assessment ξ on $\mathcal{G}$ contains in itself a completely additive probabilistic model on $\mathcal{F}$, namely that element $\bar{\xi}^e$ of $ca(\mathcal{F})$ such that $\bar{\xi}^e \big| \mathcal{G} = \xi^e$. Uniqueness of ξ^e does not imply in general that $\bar{\xi}^e$ is itself unique (unless of course $\mathcal{F} = \sigma(\mathcal{G})$): if $\mathcal{G} = \{\varnothing, \Omega\}$, any $P \in ca(\mathcal{F})$ represents an extension of ξ provided $P(\Omega) = \xi(\Omega)$. In section 4 the relationship between ξ and $\bar{\xi}^e$ will be viewed as the outcome of an inferential process.

In the context of a filtered probability space when $m \in ba(\mathcal{F})_+$ and $\tau \in \mathcal{T}$ we shall denote by m_τ the restriction of m to $\mathcal{F}_\tau$. Letting $\mathcal{G} = \mathcal{F}_\tau$ in the above Lemma 1, we obtain for each $\tau \in \mathcal{T}$ a decomposition $m_\tau = m^e_\tau + m^p_\tau$ in accordance with (2.6). Then, if $s < t$ we have

$$(m^e_s - m^e_t)|\mathcal{F}_s = (m_s - m_t)|\mathcal{F}_s + (m^p_t - m^p_s)|\mathcal{F}_s = (m^p_t - m^p_s)|\mathcal{F}_s \tag{2.7}$$

Although the decomposants m_t^e and m_t^p are orthogonal (and therefore as different as possible), (2.7) illustrates how the "processes" $\tilde{m}^e = (m_t^e : t \in \mathbb{R}_+)$ and $\tilde{m}^p = (m_t^p : t \in \mathbb{R}_+)$ exhibit mirroring behaviour, analogously to the Poisson process, a purely discontinuous process admitting a predictable compensator with continuous paths. This suggests that the expectation with respect to the m^p component may be characterized to some extent by $\tilde{m}^e$, a much more treatable object. Much of section 5 builds on this remark.

3. Arbitrage, Martingales and Bubbles: The Pricing Measure

Any sensible model of financial markets should exclude the existence of free money as, in the absence of restrictions to trade, this would contrast with the existence of equilibrium. An arbitrage opportunity occurs whenever there exists an admissible investment yielding a return which, in discounted terms, is strictly positive up to negligibility. The initial cost of such investment is fact null while its final return would provide a strict improvement of welfare for any agent with strictly increasing preferences over $\mathfrak{B}(\mathcal{F},\mathcal{N})_{++}$. In our setting, therefore, the absence of arbitrage opportunities takes the form:

$$\mathcal{K} \cap \mathfrak{B}(\mathcal{F},\mathcal{N})_+ = \{0\} \tag{3.1}$$

Many versions of the above condition appear in the literature, all considerably more restrictive than (3.1). Further to assuming a richer structure of asset returns, the concept of an arbitrage opportunity is often conveniently reinforced into that of a free lunch (see [15], [16] and the seminal paper by Kreps [43], for a discussion). In our setting the absence of free lunches may be defined via the condition

$$\overline{\mathcal{C}} \cap \mathfrak{B}(\mathcal{F},\mathcal{N})_+ = \{0\} \tag{3.2}$$

(the upper bar denotes closure in the norm topology).

Let us introduce the following quantities, where $k \in \mathcal{K}$ and $f \in \mathbb{R}^\Omega$

$$\bar{\alpha}_k(f) = \inf_{N\in\mathcal{N}} \sup_{\omega\in N^c} (k+f)(\omega) \qquad \underline{\alpha}_k(f) = \sup_{N\in\mathcal{N}} \inf_{\omega\in N^c} (k+f)(\omega) \tag{3.3}$$

and

$$\bar{\alpha}_{\mathcal{K}}(f) = \inf_{k\in\mathcal{K}} \bar{\alpha}_k(f) \qquad \underline{\alpha}_{\mathcal{K}}(f) = \sup_{k\in\mathcal{K}} \underline{\alpha}_k(f) \tag{3.4}$$

Theorem 2. *Let the market be free of arbitrage opportunities. Then*

(*i*). *there exists $m \in ba(\mathcal{F},\mathcal{N})_+$ such that $m(\Omega) = 1$ and $m[\mathcal{K}] = 0$, i.e. a pricing measure;*
(*ii*). *if markets are complete, then $F \in \mathcal{F}$ and $m(F) = 0$ imply $F \in \mathcal{N}$;*
(*iii*). *there exists a countably additive pricing measure whenever*
  (a) *for any sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}(\mathcal{F},\mathcal{N})_+$ with $\sum_n f_n \in \mathfrak{B}(\mathcal{F},\mathcal{N})_+$,*

$$\inf\left\{\sum_n \bar{\alpha}_{k_n}(f_n) : k_n \in \mathcal{K}, n \in \mathbb{N}\right\} < \infty \tag{3.5}$$

  (b) *$\mathcal{N}$ is closed under countable unions and*
  (c) *$\mathcal{C}_\sigma \cap \mathfrak{B}(\mathcal{F},\mathcal{N})_+ = \{0\}$.*

In the traditional setting, the existence of a pricing measure (often termed variously a *risk neutral* or a *martingale* measure) gets along with the existence of a state price density process or stochastic discount factor, e.g. [27, p. 47]. Such process is also often obtained from the first order conditions for portfolio choice and its crucial role in financial modelling is witnessed by the common practice of simply assuming its existence. We will show in section 6 that it is possible to recover the existence of a stochastic discount factor even in our setting but with some important differences.

The last claim provides evidence that the countable additivity property of m is related to the degree to which portfolio diversification is allowed. The set $\mathcal{K}_\sigma$ will not be considered further in the paper but it contributes here to the view that countable additivity of the pricing measure is more an artifact of the theory than a property of markets. In particular, (3.5) requires that the cost incurred into by hedging separately each of the countably many components of f is limited.

Completeness of financial markets is not likely to prevail in general (particularly so under Assumption 4) and uniqueness of the pricing measure cannot be claimed. We denote

$$\mathcal{M}(\mathcal{C}) = \left\{m \in ba(\mathcal{F}, \mathcal{N})_+ : m(\Omega) = 1, m[\mathcal{C}] \leq 0\right\} \tag{3.6}$$

3.1. **Conditional Expectation, Asset Pricing and Bubbles.** A straightforward implication of the existence of a pricing measure in the traditional setting is that investment returns obey a martingale restriction with respect to it. This is also of fundamental importance in order to establish a clear, backward pricing rule. It is not straightforward that these conclusions carry over to our model as conditional expectation is not available with respect to finitely additive probability. The construction of conditional expectation in the finitely additive setting has received due attention in the subjective approach to probability theory in which it is obtained under the requirement of conditional coherence (see, among others, [20], [36] and [49]). In the following proposition we introduce a new operator acting on finitely additive probabilities and possessing some of the properties of ordinary conditional expectation.

Proposition 1. *Let $\mathcal{G} \subset \mathcal{F}$ be a σ algebra and $\xi \in ba(\mathcal{F})_+$. Denote by $\xi_\mathcal{G}$ the restriction of ξ to $\mathcal{G}$, let $\xi_\mathcal{G} = \gamma + \eta$ be an orthogonal decomposition of $\xi_\mathcal{G}$ with $\gamma \in ca(\mathcal{G})_+$ and $\eta \in ba(\mathcal{G})_+$ and define*

$$\mathcal{I}_\eta = \{F \in \mathcal{G} : \eta(F) = 0\} \tag{3.7}$$

Then, for each $f \in L^1(\mathcal{F}, \xi)$ there exists a unique $\xi(f|\mathcal{I}_\eta) \in L^1(\mathcal{G}, \gamma)$ such that

$$\xi(fI) = \xi(\xi(f|\mathcal{I}_\eta) I) = \gamma(\xi(f|\mathcal{I}_\eta) I) \tag{3.8}$$

for each $I \in \mathcal{I}_\eta$ and that for any $G \in \mathcal{G}$

$$\xi(fG|\mathcal{I}_\eta) = \xi(f|\mathcal{I}_\eta) G \tag{3.9}$$

The mapping $\xi(\cdot|\mathcal{I}_\eta) : L^1(\mathcal{F}, \xi) \to L^1(\mathcal{G}, \gamma)$ is a positive, unitary and linear operator.

We find it convenient to call the operator $\xi(\cdot|\mathcal{I}_\eta)$ "conditional expectation" for purely terminological reasons, although it is evident from (3.8) that it does not satisfy the law of iterated expectation but locally, i.e. with respect to sets in $\mathcal{I}_\eta$. From the point of view of a statistician it is perhaps regrettable that the forecast of a forecast may differ from the direct forecast. Although we have no explicit interest here for the otherwise important statistical interpretation of conditional expectation, it should be remarked that in the subjective approach following from the work of de Finetti [20], conditioning events are determined by admissible bets the family of which, therefore, need not be an algebra.

To illustrate the use we shall make of the preceding Proposition in the current and the following sections, let $\sigma \in \mathcal{T}$. Remark that in view of (2.7)

$$m^p|\mathcal{F}_\sigma = (m^e_\sigma - m^e)|\mathcal{F}_\sigma + m^p_\sigma$$

and $(m^e_\sigma - m^e)|\mathcal{F}_\sigma \geq 0$, by Lemma 1(*iii*). Proposition 1 then applies with $\mathcal{G} = \mathcal{F}_\sigma$, $\xi = m^p$, $\gamma = (m^e_\sigma - m^e)|\mathcal{F}_\sigma$ and $\eta = m^p_\sigma$. Write $\mathcal{I}_\sigma = \mathcal{I}_{m^p_\sigma}$. If $I \in \mathcal{I}_\sigma$ and $f \in \mathfrak{B}(\mathcal{F})$ then by (3.8)

$$m^p(fI) = m^p(m^p(f|\mathcal{I}_\sigma)I) = (m^e_\sigma - m^e)(m^p(f|\mathcal{I}_\sigma)I)$$

i.e.

$$m(fI) = m^e(m^e(f|\mathcal{F}_\sigma)I) + (m^e_\sigma - m^e)(m^p(f|\mathcal{I}_\sigma)I) \tag{3.10}$$

From (3.10) we obtain

Theorem 3. *Let $K \in \mathbb{K}$. If the market is free of arbitrage opportunities then there exists an adapted stochastic process $h = (h_t : t \in \mathbb{R}_+)$ with $h_0 = m^e(\Omega)$ such that for any $\sigma, \tau \in \mathcal{T}$ with $\sigma \leq \tau$, $0 \leq h_\sigma \leq 1$ and $h_\sigma \leq m^e_\tau(h_\tau|\mathcal{F}_\sigma)$, m^e_σ a.s. and*

$$K_\sigma = h_\sigma m^e(K_\infty|\mathcal{F}_\sigma) + (1 - h_\sigma)m^p(K_\infty|\mathcal{I}_\sigma) = m(K_\infty|\mathcal{I}_\sigma) \tag{3.11}$$

Furthermore $h_\sigma = 1$ m^e_σ a.s. for each $\sigma \in \mathcal{T}$ if and only if m is countably additive.

The second equality in (3.11) establishes that, relatively to the conditioning operator introduced in Proposition 1, the pricing measure is indeed a "martingale" measure – although many analytical properties of ordinary martingales (such as convergence theorems) do not apply here. Pricing is therefore an intrinsically forward looking exercise.

Let W represent the wealth process out of some admissible investment, so that $W - W_0 \in \mathbb{K}$: (3.11) clearly translates into

$$W_\sigma = h_\sigma \phi(W)_\sigma + (1 - h_\sigma)\beta(W)_\sigma \tag{3.12}$$

where $\phi(W)_\sigma = m^e(W_\infty|\mathcal{F}_\sigma)$ and $\beta(W)_\sigma = m^p(W_\infty|\mathcal{I}_\sigma)$. Indeed (3.12) establishes that the pricing rule just described differs considerably from the traditional one. In fact the conditioning operator $m^p(\cdot|\mathcal{I}_\sigma)$ inherits, through (3.8), the property that for any $P \in ca(\mathcal{F})_+$ and $\epsilon > 0$ there exists $F \in \mathcal{F}$ such that $P(F^c) < \epsilon$ and $m^p(F|\mathcal{I}_\sigma) = 0$. In other words, the component $\beta(W)$ of W only charges the remote behavior of the wealth process W, both with respect to time and randomness. It is therefore quite natural, after the seminal work of Gilles and Leroy [30], to interpret $\phi(W)$ as the fundamental value of the investment and $\beta(W)$ as its bubble part. The noteworthy properties of (3.12) is that such decomposition is established here in conditional terms, that it applies to bounded processes over a finite horizon and that, as asset returns need not be positive throughout, bubbles may assume either sign.

3.2. **An Example.** Consider a traditional financial model with underlying probability Q in which (discounted) asset returns are turned into martingales by multiplication by positive martingale Z with $Z_0 = 1$ but non necessarily uniformly integrable[4]. As we shall see in the following section 6 this situation is quite general. Let Z_∞ be the Q a.s. limit of Z. We can associate to Z the finitely additive probability measure μ defined as

$$\mu(F) = \operatorname*{LIM}_n Q(Z_n F) \tag{3.13}$$

for $F \in \mathcal{F}$ – where LIM denotes here the Banach limit introduced in [1] (but see also [50, p. 367])[5]. It is easy to conclude that μ is a pricing measure as for $k \in \mathcal{K}$

$$\mu(k) = \mu(K_t) = \lim_n Q(Z_n K_t) = Q(Z_t K_t) = 0$$

[4]Some aspects of this example were treated in [13].

[5]Definition (3.13) may be given in terms of ordinary limits if and only if Z is a uniformly integrable martingale.

Furthermore, $\mu^e(F) = Q(Z_\infty F)$ for $F \in \mathcal{F}_t$ and $\mu^P(F) = \mathrm{LIM}_n\, Q((Z_n - Z_\infty)F)$ for $t \in \mathbb{R}_+$: in fact, μ^P vanishes on $\{\sup_s Z_s < 2^n\}$ for each n so that it is purely finitely additive [9, theorem 10.3.3, p. 244]. Moreover since $m_t = Z_t dQ$, $\mu_t^p = 0$.

Let $\sigma \in \mathcal{T}$ and $W - W_0 \in \mathbb{K}$. Recalling Assumption 4

$$\begin{aligned} Z_\sigma W_\sigma &= \lim_n Z_\sigma^n W_\sigma^n \\ &= \lim_n Q\left(Z_n W_n \middle| \mathcal{F}_\sigma \right) \\ &= Q\left(Z_\infty W_\infty \middle| \mathcal{F}_\sigma \right) + \lim_n Q\left(\left(Z_n - Z_\infty \right) W_\infty \middle| \mathcal{F}_\sigma \right) \end{aligned} \tag{3.14}$$

If we define (with the convention $\frac{0}{0} = 0$) $h_\sigma = Z_\sigma^{-1} Q\left(Z_\infty \middle| \mathcal{F}_\sigma \right)$, $\phi(W)_\sigma = Q\left(Z_\infty \middle| \mathcal{F}_\sigma \right)^{-1} Q\left(Z_\infty W_\infty \middle| \mathcal{F}_\sigma \right)$ and $\beta(W)_\sigma = \left(Z_\sigma - Q\left(Z_\infty \middle| \mathcal{F}_\sigma \right) \right)^{-1} \lim_n Q\left(\left(Z_n - Z_\infty \right) W_\infty \middle| \mathcal{F}_\sigma \right)$, then (3.14) is the exact translation of (3.12) to the present setting. It should be remarked that $h_t = 1$ Q a.s. is equivalent to the case in which Z is a uniformly integrable martingale i.e. μ is countably additive. In models of optimal consumption and portfolio selection, e.g. [5] and [28], Z represents marginal utility of consumption along the optimal path. In these models it cannot usually be established that Z is a uniformly integrable martingale nor is it clear which economically meaningful conditions could be imposed in order to obtain such property. This remark is a point in case for the finitely additive model we propose here. Of course, it would be important to see if a partial converse could be established, i.e. if in our model each pricing measure can be associated to a martingale density. An answer to this problem will be offered in section 6.

4. Probability from Asset Returns: The Representing Measure.

In this section we shall show that associated to m is a full probability measure P on $\mathcal{F}$. The role played by P may be compared to that of the objective measure in the traditional setting and we will refer to it as the *representing* measure generated by m. Although pricing is performed via the finitely additive measure m, it could still be of practical as well as of theoretical worth to establish properties of the return process for which countable additivity matters. Given the different role played, m and P may in principle be far apart not only for what concerns additivity. In section 8 we will introduce and discuss a notion of consistency between m and P.

Given the information $\mathcal{F}_t$ available at time t, the component m_t^e, by its same definition, allows to infer a completely additive measure $\bar{m}_t^e$ over the whole of $\mathcal{F}$: in other words, agents may extract from the restriction of m to $\mathcal{F}_t$ a fully additive assessment of randomness. However, the probabilistic view implicit in $\bar{m}_t^e$ has only a local meaning and is bound to change considerably as time passes by, as the effect of the arrival of new information. In fact as illustrated by Lemma 1, the decomposition (2.6) depends on the underlying information structure. The question therefore arises whether it is possible to extract from the collection $\{\bar{m}_t^e : t \in \mathbb{R}_+\}$ a global perspective P on $\mathcal{F}$ not contradicting the inference m_t^e made at each point in time. Although different, sensible criteria could be considered in order to judge whether P contrasts with m_t^e or not, a clear contradiction definitely exists between these two measures whenever, for some $F \in \mathcal{F}_t$, $m_t^e(F) > 0$ but $P(F) = 0$: it may well be that that are locally null have a positive global probability, but the opposite would indeed imply that the global assessment expressed by P implicitly disproves the one embodied in m_t^e. In the context of a model in which agents form their beliefs based on past experience, Kurz [44, axiom 2, p. 13] suggests the above criterion as a definition of individual beliefs not contradicting observable data (so called *rational beliefs*). The following result provides a positive answer to the above question.

Theorem 4. *There exist $P \in \mathbb{P}(\mathcal{F})$ such that $m^e_\tau \ll P|\mathcal{F}_\tau$ for each $\tau \in \mathcal{T}_0$. If Assumption 3 is satisfied, then we may choose $P \in \mathbb{P}(\mathcal{F}, \mathcal{N})$.*

It should be remarked that P is partly influenced by subjective elements – namely the collection $\mathcal{N}$ – and partly by the structure of markets. When confronted with a richer structure either of negligible events or of admissible trading strategies the resulting set $\mathcal{K}'$ of marketed claims would be strictly larger than $\mathcal{K}$ and both the separating measure and the probability associated to $\mathcal{K}'$ will differ from the ones arising from $\mathcal{K}$. Denote

$$\mathbb{P}(m) = \{P \in \mathbb{P}(\mathcal{F}) : m^e_\tau \ll P|\mathcal{F}_\tau,\ \tau \in \mathcal{T}_0\} \tag{4.1}$$

and $\mathbb{P}(m, \mathcal{N}) = \mathbb{P}(m) \cap \mathbb{P}(\mathcal{F}, \mathcal{N})$.

An almost immediate consequence of Theorem 4 and Lemma 1(*iii*) is the following

Corollary 1. *Let $P \in \mathbb{P}(m)$, $\tau \in \mathcal{T}_0$ and $dm^e_\tau / dP_\tau = X_\tau$. The stochastic process $X = (X_t; t \in \mathbb{R}_+)$ is a P right continuous, positive supermartingale, decomposing as*

$$X = M - A \tag{4.2}$$

where M is a positive local martingale and A an increasing, predictable process with $A_0 = 0$ and $P(A_\infty) < \infty$.

Given right continuity of A, we can define $\lambda \in ca\left(\tilde{\mathcal{F}}\right)_+$ implicitly through the equation $\lambda(F) = P \int F dA$. It is clear from (4.1) that if $P \in \mathbb{P}(m)$ and $P' \gg P$ then $P' \in \mathbb{P}(m)$. This notwithstanding, in the following sections we will treat $P \in \mathbb{P}(m)$ as fixed.

5. An Explicit Representation.

Due to finite additivity, the expectation with respect to m has a limited analytical tractability and this may represent a major drawback of the present approach, both in theory and in applications. For what concerns applications, this issue will be addressed in section 9. In this section we shall prove that the expectation $m(k)$ may receive an explicit and convenient representation whenever $k \in \mathcal{K}$.

The structure of the m^e_t component has been characterized in Corollary 1 via the supermartingale X. For what concerns the m^p_τ component we shall take advantage of the following result:

Lemma 2. *There exists a collection $\left\{\bar{m}^p_\tau \in ba(\mathcal{F})_+ : \tau \in \mathcal{T}_0\right\}$ such that $\bar{m}^p_\tau$ is an extension of m^p_τ to $\mathcal{F}$ and $\bar{m}^p_\tau \geq \bar{m}^p_\sigma$ whenever $\sigma, \tau \in \mathcal{T}_0$ and $\sigma \leq \tau$.*

We shall now investigate more deeply the properties of the pricing kernel. Define to this end the following quantities:

- The collection $\mathcal{H}$ of all pairs $H = \left(\langle t_i\rangle_{i=0}^I, \langle F_i\rangle_{i=0}^I\right)$ of finite sequences such that
  - (*i*) $t_i \in \mathcal{T}_0$ and $0 = t_0 \leq t_1 \leq \ldots \leq t_I$, P a.s.,
  - (*ii*) $F_i \in \mathcal{I}_{t_i} = \left\{G \in \mathcal{F}_{t_i} : m^p_{t_i}(G) = 0\right\}$, $F_i \subset F_{i-1}$ for $i = 1, \ldots, I$, $F_I = \varnothing$ and
  - (*iii*) M^{t_I} is a uniformly integrable martingale.
- $D^H_i(K) = F_i\left(K_{t_{i+1}} - K_{t_i}\right)$ and
- $K^H = \left(K^H_t : t \in \mathbb{R}_+\right)$ where

$$K^H_t = \sum_{i=0}^{I-1} D^H_i\left(K^t\right) \tag{5.1}$$

K_t^H is an H "approximation" of K_t and may be rewritten as $K_t^H = \sum_{i=0}^{I-1} F_i F_{i+1}^c K_{t_{i+1}\wedge t}$. The trading strategy behind (5.1) prescribes to stop at time t_i whenever F_i^c occurs. Choosing F_i appropriately, this criterion will apply very rarely according to $m_{t_i}^e$, although with certainty under $m_{t_i}^p$. In this way the role of the "irregular" component $m_{t_i}^p$ of m at the start of each investing period (t_i, t_{i+1}) may be entirely neglected[6]. On the other side, the behaviour of $m_t^p - m_{t_i}^p$ over (t_i, t_{i+1}), as remarked above, mirrors that of $m_t^e - m_{t_i}^e$, then it may be conjectured that $m_{t_{i+1}}^p$ may receive an explicit characterizations. This argument clearly hinges on the behavior of K^H when passing to the limit, provided convergence obtains in some suitable sense. It is crucial to our aims that if $K_t \in \mathcal{K}$ then $K_t^H \in \mathcal{K}$ as well. Observe that $D_i^H(K^t)$ and K_t^H are $\mathcal{F}_{t_{i+1}\wedge t}$ and $\mathcal{F}_{t_I\wedge t}$ measurable respectively and that $\bar{m}_{t_{i+1}\wedge t}^p(F_i; t_i \geq t) = 0$ (see Lemma 6 in the Appendix). We shall write $\mathcal{I}_{t_i}^t$ as short for $\mathcal{I}_{m_{t_i\wedge t}^p}$ (see (3.7)).

Let us define the following key terms:

$$J_H(K)_t = \sum_{i=0}^{I-1} \left(\bar{m}_{t_I\wedge t}^p - \bar{m}_{t_{i+1}\wedge t}^p\right)\left(D_i^H(K^t)\right) - \sum_{i=1}^{I-1} \bar{m}_{t_{i+1}\wedge t}^p\left(F_i K_{t_i}^t\right) \tag{5.2}$$

and

$$I_H(K)_t = \sum_{i=0}^{I-1} \bar{m}_{t_{i+1}\wedge t}^p\left(F_i K_{t_{i+1}}^t\right) \tag{5.3}$$

From (5.1) – (5.3) it clearly follows the decomposition

$$m_{t_I\wedge t}^p\left(K_t^H\right) = J_H(K)_t + I_H(K)_t \tag{5.4}$$

Exploiting (2.7) and Proposition 1, we show in Proposition 2 below that the terms $J_H(K)$ and $I_H(K)$ can be described explicitly. This result is based on the following intuition. First, since, as we have seen, $\bar{m}_{t_{i+1}\wedge t}^p(F_i) = m_{t_{i+1}\wedge t}^p(F_i; t_i < t)$ and $F_i\{t_i < t\} \in \mathcal{I}_{t_i}^t$, then

$$\begin{aligned}
I_H(K)_t &= \sum_{i=0}^{I-1} m_{t_{i+1}\wedge t}^p\left(F_i\{t_i < t\} K_{t_{i+1}}^t\right) \\
&= \sum_{i=0}^{I-1} m_{t_{i+1}\wedge t}^p\left(m_{t_{i+1}\wedge t}^p\left(K_{t_{i+1}}^t \middle| \mathcal{I}_{t_i}^t\right) F_i\{t_i < t\}\right) \\
&= \sum_{i=0}^{I-1} \left(m_{t_{i+1}\wedge t}^p - m_{t_i\wedge t}^p\right)\left(m_{t_{i+1}\wedge t}^p\left(K_{t_{i+1}}^t \middle| \mathcal{I}_{t_i}^t\right) F_i\{t_i < t\}\right) \\
&= P\sum_{i=0}^{I-1} \left(A_{t_{i+1}}^t - A_{t_i}^t\right) m_{t_{i+1}\wedge t}^p\left(K_{t_{i+1}}^t \middle| \mathcal{I}_{t_i}^t\right) F_i\{t_i < t\}
\end{aligned}$$

Observe that this can be rewritten more concisely as

$$I_H(K)_t = P\int_0^t f_H^t\left(K^t\right) dA = \int_0^t f_H^t\left(K^t\right) d\lambda \tag{5.5}$$

[6] In order to prove that the purely finitely additive part of equilibrium prices does not matter, Bewley assumes that production plans satisfy the exclusion property [8, p. 524 and section 5], i.e. that production plans may be made contingent on the occurence of whatever future event. This translates in our setting into the condition: $FK_\infty \in \mathcal{K}$ whenever $F \in \mathcal{F}$ and $K \in \mathbb{K}$. It is clear that this property contrasts with the general principle that investments can only be contingent on events known by the start of the investment period . This remark explains why in our framework the m^p component of the pricing measure will in general matter.

where for $Y \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$,

$$f_H^u(Y) = \sum_{i=0}^{I-1} \bar{m}^p_{t_{i+1}\wedge u}\left(Y_{t_{i+1}} \middle| \mathcal{I}^u_{t_i}\right) F_i \{t_i < u\}]]t_i, t_{i+1}]] \tag{5.6}$$

We obtain then from (5.5) that I behaves like an ordinary stochastic integral: several nice properties become thus available.

For what concerns the J term, by (2.7)

$$\begin{aligned}
\sum_{i=0}^{I-1}\left(m^p_{t_I\wedge t} - m^p_{t_{i+1}\wedge t}\right)\left(D_i^H\left(K^t\right)\right) &= P\sum_{i=0}^{I-1}\left(A^t_{t_I} - A^t_{t_{i+1}}\right) D_i^H\left(K^t\right) \\
&= P\sum_{i=0}^{I-1}\sum_{j=i+1}^{I-1}\left(A^t_{t_{j+1}} - A^t_{t_j}\right) D_i^H\left(K^t\right) \\
&= P\sum_{j=1}^{I-1}\left(A^t_{t_{j+1}} - A^t_{t_j}\right)\sum_{i=0}^{j-1} D_i^H\left(K^t\right) \\
&= P\sum_{j=1}^{I-1}\left(A^t_{t_{j+1}} - A^t_{t_j}\right) K^H_{t\wedge t_j}
\end{aligned}$$

From (5.2), $K_0 = 0$ and the fact that $\bar{m}^p_{t_{i+1}\wedge t}\left(F_i K^t_{t_i}\right) = \left(m^p_{t_{i+1}\wedge t} - m^p_{t_i\wedge t}\right)\left(F_i K^t_{t_i}\right)$ and that $K^H_{t_i} = F_i K_{t_i} + \sum_{j=0}^{i-1} F_j F^c_{j+1} K_{t_{j+1}}$ it follows that

$$\begin{aligned}
J_H(K)_t &= P\sum_{j=1}^{I-1}\left(A^t_{t_{j+1}} - A^t_{t_j}\right) K^H_{t\wedge t_j} - P\sum_{i=1}^{I-1}\left(A^t_{t_{i+1}} - A^t_{t_i}\right) F_i K^t_{t_i} \\
&= P\sum_{i=1}^{I-1}\left(A^t_{t_{i+1}} - A^t_{t_i}\right)\left(K^H_{t_i} - F_i K_{t_i}\right) \\
&= P\sum_{i=1}^{I-1}\left(A^t_{t_{i+1}} - A^t_{t_i}\right)\sum_{j=0}^{i-1} F_j F^c_{j+1} K_{t_{j+1}}
\end{aligned}$$

so that we obtain the bound

$$\left|J_H(K)_t\right| \le \|K\| P\left(A_t F^c_{I-1}\right) \tag{5.7}$$

It is natural to conjecture from (5.7) that the J term may be set so to converge to 0; a more delicate issue is that of existence of the limit for the "stochastic integral" I and of its representation. This is solved in the following

Proposition 2. *Let $\tau \in \mathcal{T}_0$ be such that X^τ is uniformly integrable. If $K \in \mathbb{K}$ and the market is free of arbitrage opportunities, then*

$$m^p_\tau(K_\tau) = \int_0^\tau \mathcal{P}_m(K)\, d\lambda \quad \textit{i.e.} \quad m(K_\tau) = P\left(X_\tau K_\tau + \int_0^\tau \mathcal{P}_m(K)\, dA\right) \tag{5.8}$$

where $\mathcal{P}_m : \mathfrak{B}\left(2^{\tilde{\Omega}}, \mathcal{N}\right) \to L^\infty(\mathcal{P}, \lambda)$ is a positive, linear operator of unitary norm and such that $\mathcal{P}_m(YU) = \mathcal{P}_m(Y)U$ whenever $Y, U \in \mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right)$ and U is càglàd (so that if $Y \in \mathfrak{B}\left(\tilde{\mathcal{F}}\right)$ is càdlàg, $\mathcal{P}_m(Y) = Y_- + \mathcal{P}_m(\Delta Y)$).

The operator $\mathcal{P}_m$ defined in Proposition 2 resembles to some extent to the P predictable projection – denoted by $\mathcal{P}_P(X)$ in the sequel – and will therefore be referred to as the m predictable projection. Remark that $\mathcal{P}_m$ is not invariant with respect to predictable processes but to *càglàd* processes only: once again the difference amounts to lack of continuity. In section 7 we will study a class larger than that of *càglàd* process but invariant with respect to $\mathcal{P}_m$. We denote $\mathcal{P}_d = \mathcal{P}_m - \mathcal{P}_P$ the "mismatch" between the two predictable projections.

It is worth noticing from the representation (5.8) that the discontinuity of the pricing rule is tantamount to that of $\mathcal{P}_m$. In other words the pricing measure m exhibits a finitely additive behaviour to the extent that there is a sequence $\langle K^n \rangle_{n\in\mathbb{N}}$ in $\mathbb{K}$ such that $\langle K^n_\infty \rangle_{n\in\mathbb{N}}$ converges pointwise but $\langle \mathcal{P}_m(K^n) \rangle_{n\in\mathbb{N}}$ does not converge for λ a.e. pairs (ω, t). The pricing rule (5.8) is however intrinsically path dependent since, regardless of the actual structure of the asset return, it is based on the whole process $\mathcal{P}_m(K)$ rather than just on K_∞.

6. The Martingale Property

In this section we will develop the relevant implications of the representation (5.8) to financial modeling, by introducing the additional assumption:

Assumption 5. *Every $K \in \mathbb{K}$ is càdlàg.*

Let us also define the stopping time

$$T = \inf \{t \in \mathbb{R}_+ : X_{t-} = 0 \text{ or } X_t = 0\} \tag{6.1}$$

We start by showing that the powerful tools of stochastic analysis apply to our setting.

Theorem 5. *Let $K \in \mathbb{K}$. In the absence of arbitrage opportunities:*

(i). the stochastic process K stopped at T, i.e. K^T, is a P semimartingale;

(ii). if financial markets are complete and Assumption 3 holds, then P may be chosen such that $P(T < \infty) = 0$ (so that K is a P semimartingale).

Theorem 5 establishes that, in some appropriate form, the absence of arbitrage opportunities implies the semimartingale nature of asset returns, a pervasive assumption in all financial models. It should be highlighted that there are predecessors to this result, particularly Ansel and Stricker [3, theorem 8, p. 383] and Stricker [53, theorem 3 p. 456 and theorem 5, p. 458] (but see also [21, theorem 7.2, p. 504]). The noticeable fact is that this property, which crucially depends on the underlying probability measure, is obtained here without explicit reference to any preassigned probability: it is therefore entirely endogenous. Of course, the behavior of K after the random time T is totally unrestricted. In fact our model contains no prediction over $]]T, \infty[[$, as the behavior of the separating measure becomes purely finitely additive over that domain. The last claim, which anticipates in its proof the main ideas of Proposition 3 below, establishes that this cannot be the case if markets are complete. Once again the implicit probabilistic model turns out to depend in a crucial way on the structure of markets. Let $M^K + V^K$ be the canonical decomposition of K^T with M^K a local martingale and V^K a predictable process of locally integrable variation.

Denote by $\mathcal{E}$ the exponential semimartingale of Doléans-Dade (and $\mathcal{L}$ its inverse, the stochastic logarithm) and the positive local martingale

$$Z = \mathcal{E}\left(\int X_-^{-1} dM\right) \tag{6.2}$$

and the predictable process of locally integrable variation

$$B = \mathcal{E}\left(\int X_{-}^{-1} dA\right) \tag{6.3}$$

(in the proof of Theorem 6 it is shown that indeed Z and B are well defined, as assumed here).

We also introduce the *augmented return process* defined as

$$\hat{K} = K^T + \int \mathcal{P}_d\left(\Delta K\right) d\mathcal{L}\left(B\right) \tag{6.4}$$

The extra return included in (6.4) is associated to the finitely additive nature of m – as $A = 0$ when $m \ll P$ – and to the discontinuities of asset returns.

Theorem 6. *Let $K \in \mathbb{K}$ and define $\hat{K}$, Z and B as in (6.4), (6.2) and (6.3) respectively. In the absence of arbitrage opportunities*

(i). *$Z\hat{K}$ is a P local martingale;*

(ii). *$V^K + \int \mathcal{P}_d\left(\Delta K\right) d\mathcal{L}\left(B\right) + \left[\mathcal{L}\left(Z\right), M^K\right]$ is a P local martingale, i.e.*

$$V^K + \int \mathcal{P}_d\left(\Delta K\right) d\mathcal{L}\left(B\right) = -\mathcal{P}_P\left(\left[\mathcal{L}\left(Z\right), M^K\right]\right) \tag{6.5}$$

It is correct to say, we believe, that two core results of modern financial theory are that there exists a stochastic discount factor or a state price process and that the equity premium of assets is equal to (the negative of) the correlation between asset returns and the market price for risk. This last conclusion admits different formulations, depending on the additional assumptions, among which the *CAPM* and the *CCAPM*. The importance of Theorem 6 emerges fully upon noting that it establishes exactly these conclusions but relatively to the augmented return process, rather than the original one. Of course it is of interest to remark that, in the special case of complete markets and continuous returns, $P\left(T < \infty\right) = 0$ and $\mathcal{P}_d\left(\Delta K\right) = 0$ so that any difference with the traditional formulation vanishes. In sections 7 and 8 such conditions will be examined under more generality.

For what concerns the first claim, in traditional models the existence of a state price process, despite being a common assumption, is not clearly related to the absence of arbitrage. It is well known, at least after the celebrated proof of the *Fundamental Theorem of Asset Pricing* due to Delbaen and Schachermayer [21], that the absence of *free lunches* is equivalent to the existence of an equivalent martingale measure and that the latter condition may be reformulated as stating the existence of a strictly positive state price process Z. However, neither implication will in general hold in the absence of the property that Z is of class D. It should be remarked that this is a demanding property and that, when not assumed right away, it is usually obtained by imposing *ad hoc* constraints on the volatility of returns (particularly in the form of some lower bound). Given the limitations to the ability of modelling the volatility process implicit in this condition, it is worth knowing that even without it the existence of a state price process is still a necessary step in the construction of a financial model free of arbitrage opportunities.

The second claim is the most important in financial terms and it has a sound economic interpretation. Adapting the *CAPM* terminology to the present context, (6.5) states that the "covariance" between asset returns and the market price of risk is not the correct explanation of the equity premium, but rather of the *augmented* equity premium. Since the difference hinges on the jump part ΔK of the return process, the extension of asset pricing models to include possible discontinuities may result in a significant innovation with respect to traditional explanations of the equity premium, a finding that contrasts blatantly with other

CAPM like models with jumps, e.g. [5] and [39]. The novel feature of (6.5) is that it contains one more factor in addition to $\mathcal{L}(Z)$, namely $\mathcal{L}(B)$, that disappears in the case in which m were countably additive. Since this second factor plays a role in the pricing of the jumps of K, it is indeed tempting to consider it as the market price of the risk implicit in the discontinuities of asset returns, examples of which are sudden drops in the market index (e.g. crashes), on the aggregate level, or idiosyncratic events such as dividend payouts or other types of corporate actions. (6.5) contributes to the view, implicit in the equity premium puzzle literature, that correlation with a unique risk factor is not enough to explain excess returns. There have in fact been several variants of the *CCAPM* which include special characteristics of preferences or beliefs in order to induce the existence of additional factors[7] and it is remarkable that this same conclusion emerges here without further assumptions.

The usual decomposition of local martingales into a continuous and a purely discontinuous part [38, theorem I.4.18, p. 42] allows to reformulate (6.5) as the joint condition:

$$V^{K,c} + \left\langle \mathcal{L}(Z), M^K \right\rangle + \int \mathcal{P}_d(\Delta K)\, d\mathcal{L}(B)^c = 0 \tag{6.6}$$

$$V^{K,d} + \sum \mathcal{P}_P\left(\Delta\mathcal{L}(Z)\,\Delta M^K\right) + \sum \mathcal{P}_d(\Delta K)\,\Delta\mathcal{L}(B) = 0 \tag{6.7}$$

Although the term $\int \mathcal{P}_d(\Delta K)\, d\mathcal{L}(B)$ is actually a function of the jumps of the return process, we cannot conclude that it is itself a jump process. The risk arising from the discontinuities of the return process may thus well affect its continuous part too, a conclusion that marks an additional difference with the more traditional version of the *CAPM* extended to include possible jumps. Of course in some examples of discontinuities (such as dividend payouts), the corresponding times are announced with due notice. This influences our previous result as follows

Corollary 2. *Let* $Y \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$ *be càdlàg and* $K \in \mathbb{K}$.

(i). *If* $\{\Delta Y \neq 0\} \in \mathcal{P}$, *then*

$$\int \mathcal{P}_m(\Delta Y)\, d\lambda = P \sum \mathcal{P}_m(\Delta Y)\,\Delta A \tag{6.8}$$

(ii). *If* θ *is a predictable process such that* $\theta.K \in \mathbb{K}$, *then*

$$P\int \mathcal{P}_m(\theta.\Delta K)\, dA = P\int \theta\mathcal{P}_m(\Delta K)\,\Delta A \tag{6.9}$$

In the special case of discontinuities occurring at predictable times then, $V^{K,c} + \left\langle \mathcal{L}(Z), M^K \right\rangle = 0$. In other words, as far as the continuous part of asset returns is concerned our model is, in this special case, indistinguishable from the traditional one. It remains true, however, that exceptional, unpredicted events, such as changes in credit ratings or market crashes, will have a deeper impact on returns. The view that events such as October '87 may have a long lasting influence on the pricing of assets has in fact obtained some consensus.

[7] In a model with habit formation, Detemple and Zapatero [22, equation (6.5), p. 1647] characterize the second factor as covariance with disutility of future standards of living. In the context of stochastic differential utility, Duffie and Epstein [24, equation (18), p. 422] recover two additional factors further to equilibrium consumption, one of which being related to market portfolio. In a model with differential information, Ziegler [60, equation (24), p. 9] obtains factors ensuing from the updating process. It should be stressed that all these papers consider a model of general equilibrium while our analysis has only a partial equilibrium flavour.

7. Predictable Returns.

A vast majority of financial models are written under the assumption that the price process is *càdlàg* and predictable – or even that it has continuous sample paths. In this section we will comply with predictability, a property that allows for a representation of the pricing kernel more explicit than (5.8). The notion of predictability has though to be partly adapted to our finitely additive set up. In fact let σ be a stopping time predictable with respect to some $Q \in \mathbb{P}(\mathcal{F})$ and $\langle\sigma^r\rangle_{r\in\mathbb{N}}$ its announcing sequence. Then, for each n and r there exists $\delta > 0$ such that $Q(\sigma_r > \sigma - \delta; \sigma > 0) < 2^{-n}Q(\sigma > 0)$: in other words, most of σ can be anticipated with fixed notice. This same property may not hold whenever Q is only finitely additive, a situation that deprives the announcing sequence of much of its economic content. Denote by $m_{\tau-}$ the restriction of m to $\mathcal{F}_{\tau-}$ when $\tau \in \mathcal{T}$ and by $m^p_{\tau-}$ and $m^e_{\tau-}$ its components.

Definition 3. *A stopping time σ is (m, P) predictable if it admits a sequence $\langle\sigma^r\rangle_{r\in\mathbb{N}}$ of stopping times such that:*

(i). $\sigma^r \uparrow \sigma$, P a.s. and $P(\sigma^r < \sigma) = P(0 < \sigma)$,
(ii). $\lim_n m(\sigma - \sigma^r \leq 2^{-n}) = 0$ and
(iii). $\lim_r \left(m^p_{\sigma-} - m^p_{\sigma^r}\right)(\Omega) = 0$.

Definition 4. *A càdlàg, adapted process K is (m, P) predictable if*

(i) *there exists a sequence $\langle\upsilon_r\rangle_{r\in\mathbb{N}}$ of (m, P) predictable stopping times such that $\{\Delta K \neq 0\} = \bigcup_r [[\upsilon_r]]$ and*
(ii) *for each r, K_{υ_r} is $\mathcal{F}_{\upsilon_r-}$ measurable.*

It is clear that continuous processes are (m, P) predictable and that the above definition of a predictable time or process is more restrictive than the usual one.

Theorem 7. *Let $K \in \mathbb{K}$ be càdlàg with $\{\Delta K \neq 0\} = \bigcup_r [[\upsilon_r]]$, let (3.1) hold and let $\tau \in \mathcal{T}$ be such that X^τ is uniformly integrable. If K is (m, P) predictable then $\mathcal{P}_m(K) = \mathcal{P}_P(K) = K$ so that*

$$m^p_\tau(K_\tau) = \int_0^\tau K d\lambda \quad \text{and} \quad m(K_\tau) = P\left\{M_\tau K_\tau - \int_0^\tau A_- dK\right\} \tag{7.1}$$

8. Consistent Pricing Measures

It is commonly believed that financial markets are incomplete. However, it is as widely shared the view that any contingent claim may be introduced and traded on the market provided its price is set fairly. The pricing measure should then not only be considered as a tool to evaluate currently traded assets, as in the preceding sections, but it should also provide reliable indications for the pricing of claims that do not yet exist on the market but that it may sensible to introduce at some later stage. Viewing the current market setting as the outcome of some equilibrium process (and borrowing from game theoretic terminology) we conclude that the pricing measure may partly depend on out of equilibrium elements.

To develop the preceding intuition with more rigour we define the extension property.

Definition 5. *Let $\pi : \mathfrak{B}(\mathcal{F}, \mathcal{N}) \to \mathbb{R}$, $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$ and $\mathcal{K}(f; \pi) = \{k + d(f - \pi(f)) : k \in \mathcal{K},\ d \in \mathbb{R}\}$. If*

$$\mathcal{K}(f; \pi) \cap \mathfrak{B}(\mathcal{F}, \mathcal{N})_+ = \{0\} \tag{8.1}$$

then we write $\pi(f) \in \mathcal{A}(f, \mathcal{K})$ *and we say that* π *is an admissible price for* f *and that* $\mathcal{K}$ *possesses the extension property with respect to* f*. If* $\mathcal{A}(f, \mathcal{K}) \neq \varnothing$ *for any* $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$ *then* $\mathcal{K}$ *is said to possess the extension property.*

Related to the preceding notion is the following one.

Definition 6. *Let* $m \in \mathcal{M}(\mathcal{C})$ *and* $P \in \mathbb{P}(m)$*. The pair* (m, P) *is consistent if* $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ *and* $P(f > 0) > 0$ *imply* $m(f) > 0$*.* m *is consistent if there exists* $P \in \mathbb{P}(m)$ *such that* (m, P) *is consistent.*

Of course if $\mathbb{P}(\mathcal{F}, \mathcal{N}) = \varnothing$ there will will be no consistent pricing measure so we shall focus on the case in which Assumption 3 holds. To understand better the connection between consistency and the extension property, let $P \in \mathbb{P}(m, \mathcal{N})$. Then $P(f > 0) > 0$ implies that f is not negligible and, as such, a potential new financial claim. However, pricing such claim by m would result in a violation of the no arbitrage principle, given that $m(f) = 0$. Therefore to the extent that market posesses the extension property, one should have a special interest for consistent pricing measures. As an example, let $P(T \leq t) > 0$ for some $t \in \mathbb{R}_+$ (where T is defined as in (6.1) with reference to m and P). Then, by Lemma 1, for each ϵ there exists a set $F \in \mathcal{F}_t$ such that $F \subset \{T \leq t\}$, $P(F) \geq (1 - \epsilon) P(T \leq t)$ and $m_t^p(F) = 0$. But then

$$m(F) = m_t^e(F) = P(X_t F \{T \leq t\}) = 0$$

It is then clear that the consistency of (m, P) requires that $P(T < \infty) = 0$.

Lemma 3. *Let* $m \in \mathcal{M}(\mathcal{C})$*,* $P \in \mathbb{P}(m, \mathcal{N})$ *and let* T *be defined as in (6.1) with reference to* m *and* P*.* (m, P) *is consistent if and only if* $P(T < \infty) = 0$*.*

It is clear that the extension property represents a reinforcement of the no arbitrage condition (3.1). The following result characterizes exactly the extent of such reinforcement (in the Appendix a more general result is proved, see Theorem 10).

Theorem 8. $\mathcal{K}$ *has the extension property if and only if the market admits no free lunches, i.e. (3.2) holds.*

The abstract absence of free lunches condition translates thus into the practical issue of whether markets may or not be extended consistently with the no arbitrage principle. This characterization helps providing economic content to the mathematical notion of free lunch, often criticized for not having a clear market interpretation (see especially [15] and [16]). In their seminal paper Harrison and Kreps [33, theorem 1, p. 386-7.] have already highlighted the relationship between the extension property and viability, i.e. the property that asset prices may support the optimal choice of an agent with regular preferences (see also [43]). It should however be remarked that our construction does not imply to restrict the underlying system of preferences to satisy convexity or continuity properties.

The extension property, however, is not sufficient to guarantee that the market could be extended to *arbitrary* set of new contracts iconsistently with the no arbitrage principle. In general, for example, it will not be possible to stretch the given pricing measure to a consistent price system for the completed financial market. The relevance of such an extension emerges from Theorem 5 (*iii*) which guarantees that the semimartingale nature of assets returns and the existence of a strictly positive martingale density would follow. In the next result we show, however, that such an extension is possible if we assume the existence of a given control measure.

Proposition 3. *Let* $P \in \bigcap_{m\in\mathcal{M}(\mathcal{C})} \mathbb{P}(m, \mathcal{N})$ *and the market admit no free lunches, i.e. (3.2) holds. Then there exists* $m \in \mathcal{M}(\mathcal{C})$ *such that the pair* (m, P) *is consistent. Consequently, every* $K \in \mathbb{K}$ *is a* P *semimartingale.*

This result should be compared to [21, theorem 7.2, p. 504 and theorem 7.6(a), p. 509]. A case in which $\bigcap_{m\in\mathcal{M}(\mathcal{C})} \mathbb{P}(m, \mathcal{N}) \neq \varnothing$ is clearly the one considered in the traditional setting, i.e. $\mathcal{N} = \mathcal{N}_Q$. Proposition 3 is however new and provides some ground for that vast stream of financial literature in which asset returns are assumed to meet the conditions of Theorem 7. For this class of models, the absence of free lunches defined as in (3.2) (i.e. with reference to trading strategies in Θ) is enough to prove that asset returns are semimartingales and that there exists a state price density process Z.

9. Applications.

A deep objection to the finitely additive model in asset pricing comes from the need for a tractable description of the distribution function of asset returns under the risk neutral measure. Statistical properties of asset returns under the pricing measure are in fact important. A major area of empirical research, originating from [11], suggests for example that the prices of derivatives may be useful to extract an estimate of such function (see [2] and [14] for samples of more recent work). It would be doubtful in the finitely additive setting that such a distribution is analytically tractable and, even more, that it may be inferred from current prices of derivative contracts. Particularly so in a model of financial markets less restrictive than the one considered under Assumption 4.

A more realistic assumption on the process of investment returns may be obtained by relaxing the condition of the existence of an upper bound.

Assumption 6. $\mathbb{K} \subset \mathbb{R}^{\tilde{\Omega}} \Big\backslash \overline{\mathfrak{L}(\mathcal{N})}$ *is such that if* $K \in \mathbb{K}$

(*i'*) K *is adapted to* $(\mathcal{F}_t : t \in \mathbb{R}_+)$, $K_0 = 0$ *and* $K^- \in \mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right)$*;*
(*ii*) *there exists* $T \in \mathbb{R}_+$ *such that* $K = K^T$*;*
(*iii'*) *if* $\theta \in \Theta$, $a, b \in \mathbb{R}$, $K_1, K_2 \in \mathbb{K}$ *then* $\theta.K, aK_1 + bK_2 \in \mathbb{K}$ *provided* $(\theta.K)^-, (aK_1 + bK_2)^- \in \mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right)$.

Furthermore, we assume that there are no arbitrage opportunities – so that (3.1) is in place. We easily deduce the analogous of Theorem 2 for the present context:

Theorem 9. *Let Assumption 6 hold. Then if there are no arbitrage opportunities there exists* $m \in ba(\mathcal{F}, \mathcal{N})_+$ *such that* $m(\Omega) = 1$ *and* $m[\mathcal{K}] \leq 0$.

It is implicit in the statement that if k is the overall, discounted return from an admissible trading strategy then necessarily k is m integrable. From this it readily follows that $m(|k| > 2^n)$ converges to 0 as n increases. Let then μ be the measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ induced by k – i.e. $\mu = m \circ k^{-1}$. Then for each ε there exists a η such that $\mu([-\eta, \eta]) < \varepsilon$ or, in other words, the measure μ is tight. Then, Dubins and Savage have proved [23, see pp. 190-191] that one may associate to μ a countably additive measure μ^* on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ – the *conventional companion* in their terminology – which is unique and has the following properties:

(1) $\int h d\mu = \int h d\mu^*$ for each $h : \mathbb{R} \to \mathbb{R}$ for which either integral is well defined;
(2) $\mu([x - \varepsilon, \infty[) \leq \mu^*([x, \infty))$ and $\mu^*([x - \varepsilon, \infty[) \leq \mu([x, \infty[)$ for each $x \in \mathbb{R}$ and $\varepsilon > 0$.

The first property ensures that μ^* may be employed to compute all moments of k along with other important quantities. The second property implies that the two distribution functions have exactly the same points of continuity and on these they agree with each other.

In particular let $K \in \mathbb{K}$ be the underlying asset return in discounted units at maturity T and let $c_t(s,T)$ be the time t price of a call option maturing at T and with discounted strike price equal to s. Then, if there are no arbitrage opportunities and Assumption 6 holds asset payoff functions are integrable with respect to the pricing measure m, by Theorem 9. By the preceding remarks, then,

$$c_t(s,T) = \int (K_T - s)^+ \, dm(\omega) = \int (x-s)^+ \, d\mu(x) = \int_s^\infty (x-s) \, d\mu^*(x)$$

Then, as in the case of a countably additive risk neutral measure, we deduce the inequalities

$$\varepsilon\mu^*(]s+\varepsilon,\infty[) \leq c_t(s,T) - c_t(s+\varepsilon,T) \leq \varepsilon\mu^*(]s,\infty[)$$

from which it follows that the right derivative of the call price with respect to the strike price, i.e.

$$\lim_n \varepsilon^{-1} [c_t(s,T) - c_t(s+\varepsilon,T)]$$

exists and coincides with $\mu^*(]s,\infty[)$. Of course this quantity may differ from $\mu(]s,\infty[)$ and, more precisely, $\mu(]s,\infty[) \leq \mu^*(]s,\infty[)$ unless s is a point of continuity.

This analysis suggests that the significance of derivative prices for evaluating the pricing measure carries over to the case in which such measure lacks countable additivity. Nevertheless it is implicit that allowing for the existence of points of discontinuity receives now greater significance since these are the only points in which the pricing measure may differ from its conventional companion. In particular, we conclude that the standard approach may induce an over-estimate of the mass assigned by the risk neutral measure to the right hand tail. The importance of discontinuities contrasts with many empirical works, e.g. [2], in which it is common to assume that μ is absolutely continuous and to estimate the density function thus ruling the case for finitely additive pricing measures out.

Appendix A. Proofs from Section 2.

If $f \in \mathfrak{B}(\mathcal{F})$ define the pseudo norm $\|f\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} = \inf_{N\in\mathcal{N}} \|fN^c\|$. $\|f\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} = 0$ if and only if f is negligible which, given Assumption 2, is in turn equivalent to $f \in \overline{\mathfrak{L}(\mathcal{N})}$. Moreover, $\|f\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} \leq \|f\|$ and $\|\cdot\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})}$ becomes a norm if we identify elements in $\mathfrak{B}(\mathcal{F})$ which coincide up to negligibility. Let $\langle f_k\rangle_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{B}(\mathcal{F})$ such that $\sum_k \|f_k\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} < \infty$ and let $N_k \in \mathcal{N}$ be such that $\|f_k N_k^c\| \leq \|f_k\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} + 2^{-k}$: the sequence $\langle f_k N_k^c\rangle_{k\in\mathbb{N}}$ has sum f in $\mathfrak{B}(\mathcal{F})$. Thus

$$\left\|\sum_{k=1}^r f_k - f\right\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} = \left\|\sum_{k=1}^r (f_k N_k^c - f)\bigcap_{k=1}^r N_k^c\right\|_{\mathfrak{B}(\mathcal{F},\mathcal{N})} \leq \left\|\sum_{k=1}^r f_k N_k^c - f\right\|$$

so that $\langle f_k\rangle_{k\in\mathbb{N}}$ is summable and $\mathfrak{B}(\mathcal{F},\mathcal{N})$ is complete. Denote by $\varkappa(f) = \left\{f + h : h \in \overline{\mathfrak{L}(\mathcal{N})}\right\}$ the equivalence class in $\mathfrak{B}(\mathcal{F},\mathcal{N})$ associated to some $f \in \mathfrak{B}(\mathcal{F})$. We shall exploit the following result in which $ba(\mathcal{F},\mathcal{N})$ denotes the set of bounded, finitely additive set functions vanishing on $\mathcal{N}$.

Lemma 4. *There exists an isometric isomorphism between $\mathfrak{B}(\mathcal{F},\mathcal{N})^*$ and $ba(\mathcal{F},\mathcal{N})$ defined implicitly via the equation*

$$\phi(\varkappa(y)) = \int y d\mu \tag{A.1}$$

Proof. Indeed the right hand side of (A.1) defines a linear functional over $\mathfrak{B}(\mathcal{F},\mathcal{N})$. If $\varkappa^* : \mathfrak{B}(\mathcal{F},\mathcal{N})^* \to \mathfrak{B}(\mathcal{F})^*$ is the adjoint of $\varkappa$ and $\phi \in \mathfrak{B}(\mathcal{F},\mathcal{N})^*$, then $\varkappa^*\phi \in \mathfrak{B}(\mathcal{F})^*$ and is therefore isometrically isomorphic to some $\mu \in ba(\mathcal{F})$ via $\int y d\mu = (\varkappa^*\phi)(y) = \phi(\varkappa(y))$. Given that $\overline{\mathfrak{L}(\mathcal{N})} = \varkappa(0)$, if $N \in \mathcal{N}$ then $\mu(N) = \phi(\varkappa(0)) = 0$ and, as $\mathcal{N}$ is closed with respect to intersection, it follows that $|\mu|(N) = 0$ for each $N \in \mathcal{N}$: in other words, $\mu \in ba(\mathcal{F},\mathcal{N})$. (A.1) establishes then an isomorphism. If $N \in \mathcal{N}$, $|\phi(\varkappa(y))| = \left|\int yN^c d\mu\right| \leq \|\mu\|\,\|yN^c\|$ so that $|\phi(\varkappa(y))| \leq \|\mu\|\,\|y\|_{\mathfrak{B}(\mathcal{F})}$; $\|\mu\| \leq \|\phi\|$ follows from $\|\varkappa^*\| \leq 1$. □

Let $\mathcal{R}(\mathcal{N})$ be the σ ring generated by the collection $\mathcal{N}$ and $\mathcal{R}(\mathcal{N})^\perp = \{F \subset \Omega : F^c \in \mathcal{R}(\mathcal{N})\}$. By Assumption 2 it follows that $\mathcal{R}(\mathcal{N}) = \left\{\bigcup_{k\in\mathbb{N}} N_k : N_k \in \mathcal{N}, k \in \mathbb{N}\right\}$; it is well known that $\sigma(\mathcal{N}) = \mathcal{R}(\mathcal{N}) \cup \mathcal{R}(\mathcal{N})^\perp$.

Proof of Theorem 1. $\mathfrak{B}(\mathcal{F},\mathcal{N})$ is a Banach space and Ω an inner point for $\mathcal{B}(\mathcal{F},\mathcal{N})_+$ as $\|f - \Omega\|_{\mathcal{B}(\mathcal{F},\mathcal{N})} < \eta$ implies $\{f < 1 - \eta\} \in \mathcal{N}$. The linear functional ϕ separating the convex sets $\mathcal{B}(\mathcal{F},\mathcal{N})_+$ and $\{0\}$ will therefore be bounded and non trivial, i.e. $\phi(\Omega) > 0$, and such that $\phi\left[\mathcal{B}(\mathcal{F},\mathcal{N})_+\right] \geq \phi(0) = 0$. By Lemma 4 ϕ is associated to some $m \in ba(\mathcal{F},\mathcal{N})_+$ that can be normalized so that $m(\Omega) = 1$.

Assume that (b) holds and let $\mathcal{N}' = \{F \cup G : F \in \mathcal{N}, G \in \mathcal{N}_Q\}$. $\mathcal{R}(\mathcal{N}')$ satisfies the properties listed in Assumption 2 so that, by the first claim of this Lemma, there exists $\lambda \in ba(\sigma(\mathcal{N}'), \mathcal{R}(\mathcal{N}'))_+$ with $\lambda(\Omega) = 1$. If $F, G \in \mathcal{R}(\mathcal{N}')^\perp$ are disjoint, then $\Omega \in \mathcal{R}(\mathcal{N}')$ a contradiction; if $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of $\sigma(\mathcal{N}')$ measurable sets, then it has at most one element in $\mathcal{R}(\mathcal{N})^\perp$, say F_1. Since $\bigcup_{n>1} F_n \in \mathcal{R}(\mathcal{N}')$, it follows that

$$\lambda\left(\bigcup_n F_n\right) = \lambda(F_1) + \lambda\left(\bigcup_{n>1} F_n\right) = \lambda(F_1) = \lambda(F_1) + \sum_{n>1} \lambda(F_n)$$

Since each two versions of $Q(F|\,\sigma(\mathcal{N}))$ differ on some set in $\mathcal{N}'$, there is no ambiguity defining $P(F) = \lambda(Q(F|\,\sigma(\mathcal{N})))$. P is positive and $P(\Omega) = 1$; furthermore, P vanishes on $\mathcal{N}$. If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of $\mathcal{F}$ measurable sets, then $Q\left(\bigcup_n F_n \middle| \sigma(\mathcal{N})\right) = \sum_n Q\left(F_n \middle| \sigma(\mathcal{N})\right)$ up to a λ null set and since $\lambda \in \mathbb{P}(\sigma(\mathcal{N}'))$,

$$P\left(\bigcup_n F_n\right) = \lambda\left(Q\left(\bigcup_n F_n \middle| \sigma(\mathcal{N})\right)\right) = \lambda\left(\sum_n Q\left(F_n \middle| \sigma(\mathcal{N})\right)\right) = \sum_n \lambda\left(Q\left(F_n \middle| \sigma(\mathcal{N})\right)\right) = \sum_n P(F_n)$$

(a) follows and the reverse implication is obvious. □

Lemma 5. *Let $\hat{\mathcal{F}}_t$ be defined as in the text (see p. 6). Then $\hat{\mathcal{F}}_0 = \sigma(\mathcal{N})$.*

Proof. It is easy to see that $\sigma(\mathcal{F}_t \cup \mathcal{N}) = \{F \Delta N : F \in \mathcal{F}_t,\ N \in \mathcal{R}(\mathcal{N})\}$. Let $t_n < 2^{-n}$. If $F \in \hat{\mathcal{F}}_0$ then $F \in \sigma(\mathcal{F}_{t_n} \cup \mathcal{N})$ for each n and we may therefore write $F = F_n \Delta N_n$ or even

$$F = \bigcup_k \bigcap_{n>k} (F_n \Delta N_n) = \bigcup_k \left\{\left(\bigcap_{n>k} F_n N_n^c\right) \cup N_k^*\right\} = \left(\bigcup_k \bigcap_{n>k} F_n \cap \bigcup_k \bigcap_{n>k} N_n^c\right) \cup \bigcup_k N_k^*$$

It is clear that $N_k^* \subset \bigcup_{n>k} N_n$ so that $N_k^*, \bigcup_k N_k^* \in \mathcal{R}(\mathcal{N})$ by Assumption 2. On the other hand, $F_0 = \bigcup_k \bigcap_{n>k} F_n \in \mathcal{F}_0$ by right continuity: if $F_0 = \Omega$, $F = \left(\bigcup_k \bigcap_{n>k} N_n^c\right) \cup \left(\bigcup_k N_k^*\right)$; if $F_0 = \varnothing$, $F = \bigcup_k N_k^*$. In either case, $F \in \sigma(\mathcal{N})$. □

Appendix B. Proofs from Section 3.

Proof of Theorem 2. Since Ω is an internal point of $\mathfrak{B}(\mathcal{F}, \mathcal{N})_{++}$, there exists a non trivial, continuous linear functional ϕ that separates $\mathfrak{B}(\mathcal{F}, \mathcal{N})_{++}$ and $\mathcal{K}$. Since $\phi[\mathcal{K}]$ is a linear space and $\phi\left[\mathfrak{B}(\mathcal{F}, \mathcal{N})_{++}\right]$ a convex cone and since $\phi[\mathcal{K}] \cap \phi\left[\mathfrak{B}(\mathcal{F}, \mathcal{N})_{++}\right] \subset \{0\}$, it must be that, up to a change of sign, $\phi[\mathcal{K}] = 0 \leq \phi\left[\mathfrak{B}(\mathcal{F}, \mathcal{N})_{++}\right]$ and $\phi(\Omega) > 0$. By Lemma 4 and normalization we may represent ϕ via $m \in ba(\mathcal{F}, \mathcal{N})_+$ with $m(\Omega) = 1$. As $\mathcal{F} = \left\{F \Delta N : F \in \bigcup_{t \in \mathbb{R}_+} \mathcal{F}_t, N \in \mathcal{N}\right\}$, if $H \in \mathcal{F}$ then $H \in \mathfrak{B}(\mathcal{F}_t, \mathcal{N})_+$ for some $t \in \mathbb{R}_+$. If $H - \gamma(H) \in \mathcal{K}$ then necessarily $\gamma(H) = m(H)$ so that $m(H) = 0$ implies $H \in \mathcal{K} \cap \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ which contradicts (3.1) unless $H \in \mathcal{N}$.

Let $F \in \mathcal{F}_t$, $N \in \mathcal{N}$ and $K \in \mathbb{K}$. Then, $F(K_\infty N^c - K_t) \in \mathcal{K}$ and

$$(K_\infty N^c - K_t) F \geq -\left(\left\|K_\infty^- N^c\right\| + K_t\right) F$$

(3.1) implies that for any $\eta > 0$, $\{-K_t \geq \|K_\infty^- N^c\| + \eta\} \in \mathcal{N}$ i.e. that $\left\|K_t^-\right\|_{\mathfrak{B}(\mathcal{F}, \mathcal{N})} \leq \|K_\infty^-\|_{\mathfrak{B}(\mathcal{F}, \mathcal{N})}$.

For $n \geq 1$ let $k_n, f_n \in \mathbb{R}^\Omega$, $f_n \geq 0$, $N_n \in \mathcal{N}$ and define $f = \sum_n f_n$ and $N_0 = \bigcup_n N_n$. Assume that $\sum_n k_n^- \in \mathbb{R}^\Omega$ and let $k = -\sum_n k_n$. Then,

$$N_0^c (f - k) = N_0^c \sum_n (f_n + k_n) \leq \sum_n \sup_{\omega \in N_n^c} (f_n + k_n)(\omega)$$

Let $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$, $k_n \in \mathcal{K}$ and N_n be such that $\bar{\alpha}_{k_n}(f_n) \geq \sup_{\omega \in N_n^c} (f_n + k_n)(\omega) - \eta 2^{-n}$. Assume that $\sum_n \bar{\alpha}_{k_n}(f_n) < \infty$. Then for $\omega \in N_0^c$,

$$\bar{\alpha}_{k_n}(f_n) + \eta 2^{-n} > k_n(\omega)$$

Therefore, if $k_n = K_\infty^n$ u.n. and given that $\|K^{n-}\|_{\mathfrak{B}(\tilde{\mathcal{F}}, \mathcal{N})} \leq \|k_n^-\|_{\mathfrak{B}(\mathcal{F}, \mathcal{N})}$ we obtain that $k \in \mathcal{K}_\sigma$ (see (2.5)). Moreover, the absence of arbitrage opportunities implies $\sup_{\omega \in N_n^c} (f_n + k_n)(\omega) \geq 0$ for each n so that

$$N_0^c f \leq \sum_n \sup_{\omega \in N_n^c} (f_n + k_n)(\omega) + N_0^c (k \wedge \|f\|) \leq \sum_n \bar{\alpha}_{k_n}(f_n) + \eta + N_0^c (k \wedge \|f\|)$$

Remark that $k \wedge \|f\| \in \mathcal{C}_\sigma$. The first claim, applied to $\mathcal{C}_\sigma$ rather than $\mathcal{C}$, shows that we can pick $m \in \mathcal{M}(\mathcal{C}_\sigma)$. Then, $m(f) \leq \sum_n \bar{\alpha}_{k_n}(f_n) + \eta$ for each $\eta \geq 0$ i.e. $m(f) \leq \sum_n \bar{\alpha}_{k_n}(f_n)$. Replacing f by $\sum_{n>N} f_n$ we obtain likewise the inequality $m\left(\sum_{n>N} f_n\right) \leq \sum_{n>N} \bar{\alpha}_{k_n}(f_n)$ from which we deduce that

$$\lim_N m\left(\sum_{n>N} f_n\right) \leq \lim_N \sum_{n>N} \bar{\alpha}_{k_n}(f_n) = 0$$

i.e. $m(f) = \sum_n m(f_n)$. □

Appendix C. Proofs from Section 3.1.

Proof of Proposition 1. By proving the statement separately for f^+ and f^- we can reduce to the case where $f \in L^1(\mathcal{F}, \xi)_+$. Let $I \in \mathcal{I}_\eta$ and $G \in \mathcal{G}$. The set function $\phi_{fI}(G) = \xi(fIG)$ on $\mathcal{G}$ is positive and additive. Let $\langle G_n \rangle_{n\in\mathbb{N}}$ be a sequence of $\mathcal{G}$ measurable sets such that $\lim_n \gamma(G_n) = 0$, fix $\varepsilon, \delta > 0$ and let $h_n = fIG_n$. As $\{h_n > \varepsilon\} \subset IG_n \in \mathcal{I}_\eta$, $\xi(h_n > \varepsilon) \leq \xi(IG_n) \leq \gamma(G_n)$ so that h_n converges to 0 in ξ measure and, since $|h_n| \leq f$ [26, theorem III.3.7, p. 124], in $L^1(\mathcal{F}, \xi)$ as well. Therefore

$$\lim_n \phi_{fI}(G_n) = \lim_n \xi(h_n) = 0$$

and ϕ_{fI} is then absolutely continuous with respect to γ. Denote by $\xi(fI|\mathcal{I}_\eta)$ the corresponding Radon Nikodym derivative. If $G \in \mathcal{G}$, $I \subset I' \in \mathcal{I}_\eta$ then

$$\gamma(\xi(fI|\mathcal{I}_\eta)G) = \xi(fIG) \leq \xi(fI'G) = \gamma(\xi(fI'|\mathcal{I}_\eta)G)$$

i.e. $0 \leq \xi(fI|\mathcal{I}_\eta) \leq \xi(fI'|\mathcal{I}_\eta)$, up to a γ null set. Let $\langle I_n \rangle_{n\in\mathbb{N}}$ be an increasing sequence of sets in $\mathcal{I}_\eta$ with the property that $\lim_n \xi(fI_n) = \sup_{I\in\mathcal{I}_\eta} \xi(fI)$: then the sequence $\langle \xi(fI_n|\mathcal{I}_\eta) \rangle_{n\in\mathbb{N}}$ increases γ a.s. and we may thus define $\xi(f|\mathcal{I}_\eta) = \lim_n \xi(fI_n|\mathcal{I}_\eta)$ outside some γ null set. Monotone convergence and the inequality $\gamma(\xi(fI_n|\mathcal{I}_\eta)) \leq \xi(f)$ imply that $\xi(f|\mathcal{I}_\eta) \in L^1(\mathcal{G}, \gamma)_+$. Let $I \in \mathcal{I}_\eta$. If $\lim_n \xi(fI_nI) < \xi(fI)$, then

$$\sup_{I'\in\mathcal{I}_\eta} \xi(fI') = \lim_n \xi(fI_n) = \lim_n [\xi(fI_nI) + \xi(fI_nI^c)] < \xi(fI) + \lim_n \xi(fI_nI^c) = \lim_n \xi(f(I_nI^c \cup I))$$

which is contradictory since $(I^cI_n \cup I) \in \mathcal{I}_\eta$. But then,

$$\gamma(\xi(f|\mathcal{I}_\eta)I) = \lim_n \gamma(\xi(fI_n|\mathcal{I}_\eta)I) = \lim_n \xi(fI_nI) = \xi(fI)$$

and (3.8) holds. To prove uniqueness remark that, since $\mathcal{G}$ is a σ algebra and γ is countably additive, for any k there exists a set $I_k \in \mathcal{I}_\eta$ such that $\gamma(I_k^c) < 2^{-k}$. Let $y \in L^1(\mathcal{G}, \gamma)_+$ satisfy (3.8) and $G \in \mathcal{G}$. Then, $y = \xi(f|\mathcal{I}_\eta)$ up to a γ null set as

$$\gamma(yG) = \lim_k \gamma(yGI_k) = \lim_k \xi(fGI_k) = \lim_k \gamma(\xi(f|\mathcal{I}_\eta)GI_k) = \gamma(\xi(f|\mathcal{I}_\eta)G)$$

Given uniqueness and additivity of ξ, $\xi(f+g|\mathcal{I}_\eta) = \xi(f|\mathcal{I}_\eta) + \xi(g|\mathcal{I}_\eta)$; (3.9) is a consequence of the fact that $IG \in \mathcal{I}_\eta$ whenever $I \in \mathcal{I}_\eta$ and $G \in \mathcal{G}$. For $f \in \mathcal{I}_\eta$ we deduce from (3.9) and (3.8) that $\gamma(|\xi(f|)|) = \xi(|f|)$ while, in the generale case,

$$\gamma(|\xi(f|\mathcal{I}_\eta)|) \leq \gamma(\xi(|f||\mathcal{I}_\eta)) = \lim_k \xi(\xi(|f||\mathcal{I}_\eta)I_k) \leq \xi(|f|)$$

It follows that$\|\xi(\cdot|\mathcal{I}_\eta)\| = 1$. □

Proof of Theorem 3. Let $K \in \mathbb{K}$ and apply (3.10) to $f = G(K_\tau - K_\sigma)$ with $\sigma, \tau \in \mathcal{T}$, $\tau \geq \sigma$ and $G \in \mathcal{F}_\sigma$. We obtain

$$m_\sigma^e(GK_\sigma I) = m^e(m^e(GK_\tau | \mathcal{F}_\sigma) I) + (m_\sigma^e - m^e)(m^p(GK_\tau | \mathcal{I}_\sigma) I)$$

Observe that, since $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$ then by Lemma 1(*iii*), $0 \leq m^e | \mathcal{F}_\sigma \leq m_\tau^e | \mathcal{F}_\sigma \leq m_\sigma^e$. We can therefore write $m^e(F) = m_\sigma^e(h_\sigma F)$ for each $F \in \mathcal{F}_\sigma$, where $0 \leq h_\sigma \leq 1$, m_σ^e a.s..We obtain

$$m_\sigma^e(h_\sigma F) = m^e(F) = m_\tau^e(h_\tau F) = m_\tau^e(m_\tau^e(h_\tau | \mathcal{F}_\sigma) F) \leq m_\sigma^e(m_\tau^e(h_\tau | \mathcal{F}_\sigma) F)$$

i.e. $h_\sigma \leq m_\tau^e(h_\tau | \mathcal{F}_\sigma)$, m_σ^e a.s.. Moreover,

$$m_\sigma^e(K_\sigma GI) = m_\sigma^e(h_\sigma m^e(K_\tau | \mathcal{F}_\sigma) GI) + m_\sigma^e((1 - h_\sigma) m^p(K_\tau | \mathcal{I}_\sigma) GI)$$

Since $G \in \mathcal{F}_\sigma$ is arbitrary and I may be so chosen that $m_\sigma^e(I^c) < \epsilon$ for each ϵ, we obtain that

$$K_\sigma = h_\sigma m^e(K_\tau | \mathcal{F}_\sigma) + (1 - h_\sigma) m^p(K_\tau | \mathcal{I}_\sigma)$$

up to a m_σ^e null set. By Assumption 4 $K_\infty = K_T$ for some $T \in \mathbb{R}_+$: choosing $\tau = T$ proves (3.11). $h_\sigma = 1$ up to a m_σ^e null set is equivalent to $m^e | \mathcal{F}_\sigma = m_\sigma^e$ i.e. $m^p(\Omega) = m_0^p(\Omega)$. However, since $\mathcal{F}_0$ is trivial and given Assumption 1, m_0 clearly admits a countably additive extension to $\mathcal{F}$ or, in other terms, $m_0 = m_0^e$ i.e. $m_0^p = 0$: $m^p(\Omega) = m_0^p(\Omega)$ is then in turn equivalent to $m^p = 0$ i.e. to m being countably additive. The second equality in (3.11) follows from $m_\sigma^e(((m(K_\tau | \mathcal{I}_\sigma) - K_\sigma)) GI) = 0$. □

Appendix D. Proofs from Section 4.

Proof of Theorem 4. It is proved in [12, theorem 2] that $P' = \sum_n 2^{-n} \bar{m}_{t_n}^e$ – where $\langle t_n \rangle_{n \in \mathbb{N}}$ is a given sequence in $\mathbb{R}_+$ and $\bar{m}_{t_n}^e$ is the countably additive extension of $m_{t_n}^e$ to $\mathcal{F}$ – is such that $P' | \mathcal{F}_t \gg m_t^e$. The same argument applies if $m_{t_n}^e$ is replaced by $m_{\hat{\mathcal{F}}_{t_n}}^e$ and let $\hat{P}'$ be the measure thus obtained: thus necessarily $\hat{P}' \in ca(\mathcal{F}, \mathcal{N})_+$. If $P' > 0$ (resp. $\hat{P}' > 0$) let $P = P'(\Omega)^{-1} P'$ (resp. $\hat{P} = \hat{P}'(\Omega)^{-1} \hat{P}'$) otherwise choose $P \in \mathbb{P}(\mathcal{F})$ arbitrarily (resp. $\hat{P} \in \mathbb{P}(\mathcal{F}, \mathcal{N})$, given Assumption 3). □

If Assumption 1 holds there exists a probability on $\mathcal{F}$ while, by lemmas **??** and 5, the restriction to $\hat{\mathcal{F}}_0$ coincides necessarily with $\hat{m}_0$. In other words, $\hat{m}_0^e(\Omega) = m(\Omega)$ so that necessarily $\hat{P}'(\Omega) > 0$. $\hat{P}$ is then obtained, as before, by normalization.

Proof of Corollary 1. The supermartingale nature of X is obvious given Lemma 1(*iii*). To prove right continuity, let $\Delta m_t^e = \lim_{u > t, u \downarrow t} (m_t^e - m_u^e) | \mathcal{F}_t = \lim_{u > t, u \downarrow t} (m_u^p - m_t^p) | \mathcal{F}_t$. By a cornerstone result of Meyer [47, p. 95], it is enough to show that the function $t \to P(Y_t) = \| m_t^e \|$ is right continuous which is in turn equivalent, in our setting, to prove that $\Delta m_t^e = 0$. Let $\langle u_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+$ decreasing to t and such that $u_n > t$. Observe that for each n,

$$0 \leq \Delta m_t^e \leq m_{u_n}^p \big| \mathcal{F}_t \tag{D.1}$$

Define ϕ to be the countably additive extension of Δm_t^e to $\mathcal{F}$ obtained by setting

$$\phi(F) = \Delta m_t^e(P(F | \mathcal{F}_t))$$

for any $F \in \mathcal{F}$. The proof will be accomplished if we show that $\phi = 0$. Define the collection

$$\mathcal{V} = \{F \subset \Omega : \Delta m_t^e(E) = 0 \text{ when } E \subset F,\ E \in \mathcal{F}_t\}$$

of Δm_t^e null sets. Inequality (D.1) implies that $\mathcal{I}_{u_n} \subset \mathcal{V}$; on the other hand, $\mathcal{V}$ contains all ϕ null sets such as $E \cap \{\phi(E|\mathcal{F}_t) = 0\}$ for $E \in \mathcal{F}$. It is well known that Δm_t^e admits a countably additive extension to the σ algebra $\hat{\mathcal{F}}_t = \sigma(\mathcal{F}_t \cup \mathcal{V})$ that we shall denote by ψ.

Let $F_n \in \mathcal{I}_{u_n}$ be such that $P(F_n^c) < 2^{-n}$, put $G_n = \{\phi(F_n^c|\mathcal{F}_t) = 0\}$ and observe that, since $G_n \subset (F_n \cup (F_n^c \cap \{\phi(F_n^c|\mathcal{F}_t) = 0\}))$, then $G_n \in \mathcal{V}$. Therefore, $\phi(\Omega) = \psi(\Omega) = \psi(G_n^c) = \phi(G_n^c)$ i.e. $\phi(F_n^c|\mathcal{F}_t) > 0$, ϕ a.s. and *a fortiori* the same is true with respect to $\phi\left(\bigcup_{n>m} F_n^c\middle|\mathcal{F}_t\right)$. Let $H = \bigcap_m \left\{\phi\left(\bigcup_{n>m} F_n^c\middle|\mathcal{F}_t\right) > 0\right\}$ and define

$$y_m = \phi\left(\bigcup\nolimits_{n>m} F_n^c\middle|\mathcal{F}_t\right)^{-1} H \cap \bigcup_{n>m} F_n^c$$

The sequence $\langle y_m\rangle_{m\in\mathbb{N}}$ converges ϕ a.s. to (the indicator of) $\bigcap_m \bigcup_{n>m} F_n^c \in \mathcal{F}_t$: plainly, $y_m = 1$ on $\bigcap_m \bigcup_{n>m} F_n^c$ while if $\omega \in \bigcap_{n>n_\omega} F_n$ for some n_ω and $m > n_\omega$, then $y_m(\omega) = 0$. So $y^* = \sup_n y_n < \infty$, ϕ a.s.. If $F \in \mathcal{F}_t$

$$\begin{aligned}
\phi(F) &= \lim_m \phi\left(\frac{\phi\left(\bigcup_{n>m} F_n^c\middle|\mathcal{F}_t\right)}{\phi\left(\bigcup_{n>m} F_n^c\middle|\mathcal{F}_t\right)} H \cap F\right)\\
&= \lim_m \phi(y_m F)\\
&= \lim_r \lim_m \phi(F(y_m \wedge r)) + \lim_r \lim_m \phi\left(F(y_m - r)^+\right)\\
&= \phi\left(F \cap \bigcap_m \bigcup_{n>m} F_n^c\right) + \lim_r \lim_m \phi\left(F(y_m - r)^+\right)\\
&= \lim_r \lim_m \phi\left(F(y_m - r)^+\right)
\end{aligned}$$

where the last line follows from the fact that $\phi \ll P$ and

$$P\left(\bigcap_m \bigcup_{n>m} F_n^c\right) \le \lim_m \sum_{m>n} P(F_n^c) \le \lim_m 2^{-m} = 0$$

In other words, ϕ vanishes outside the set $G_r = \{r < y^* < \infty\}$ for each $r \in \mathbb{R}_+$. Since $\bigcap_r G_r = \varnothing$ it follows that ϕ is purely finitely additive too, i.e. $\phi = 0$. □

Appendix E. Proofs from Section 5.

Proof of Lemma 2. Let $n_t = (m^p - m_t^p)|\mathcal{F}_t$ and $\tilde{n} = (n_t : t \in \mathbb{R}_+)$. As $n_s \ge n_t|\mathcal{F}_s \ge 0$ (by Lemma 1(*iii*)), $\tilde{n}$ describes a positive, bounded finitely additive supermartingale. Let ϕ be the positive, bounded measure on $\{(F_0 \times \{0\}) \cup]]\sigma, \infty[[: \sigma \in \mathcal{T},\ F_0 \in \mathcal{F}_0\}$ (an additive class) defined by letting

$$\phi((F_0 \times \{0\}) \cup]]\sigma, \infty[[) = n_0(F_0) + n_\sigma(\sigma < \infty)$$

By lemma 2 in [12], ϕ admits a positive, bounded extension $\bar{\phi}$ to the whole of $2^{\tilde{\Omega}}$. For $\tau \in \mathcal{T}$, define $\bar{m}_\tau^p(F) = m^p(F) - \bar{\phi}(F \times \mathbb{R}_+]]\tau, \infty[[)$ for $F \subset \Omega$. This is clearly an extension of m_τ^p satisfying the claim. □

We shall repeatedly exploit the following lemma.

Lemma 6. *If $\sigma, \tau \in \mathcal{T}_0$ and $F \in \mathcal{I}_\tau$, then $\bar{m}_{\sigma\vee\tau}^p(F; \tau \ge \sigma) = 0$.*

Proof. The claim follows from $\{F; \tau \ge \sigma\} \in \mathcal{I}_\tau$ and

$$m_{\sigma\vee\tau}^p(F; \tau \ge \sigma) = (m_{\sigma\vee\tau}^p - m_\tau^p)(F; \tau \ge \sigma) = (m_\tau^e - m_{\sigma\vee\tau}^e)(F; \tau \ge \sigma) = P((X_\tau - X_{\sigma\vee\tau}) F\{\tau \ge \sigma\}) = 0$$

□

We start the proof of Proposition 2 by defining a net in $\mathcal{H}$. Let $\mathbb{D}$ be the set of all *càdlàg* processes with respect to P; $\mathcal{D}$ the collection of all finite sets D in $\mathbb{D}$ which include the processes Z where $Z_t = t$; $\mathbf{A} = \mathcal{D} \times \mathbb{R}^3_+$. Despite the potential incompleteness of the filtration [38, lemma I.1.28, p. 7], to any $\alpha \in \mathbf{A}$, with $\alpha = (D_\alpha; (t_\alpha, \eta_\alpha, \epsilon_\alpha))$, we can associate the sequence $\langle t_i^\alpha \rangle_{i\in\mathbb{N}}$ in $\mathcal{T}_0$ defined recursively as follows (with $\inf \varnothing = \infty$): $t_0^\alpha = 0$;

$$t_i^\alpha = \inf\left\{t > t_{i-1}^\alpha : \bigvee_{X\in D_\alpha} \left|X_t - X_{t_{i-1}^\alpha}\right| > \eta_\alpha\right\} \wedge \inf\left\{t : M_t^* > i\eta_\alpha\right\} \tag{E.1}$$

By construction, $t_i^\alpha \le t_{i-1}^\alpha + \eta_\alpha \le i\eta_\alpha$ and $M_{t\wedge t_i^\alpha} \le 2M_{t_i^\alpha-} + M_{t_i^\alpha} \le 2i\eta_\alpha + M_{t_i^\alpha}$. It follows that $M^{t_i^\alpha}$ is uniformly integrable. On the set $\{\lim_i t_i^\alpha < t_\alpha\}$ there either exist one process $X \in D_\alpha$ which has infinitely many oscillations larger than η_α or $\sup_{t\le t_\alpha} M_t = \infty$: since both are P null events $0 = P(\lim_i t_i^\alpha < t_\alpha) = \lim_i P(t_i^\alpha < t_\alpha)$. Define then

$$I_\alpha = \min\left\{i \in \mathbb{N} : P(t_i^\alpha < t_\alpha) \le \epsilon_\alpha\right\}$$

Let $\mathbf{A}$ be directed with respect to the partial order defined implicitly by letting $\alpha \ge \beta$ whenever $D_\beta \subset D_\alpha$, $t_\beta \le t_\alpha$, $\eta_\beta \ge \eta_\alpha$ and $\epsilon_\beta \ge \epsilon_\alpha$. For each $\alpha \in \mathbf{A}$ the set of elements $\left(\langle t_i^\alpha\rangle_{i=0}^{I^\alpha}, \langle F_i^\alpha\rangle_{i=0}^{I^\alpha}\right)$ in $\mathcal{H}$ such that $P\left(F_{I_\alpha-1}^{\alpha c}\right) \le \epsilon_\alpha$ is non empty. Invoking the axiom of choice, we can select for each $\alpha \in \mathbf{A}$ an element $H_\alpha = \left(\langle t_i^\alpha\rangle_{i=0}^{I^\alpha}, \langle F_i^\alpha\rangle_{i=0}^{I^\alpha}\right) \in \mathcal{H}$ with the above properties. $\langle H_\alpha\rangle_{\alpha\in\mathbf{A}}$ will be a fixed net throughout this section; let $I_\alpha(Y)_t = I_{H_\alpha}(Y)_t$, $J_\alpha(Y)_t = J_{H_\alpha}(Y)_t$ and $K^\alpha = K^{H_\alpha}$.

We start with a preliminary result. Let $\tilde\Omega_0 = [[0]]$, $\tilde\Omega_j^\alpha =]]t_{j-1}^\alpha, t_j^\alpha]]$ for $0 < j \le I_\alpha$ and $\tilde\Omega_{I_\alpha+1}^\alpha =]]t_{I_\alpha}^\alpha, \infty[[$ and define

$$\mathcal{P}_j^\alpha = \left\{(F_j \times \mathbb{R}_+)\,\tilde\Omega_j^\alpha : F_j \in \mathcal{F}_{t^\alpha_{(j-1)\vee 0}}\right\}$$

for $j = 0, \ldots, I_\alpha + 1$. It is clear that $\mathcal{P}_j^\alpha$ is a σ algebra of subsets of $\tilde\Omega_j^\alpha$.

Lemma 7. *For each $\alpha \in \mathbf{A}$ define*

$$\mathcal{P}^\alpha = \left\{\bigcup_{j=0}^{I_\alpha+1} \tilde E_j : \tilde E_j \in \mathcal{P}_j^\alpha, 0 \le j \le I_\alpha + 1\right\} \tag{E.2}$$

$\mathcal{P}^\alpha$ is a σ algebra of subsets of $\tilde\Omega$ and any $g : \tilde\Omega \to \mathbb{R}$ is $\mathcal{P}^\alpha$ measurable if and only if it is of the form

$$g = g_0\,[[0]] + \sum_{i=0}^{I_\alpha-1} g_i\,]]t_i^\alpha, t_{i+1}^\alpha]] + g_{I_\alpha}\,]]t_{I_\alpha}^\alpha, \infty[[\tag{E.3}$$

with g_i measurable with respect to $\mathcal{F}_{t_i^\alpha}$.

Proof. Since $\mathcal{P}^\alpha = \left\{\tilde E \subset \tilde\Omega : \tilde E\tilde\Omega_j^\alpha \in \mathcal{P}_j^\alpha,\ 0 \le j \le I_\alpha + 1\right\}$ and $\left\{\tilde\Omega_j^\alpha : j = 0, \ldots, I_\alpha + 1\right\}$ is a partition of $\tilde\Omega$ then $\mathcal{P}^\alpha$ is a σ algebra [50, problem 2, p. 257]. If g is of the form (E.3) it is clearly $\mathcal{P}^\alpha$ measurable. On the other hand the class of all bounded processes of the form (E.3) is a vector space which contains the indicators of all $\mathcal{P}^\alpha$ measurable sets and is closed with respect to increasing, bounded sequences. The claim then follows by a monotone class argument. □

Lemma 8. *There exists a mapping $\mathcal{P}_m : \mathfrak{B}\left(2^{\tilde\Omega}\right) \to L^\infty(\mathcal{P}, \lambda)$ such that for each $Y \in \mathfrak{B}\left(2^{\tilde\Omega}\right)$ and $\upsilon \in \mathcal{T}$,*

$$\operatorname*{LIM}_\alpha I_\alpha(Y)_\upsilon = P\int_0^\upsilon \mathcal{P}_m(Y)\,dA$$

$\mathcal{P}_m$ is a positive, linear operator such that $\|\mathcal{P}_m\| = 1$ and $\mathcal{P}_m(YZ) = \mathcal{P}_m(Y)Z$ whenever $Y, YZ \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$ and $Z \in \mathfrak{B}\left(\tilde{\mathcal{F}}, \mathcal{N}\right)$ is càglàd.

Proof. Adapting (5.6) to the present setting we have, for $\upsilon \in \mathcal{T}$ and $Y \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$,

$$f_\alpha^\upsilon(Y) = \sum_{i=0}^{I_\alpha - 1} \bar{m}^p_{t^\alpha_{i+1}\wedge\upsilon}\left(Y_{t^\alpha_{i+1}} \middle| \mathcal{I}^\upsilon_{t^\alpha_i}\right) F_i^\alpha \{t_i^\alpha < \upsilon\}]]t_i^\alpha, t_{i+1}^\alpha]] \tag{E.4}$$

Remark that $f_\alpha^\upsilon(Y) \in L^\infty(\mathcal{P}^\alpha, \lambda)$ by Lemma 7 with $\|f_\alpha^\upsilon(Y)\| \le \|Y\|$ and that $\{t_i^\alpha < \upsilon\} \subset \left\{t_{i+1}^\alpha < \upsilon + \eta_\alpha\right\}$ by (E.1) so that $f_\alpha^\upsilon(Y)$ vanishes then on $]]\upsilon + \eta_\alpha, \infty[[$. If $g \in L^1(\mathcal{P}, \lambda)$ then $\left|\int g f_\alpha^\upsilon(Y) d\lambda\right| \le \|Y\| \|g\|$ and the quantity $\phi_Y^\upsilon(g) = \mathrm{LIM}_\alpha \int g f_\alpha^\upsilon(Y) d\lambda$ is well defined and finite. The functional $\phi_Y^\upsilon : L^1(\mathcal{P}, \lambda) \to \mathbb{R}$, is linear and $\|\phi_Y^\upsilon\| \le \|Y\|$: by standard representation theorems [26, IV.8.5], we can associate to ϕ_Y^υ an element $f^\upsilon(Y) \in L^\infty(\mathcal{P}, \lambda)$ such that

$$\mathop{\mathrm{LIM}}_\alpha \int g f_\alpha^\upsilon(Y) d\lambda = \int g f^\upsilon(Y) d\lambda$$

holds for any $g \in L^1(\mathcal{P}, \lambda)$. Furthermore, by the remark following (E.4), $f^\upsilon(Y)$ vanishes outside $[[0, \upsilon]]$; eventually, f^υ is additive since f_α^υ is.

By Lemma 7 we can define $g^\alpha = \lambda(g | \mathcal{P}_\alpha)$ and we deduce from (E.4) and (E.3) that $f_\alpha^\upsilon(Y) g^\alpha = f_\alpha^\upsilon(Y g^\alpha)$. Therefore,

$$\int_0^\upsilon f_\alpha^\upsilon(Y) g d\lambda = \int_0^\upsilon f_\alpha^\upsilon(Y) g^\alpha d\lambda = \int_0^\upsilon f_\alpha^\upsilon(Y g^\alpha) d\lambda = \sum_{i=0}^{I_\alpha - 1} \bar{m}^p_{t^\alpha_{i+1}\wedge\upsilon}\left(Y_{t^\alpha_{i+1}} g^\alpha_{t^\alpha_{i+1}} F_i^\alpha\right)$$

and consequently

$$\int f_\alpha^\upsilon(Y) g d\lambda = \sum_{i=0}^{I_\alpha - 1} \bar{m}^p_{t^\alpha_{i+1}\wedge\upsilon}\left(Y_{t^\alpha_{i+1}} g^\alpha_{t^\alpha_{i+1}} F_i^\alpha\right) + \int_\upsilon^{\upsilon+\eta_\alpha} f_\alpha^\upsilon(Y g^\alpha) d\lambda$$

Let now $\sigma \in \mathcal{T}$ be such that $\upsilon \ge \sigma$. Since $\mathrm{LIM}_\alpha \int_\upsilon^{\upsilon+\eta_\alpha} g d\lambda = 0$ as λ is countably additive,

$$\mathop{\mathrm{LIM}}_\alpha \left| \int \left[f_\alpha^\upsilon(Y) - f_\alpha^\sigma(Y)\right] g d\lambda \right| \le 2\|Y\| \mathop{\mathrm{LIM}}_\alpha \sum_{i=0}^{I_\alpha - 1} \left| \left(\bar{m}^p_{t^\alpha_{i+1}\wedge\upsilon} - \bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\right)\left(g^\alpha_{t^\alpha_{i+1}} F_i^\alpha\right)\right| \tag{E.5}$$

Remark now that

$$\begin{aligned}
\sum_{i=0}^{I_\alpha - 1} \left| \left(\bar{m}^p_{t^\alpha_{i+1}\wedge\upsilon} - \bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\right)\left(F_i^\alpha \{t_i^\alpha < \sigma\}\right)\right| &\le P \sum_{i=0}^{I_\alpha - 1} \left(A^\upsilon_{t^\alpha_{i+1}} - A^\sigma_{t^\alpha_{i+1}}\right)\left\{t_i^\alpha < \sigma \le t_{i+1}^\alpha\right\} \\
&\le P \sum_{i=0}^{I_\alpha - 1} \left(A_{\sigma+\eta_\alpha} - A_\sigma\right)\left\{t_i^\alpha < \sigma \le t_{i+1}^\alpha\right\} \qquad \text{(E.6)} \\
&\le P\left(A_{\sigma+\eta_\alpha} - A_\sigma\right)
\end{aligned}$$

Let $h_\alpha = \sum_{i=0}^{I_\alpha-1}\{\sigma \le t_i^\alpha < \upsilon\}\,]]t_i^\alpha, t_{i+1}^\alpha]]$, recall (from Lemma 6) that $\bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\left(F_i^\alpha \{t_i^\alpha \ge \sigma\}\right) = 0$ and that $g^\alpha_{t^\alpha_{i+1}}$ is $\mathcal{F}_{t_i^\alpha}$ measurable. Then,

$$\begin{aligned}
\sum_{i=0}^{I_\alpha-1}\left|\left(\bar{m}^p_{t^\alpha_{i+1}\wedge\upsilon} - \bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\right)\left(F_i^\alpha g^\alpha_{t^\alpha_{i+1}}\{t_i^\alpha \ge \sigma\}\right)\right| &\le \sum_{i=0}^{I_\alpha-1} m^p_{t^\alpha_{i+1}\wedge\upsilon}\left(F_i^\alpha\left|g^\alpha_{t^\alpha_{i+1}}\right|\{\sigma \le t_i^\alpha < \upsilon\}\right)\\
&\le P\sum_{i=0}^{I_\alpha-1}\left(A^\upsilon_{t^\alpha_{i+1}} - A^\upsilon_{t^\alpha_i}\right)\left|g^\alpha_{t^\alpha_{i+1}}\right|\{\sigma \le t_i^\alpha < \upsilon\} \qquad \text{(E.7)}\\
&= P\int_0^\upsilon |g^\alpha|\, h_\alpha d\lambda
\end{aligned}$$

It is clear that h^α is $\mathcal{P}_\alpha$ measurable and vanishes on $[[0,\sigma]]$. Joining (E.5), (E.6) and (E.7) we obtain that if $g \in L^\infty(\mathcal{P},\lambda)_+$ vanishes on $[[0,\sigma]]^c$, then

$$\begin{aligned}
\left|\int [f^\upsilon(Y) - f^\sigma(Y)]\, g d\lambda\right| &\le \operatorname*{LIM}_\alpha \left|\int [f^\upsilon_\alpha(Y) - f^\sigma_\alpha(Y)]\, g d\lambda\right|\\
&\le 2\|Y\| \operatorname*{LIM}_\alpha \sum_{i=0}^{I_\alpha-1}\left(\bar{m}^p_{t^\alpha_{i+1}\wedge\upsilon} - \bar{m}^p_{t^\alpha_{i+1}\wedge\sigma}\right)\left(g^\alpha_{t^\alpha_{i+1}} F_i^\alpha\right)\\
&\le 2\|Y\| \operatorname*{LIM}_\alpha P\int_0^\upsilon g^\alpha h^\alpha d\lambda\\
&= 2\|Y\| \operatorname*{LIM}_\alpha P\int_0^\upsilon g h^\alpha d\lambda\\
&= 0
\end{aligned}$$

i.e. that $f^\upsilon(Y) = f^\sigma(Y)$ on $[[0,\sigma]]$ up to a λ null set. We can then define $f : \mathfrak{B}\left(2^{\tilde{\Omega}}\right) \to L^\infty(\mathcal{P},\lambda)$ by setting

$$\mathcal{P}_m(Y) = \sum_n f^n(Y)\,]n-1,n] \qquad \text{(E.8)}$$

$\mathcal{P}_m$ is positive and linear since $f^n(Y)$ is for each n; furthermore, up to a λ null set, $\mathcal{P}_m(Y) = f^\upsilon(Y)$ on $[[0,\upsilon]]$ for each $\upsilon \in \mathcal{T}$.

Let now U be *càdlàg* and bounded and define $\pi_\alpha(U) = U_0\{0\} + \sum_{i=0}^{I_\alpha-1} U_{t_i^\alpha}\,]]t_i^\alpha, t_{i+1}^\alpha]]$. Since $U \in \mathbb{D}$, then for $\alpha \in \mathbf{A}$ sufficiently large, $|U_- - \pi_\alpha(U)| \le \eta_\alpha$ on $\left[[0, t^\alpha_{I_\alpha}]\right]$. Therefore, since $f^T_\alpha(YU)\left[[0,t^\alpha_{I_\alpha}]\right] = f^T_\alpha\left(YU\left[[0,t^\alpha_{I_\alpha}]\right]\right)$ and $\pi_\alpha(U)$ is $\mathcal{P}_\alpha$ measurable, we obtain for $Y \in \mathfrak{B}\left(2^{\tilde{\Omega}}\right)$

$$\begin{aligned}
\int |\mathcal{P}_m(YU_-) - \mathcal{P}_m(Y)U_-|\, d\lambda &= \lim_n \int_0^n |f^n(YU_-) - f^n(Y)U_-|\, d\lambda\\
&= \lim_n \operatorname*{LIM}_\alpha \int_0^{n\wedge t^\alpha_{I_\alpha}} |f^n_\alpha(YU_-) - f^n_\alpha(Y)U_-|\, d\lambda\\
&= \lim_n \operatorname*{LIM}_\alpha \int_0^{n\wedge t^\alpha_{I_\alpha}} |f^n_\alpha(YU_-) - f^n_\alpha(Y)\pi_\alpha(U)|\, d\lambda\\
&= \lim_n \operatorname*{LIM}_\alpha \int_0^{n\wedge t^\alpha_{I_\alpha}} |f^n_\alpha(Y(U_- - \pi_\alpha(U)))|\, d\lambda\\
&\le \lim_\alpha 2\eta_\alpha \|Y\|\, P(A_\infty)\\
&= 0
\end{aligned}$$

so that, by linearity,

$$\int \mathcal{P}_m(Y)\, d\lambda = \int \mathcal{P}_m(\Delta Y)\, d\lambda + \int \mathcal{P}_m(Y_-)\, d\lambda = \int \mathcal{P}_m(\Delta Y)\, d\lambda + \int Y_- d\lambda$$

It is clear that $\|\mathcal{P}_m\| \leq 1$ but we have just considered a case in which $\mathcal{P}_m(Y) = Y$. $\square$

Proof of Proposition 2. For any $\tau \in \mathcal{T}$ and $K \in \mathbb{K}$

$$\begin{aligned} m^p_\tau(K_\tau) &= m^p_\tau(K_\tau - K^\alpha_\tau) + m^p_{\tau\wedge t^\alpha_{I_\alpha}}(K^\alpha_\tau) + \left(m^p_\tau - m^p_{\tau\wedge t^\alpha_{I_\alpha}}\right)(K^\alpha_\tau) \\ &= m^p_\tau(K_\tau - K^\alpha_\tau) + J_\alpha(K)_\tau + \left(m^p_\tau - m^p_{\tau\wedge t^\alpha_{I_\alpha}}\right)(K^\alpha_\tau) + I_\alpha(K)_\tau \end{aligned}$$

As for the first term,

$$\begin{aligned} |K_\tau - K^\alpha_\tau| &\leq \left|K_{\tau\wedge t^\alpha_{I_\alpha}} - K^\alpha_\tau\right| + \left|K_\tau - K_{\tau\wedge t^\alpha_{I_\alpha}}\right| \\ &\leq \sum_{i=0}^{I_\alpha-1} F^{\alpha c}_i \left|K_{\tau\wedge t^\alpha_{i+1}} - K_{\tau\wedge t^\alpha_i}\right| + \left|K_\tau - K_{\tau\wedge t^\alpha_{I_\alpha}}\right| \\ &\leq 2\|K\| \left\{F^{\alpha c}_{I_\alpha-1} \cup \left\{\tau > t^\alpha_{I_\alpha}\right\}\right\} \end{aligned}$$

and given that $K^\alpha_\tau, K_\tau \in \mathcal{K}$

$$\begin{aligned} |m^p_\tau(K^\alpha_\tau - K_\tau)| &\leq |m_\tau(K^\alpha_\tau - K_\tau)| + |m^e_\tau(K^\alpha_\tau - K_\tau)| \\ &\leq m^e_\tau(|K^\alpha_\tau - K_\tau|) \\ &\leq 2\|K\| P\left(X_\tau\left(F^{\alpha c}_{I_\alpha-1} + \left\{\tau > t^\alpha_{I_\alpha}\right\}\right)\right) \end{aligned} \tag{E.9}$$

i.e. (i) $\mathrm{LIM}_\alpha |m^p_\tau(K^\alpha_\tau - K_\tau)| = 0$. On the other hand, from (5.7) we conclude that (ii) $\lim_\alpha J_\alpha(K)_\tau = 0$. Eventually, by (2.7),

$$\left|\left(m^p_\tau - m^p_{\tau\wedge t^\alpha_{I_\alpha}}\right)(K^\alpha_\tau)\right| \leq \|K\| P\left(X_{\tau\wedge t^\alpha_{I_\alpha}} - X_\tau\right)$$

so that (iii) $\lim_\alpha \left(m^p_\tau - m^p_{\tau\wedge t^\alpha_{I_\alpha}}\right)(K^\alpha_\tau) = 0$ whenever τ is such that X^τ is uniformly integrable. We then conclude that, when $\tau \in \mathcal{T}$ and X^τ is uniformly integrable, $m^p_\tau(K_\tau) = \lim_\alpha I_\alpha(K)_\tau$. The claim then follows from Lemma 8. $\square$

Appendix F. Proofs from Section 6.

Proof of Theorem 5. By localization, we can assume temporarily that (5.8) holds for every $\tau \in \mathcal{T}$. Observe that the process $Y_t = X_t K_t + \int_0^t \mathcal{P}_m(K)\, dA$ is right continuous, admits a terminal variable and $Y_0 = 0$. Then, [38, lemma I.1.44], Y is a uniformly integrable martingale, i.e. $X_t K_t = Y_t - \int_0^t \mathcal{P}_m(K)\, dA$ a special semimartingale, given that $\int \mathcal{P}_m(K)\, dA$ is predictable. If W is a bounded process, then $W^T[[T,\infty[[$ consists of a bounded jump at time T and is therefore *càdlàg* and of integrable variation, i.e. a semimartingale. It follows that $K^T[[T,\infty[[$ and $[[T,\infty[[$ are semimartingales as well as the process

$$XK + K^T[[T,\infty[[= (X + [[T,\infty[[)\, K^T = UK^T$$

The process U is a strictly positive semimartingale, as $P(X_t = 0; t < T) = 0$. Let

$$R_n = \inf\left\{t \in \mathbb{R}_+ : \sup_{s\leq t} X_s > 2^n \text{ or } X_t \leq 2^{-n}\right\}$$

$D = [[T,\infty[[$ and let superscript n denote a process stopped before time R_n, i.e. $U^n = U^{r_n-}$. U^n takes its values in the compact set $[2^{-n}, 2^n]$ on which the inverse function h is well defined and, being convex, admits a Lipschitz constant c_n. Let $F \in \mathcal{F}_s$ and $s < t$. Then $|h(X_t^n + D_t^n) - h(X_t^n + D_s^n)| \leq c_n (D_t^n - D_s^n)$ so that

$$\begin{aligned} P(h(X_t^n + D_t^n) + c_n (D_t^n - D_s^n) | \mathcal{F}_s) &\geq P(h(X_t^n + D_s^n) | \mathcal{F}_s) \\ &\geq h(P(X_t^n + D_s^n | \mathcal{F}_s)) \\ &\geq h(X_s^n + D_s^n) \end{aligned}$$

In other words, $h(U^n) + c_n D^n$ is a submartingale therefore $h(U^n) = h(U)^{r_n -}$ is a semimartingale. As the sequence $\langle R_n \rangle_{n \in \mathbb{N}}$ increases to ∞, P a.s. it follows [48, theorem 6, p. 46] that $h(U) = U^{-1}$ is a semimartingale. But then K^T, being the product of two semimartingales, is itself a semimartingale by Ito's lemma.

If $F \in \mathcal{I}_t$ is such that $m_t^p(F) = 0$ then

$$m(F; T \leq t) = m_t^e(F; T \leq t) \leq P(X_t \{T \leq t\}) = 0$$

so that if market are complete $\{F; T \leq t\} \in \mathcal{N}$, by Theorem 2. But under the current assumptions, by Theorem 4, we conclude that P vanishes on $\mathcal{N}$. Then, $P(F; T \leq t) = 0$ for each $F \in \mathcal{I}_t$ i.e. $P(T \leq t) = 0$ by Lemma 1(ii) and the claim follows. □

Proof of Theorem 6. First remark that $M = A$ on $\{X_- = 0\}$ up to indistinguishability and therefore, by Doob Meyer theorem, M and A remain constant over that stochastic interval so that the stochastic integrals $\int X_-^{-1} dM$ and $\int X_-^{-1} dA$ are well defined. The first statement is a fairly obvious consequence of integration by parts and (5.8) from which we obtain that the process Y, where $Y_t = X_t K_t + \int_0^t \mathcal{P}_m(K) dA$, is a local martingale (see the proof of Theorem 5). Integration by parts implies

$$Y = \int X_- dK^T + \int K_-^T dM + \int \mathcal{P}_d(K) dA + [K^T, M]$$

i.e., given that $\int \{X_- = 0\} dY = 0$,

$$\begin{aligned} \int X_-^{-1} dY - \int \frac{K_-^T}{X_-} dM &= K + \int \mathcal{P}_d(\Delta K) d\mathcal{L}(B) + [K^T, \mathcal{L}(Z)] \\ &= \hat{K} + [K^T, \mathcal{L}(Z)] \\ &= \hat{K} + \left[\hat{K}^T, \mathcal{L}(Z)\right] - \left[\int \mathcal{P}_d(K) dA, \mathcal{L}(Z)\right] \end{aligned}$$

One deduces [38, proposition I.4.49.c, p. 52] that $\hat{K} + \left[\hat{K}, \mathcal{L}(Z)\right]$ is a local martingale. The first claim then follows from

$$Z\hat{K} = \int \hat{K}_- dZ + \int Z_- d\hat{K} + \left[Z, \hat{K}\right] = \int \hat{K}_- dZ + \int Z_- d\left(\hat{K} + \left[\hat{K}, \mathcal{L}(Z)\right]\right)$$

while (6.5) is a consequence of [38, lemma I.3.22, p. 33] and of the decomposition

$$\hat{K} + \left[\hat{K}, \mathcal{L}(Z)\right] - M^K = V^K + \int_0^t \mathcal{P}_d(\Delta K) dA + \left[\hat{K}, \mathcal{L}(Z)\right]$$

according to which the right hand side is a local martingale of locally integrable variation. □

Proof of Corollary 2. Let $\eta > 0$ and $U^\eta(\omega, t) = \sum_{s \leq t} \{|\Delta Y_s(\omega)| > \eta\}$. Given that Y is bounded and *càdlàg,* U^η is an increasing, right continuous, process of finite variation and it is predictable by Fubini's theorem. We may thus define [38, proposition I.2.13, p. 18] the predictable stopping times $\sigma_0 = 0$ and

$$\sigma_k = \inf\left\{t > \sigma_{k-1} : U_t - U_{\sigma_{k-1}} \geq 1\right\}$$

and conclude that the sequence $\langle \sigma_k \rangle_{k\in\mathbb{N}}$ increases P a.s. to ∞. Clearly, $\{|\Delta Y| > \eta\} = \bigcup_k [[\sigma_k]]$. Letting $\Delta^\eta Y = \Delta Y \{|\Delta Y| > \eta\}$ and $\langle \sigma_k^n \rangle_{n\in\mathbb{N}}$ be the announcing sequence for σ_k we obtain

$$\begin{aligned}\int \mathcal{P}_d (\Delta^\eta Y) \, d\lambda &= \lim_j \int_0^{\sigma_j} \mathcal{P}_d (\Delta^\eta Y) \, d\lambda \\ &= \lim_j \lim_n \int_0^{\sigma_j} \mathcal{P}_d \left(\sum_{k=1}^{j} \Delta^\eta Y_{\sigma_k} \,]]\sigma_{k-1} \vee \sigma_k^n, \sigma_k]] \right) d\lambda \\ &= \lim_j \lim_n \sum_{k=1}^{j} \int_{\sigma_k^n \vee}^{\sigma_k} \mathcal{P}_d (\Delta^\eta Y) \, d\lambda \\ &= P \sum_k \mathcal{P}_d (\Delta^\eta Y)_{\sigma_k} \, \Delta A_{\sigma_k}\end{aligned}$$

Moreover, $\left|\int \mathcal{P}_d (\Delta Y - \Delta^\eta Y) \, d\lambda\right| + \left|P \sum_k \mathcal{P}_d (\Delta Y - \Delta^\eta Y) \, \Delta A\right| \le 4\eta P(A_\infty)$ and the first claim follows by letting η tend to 0.

If $\theta.K \in \mathbb{K}$ then by (6.5), uniqueness of the canonical decomposition of the special semimartingale K^T and ordinary properties of the stochastic integral

$$\begin{aligned}\int \theta \mathcal{P}_d (\Delta K) \, d\mathcal{L}(B) &= -\theta.V^K - \theta.\mathcal{P}_P \left(\left[\mathcal{L}(Z), M^K \right] \right) \\ &= -V^{\theta.K} - \mathcal{P}_P \left(\left[\mathcal{L}(Z), M^{\theta.K} \right] \right) \\ &= \int \mathcal{P}_d (\theta.\Delta K) \, d\mathcal{L}(B)\end{aligned}$$

□

Appendix G. Proofs from Section 7.

Proof of Theorem 7. The same argument used in the proof of Corollary 2 may be employed to obtain that, under the current assumptions, $\mathcal{P}_m(\Delta K) = \sum_n \mathcal{P}_m(\Delta K_{\tau_n}) [[\tau_n]]$. Therefore, by definition of the operator $\mathcal{P}_m$,

$$\begin{aligned}\int \mathcal{P}_m (\Delta K_{\tau_n}) \, d\lambda &= \operatorname*{LIM}_\alpha \int f_\alpha^{\tau_n} (\Delta K_{\tau_n}) \, d\lambda \\ &= \operatorname*{LIM}_\alpha \sum_{i=0}^{I_\alpha - 1} m^p_{t^\alpha_{i+1} \wedge \tau_n} \left(\Delta K_{\tau^n} \left\{ t^\alpha_{i+1} = \tau_n \right\} F_i^\alpha \right) \\ &= \operatorname*{LIM}_\alpha m^p_{\tau_n} \left(\Delta K_{\tau^n} \bigcup_{i=0}^{I_\alpha - 1} \left\{ t^\alpha_{i+1} = \tau_n \right\} F_i^\alpha \right)\end{aligned}$$

Under the current assumptions, however,

$$\operatorname*{LIM}_\alpha \sum_{i=0}^{I_\alpha - 1} m^p_{\tau_n} \left(\left\{ t_i^\alpha < \tau_n^r < \tau_n \le t^\alpha_{i+1} \right\} \right) \le \lim_\alpha m(\tau_n - \tau_n^r < \eta_\alpha) = 0 \tag{G.1}$$

and, for r sufficiently large, $\left\| \bar{m}^p_{\tau_n^r} - \bar{m}^p_{\tau_n -} \right\| = \left(m^p_{\tau_n^r} - m^p_{\tau_n -} \right)(\Omega) < \epsilon$. For $F \in \bigcup \mathcal{F}_{\tau_n^r}$ we deduce from (2.7)

$$\begin{aligned}m^e_{\tau_n -}(F) &= \lim_r m^e_{\tau_n^r}(F) + \lim_r \left(m^p_{\tau_n^r} - m^p_{\tau_n -} \right)(F) \\ &= \lim_r P\left(A_{\tau_n^r} F \right) \\ &= P(A_{\tau_n -} F)\end{aligned}$$

and since $\mathcal{F}_{\tau_n-} = \sigma\left(\bigcup \mathcal{F}_{\tau_n^r}\right)$ we conclude that $dm^e_{\tau_n-} \big/ dP_{\mathcal{F}_{\tau_n-}} = A_{\tau_n-}$ by uniqueness of the Carathéodory extension. Moreover,

$$\begin{aligned}
\sum_{i=0}^{I_\alpha-1} m^p_{\tau_n-}\left(\left\{t^\alpha_{i+1} = \tau_n; t^\alpha_i > \tau^r_n\right\} F^\alpha_i\right) &\leq \sum_{i=0}^{I_\alpha-1}\left(m^p_{\tau_n-} - \bar{m}^p_{\tau^r_n}\right)\left(t^\alpha_{i+1} = \tau_n\right) \\
&= \left(m^p_{\tau_n-} - m^p_{\tau^r_n}\right)\left(\bigcup_{i=1}^{I_\alpha-1}\left\{t^\alpha_{i+1} = \tau_n\right\}\right) \qquad \text{(G.2)} \\
&\leq \epsilon
\end{aligned}$$

We can now remark that $\Delta K_{\tau^n}\left\{t^\alpha_{i+1} = \tau_n\right\} F^\alpha_i$ is $\mathcal{F}_{\tau_n-}$ measurable by assumption, that $\left(m^p_{\tau_n} - m^p_{\tau_n-}\right)\big| \mathcal{F}_{\tau_n-} = \left(m^e_{\tau_n-} - m^e_{\tau_n}\right)\big| \mathcal{F}_{\tau_n-}$ and that $\left\{\Delta^k K^{t_{I_\alpha}-1} \neq 0\right\} \subset \bigcup_{i=1}^{I_\alpha-1}[[t^\alpha_i]]$ whenever α is large enough. Eventually we conclude that

$$\begin{aligned}
\int \mathcal{P}_m\left(\Delta K_{\tau_n}\right) d\lambda &= \operatorname*{LIM}_{\alpha} \sum_{i=0}^{I_\alpha-1}\left(m^p_{\tau_n} - m^p_{\tau_n-}\right)\left(\Delta K_{\tau^n}\left\{t^\alpha_{i+1} = \tau_n\right\} F^\alpha_i\right) \\
&= \operatorname*{LIM}_{\alpha} P\left(\Delta K_{\tau^n} \bigcup_{i=1}^{I_\alpha-1}\left\{t^\alpha_{i+1} = \tau_n\right\}\Delta A_{\tau_n}\right) \\
&= \lim_k \operatorname*{LIM}_{\alpha} P\left(\Delta^k K^{t_{I_\alpha-1}}_{\tau^n}\Delta A_{\tau_n}\right) \\
&= P\left(\Delta K_{\tau^n}\Delta A_{\tau_n}\right)
\end{aligned}$$

i.e. that $\int \mathcal{P}_m(\Delta K)\, d\lambda = P\sum \Delta K_{\tau^n}\Delta A_{\tau_n}$. The last claim is obvious given Theorem 6.2 and that fact that in the current context $\mathcal{P}_d(\Delta K) = 0$. □

Appendix H. Proofs from Section 8.

Proof of Lemma 3. Necessity is obvious given the remark preceding the lemma; the inequality, $m(F) \geq m^e_t(F) = P_m(X_tF) = P_m(X_tF\{T_m > t\})$ for $F \in \mathcal{F}_t$ implies that this is sufficient as well. □

Lemma 9. *Let $f \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$, $v \in ba(\mathcal{F}, \mathcal{N})_+$ with $v(\Omega) = 1$ and $k \in \mathcal{K}$, then*

$$\overline{\alpha}_k(f) \geq v(k+f) \geq \underline{\alpha}_k(f) \qquad \text{(H.1)}$$

If there are no arbitrage opportunities

1. $(\underline{\alpha}_{\mathcal{K}}(f); \overline{\alpha}_{\mathcal{K}}(f)) \subset \mathcal{A}(f, \mathcal{K}) \subset [\underline{\alpha}_{\mathcal{K}}(f); \overline{\alpha}_{\mathcal{K}}(f)]$*;*
2. $\overline{\alpha}_{\mathcal{K}}(f) = \underline{\alpha}_{\mathcal{K}}(f)$ *if and only if* $f - \frac{1}{2}\left(\overline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(f)\right) \in \bar{\mathcal{C}}$ *.*

Proof. Fix $k \in \mathcal{K}$ and let $N_+, N_- \in \mathcal{N}$ be such that $\sup_{\omega \in N^c_+}(k+f)(\omega) \leq \overline{\alpha}_k(f) + 2^{-n}$ and $\inf_{\omega \in N^c_-}(k+f)(\omega) + 2^{-n} \geq \underline{\alpha}_k(f)$. Then on $N^c = N^c_+ N^c_-$

$$2^{-n} + \overline{\alpha}_k(f) \geq k + f \geq \underline{\alpha}_k(f) - 2^{-n} \qquad \text{(H.2)}$$

The first claim follows from $N \in \mathcal{N}$.

(1). Suppose that $\pi(f) \notin \mathcal{A}(f, \mathcal{K})$: then for some $k \in \mathcal{K}$ and $d \in \mathbb{R}$ we would have $k + d(f - \pi(f)) \in \mathfrak{B}(\mathcal{F}, \mathcal{N})_+$ or, equivalently, $N_{k,\eta} = \{k + d(f - \pi(f)) < -\eta\} \in \mathcal{N}$ for all $\eta > 0$. Take the case $d > 0$, then

$$\underline{\alpha}_{d^{-1}k}(f) \geq \inf_{\omega \in N^c_{k,\eta}}\left(d^{-1}k + f\right)(\omega) = d^{-1}\inf_{\omega \in N^c_{k,\eta}}(k + df)(\omega) \geq \pi(f) - \eta$$

If $d < 0$ we likewise deduce $\bar{\alpha}_{-d^{-1}k}(f) \leq \pi(f) + \eta$. Of course, η being arbitrary, we conclude there exists $k \in \mathcal{K}$ such that either $\underline{\alpha}_k(f) \geq \pi(f)$ or $\bar{\alpha}_k(f) \leq \pi(f)$. We deduce that $(\underline{\alpha}_{\mathcal{K}}(f); \overline{\alpha}_{\mathcal{K}}(f)) \subset \mathcal{A}(f, \mathcal{K})$. If

$\pi(f) \in \mathcal{A}(f,\mathcal{K})$, $\mathcal{K}(f;\pi)$ and $\mathfrak{B}(\mathcal{F},\mathcal{N})_+$ are separated and there exists therefore $m_f \in \mathcal{M}(\mathcal{C})$ such that $m_f(f) = \pi(f)$. By (H.2), we conclude that $\pi(f) \in [\underline{\alpha}_{\mathcal{K}}(f); \overline{\alpha}_{\mathcal{K}}(f)]$.

(2). Let $\hat{f} = f - \frac{1}{2}(\overline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(f))$ and suppose that $\hat{f} \notin \bar{\mathcal{C}}$. Then $\left\{\hat{f}\right\}$ and $\bar{\mathcal{C}}$ may be separated by a finitely additive probability m_f vanishing on $\mathcal{N}$ and on $\mathcal{K}$ and such that $m_f\left(\hat{f}\right) > 0$. Since m_f is a separating measure for $\mathcal{K}$, by (H.2) it follows that $\overline{\alpha}_{\mathcal{K}}\left(\hat{f}\right) \geq m_f\left(\hat{f}\right) \geq \underline{\alpha}_{\mathcal{K}}\left(\hat{f}\right)$. Given that both functionals, $\overline{\alpha}_{\mathcal{K}}(\cdot)$ and $\underline{\alpha}_{\mathcal{K}}(\cdot)$, are linear with respect to constants the preceding double inequality translates into

$$\overline{\alpha}_{\mathcal{K}}(f) \geq m_f\left(\hat{f}\right) + \frac{1}{2}(\overline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(f)) > \frac{1}{2}(\overline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(f)) \geq \underline{\alpha}_{\mathcal{K}}(f)$$

i.e. $\overline{\alpha}_{\mathcal{K}}(f) > \underline{\alpha}_{\mathcal{K}}(f)$. On the other hand, if $\hat{f} \in \bar{\mathcal{C}}$ then $m\left(\hat{f}\right) \leq 0$ for each $m \in \mathcal{M}(\mathcal{C})$ so that $\mathcal{A}\left(\hat{f},\mathcal{K}\right) \subset \mathbb{R}_-$ and therefore, by the first claim, $0 \geq \overline{\alpha}_{\mathcal{K}}\left(\hat{f}\right)$ i.e. $\overline{\alpha}_{\mathcal{K}}(f) \leq \underline{\alpha}_{\mathcal{K}}(f)$. □

We shall now prove a theorem more general that Theorem 8. Let us introduce the following definition.

Definition 7. *Let $\mathcal{U} \subset ba(\mathcal{F},\mathcal{N})$ and $\mathcal{J} \subset \mathfrak{B}(\mathcal{F},\mathcal{N})$. $\mathcal{U}$ is norm attaining for $\mathcal{J}$ if for each $f \in \mathcal{J}$, $\|f\| = \sup_{\upsilon \in \mathcal{U}} \upsilon(f)$.*

Theorem 10. *The following properties are mutually equivalent:*

- (a) *there exists a subset $\mathcal{U}$ of finitely additive probabilities vanishing on $\mathcal{N}$ which is (i) norm attaining for $\mathfrak{B}(\mathcal{F},\mathcal{N})_+$ and such that (ii) if $\upsilon \in \mathcal{U}$ and $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}$ such that $\|h_n^-\| \to 0$ then h_n converges to 0 in υ measure;*
- (b) *for every $k \in \mathcal{K}$ and $f \in \mathfrak{B}(\mathcal{F},\mathcal{N})$, $\bar{\alpha}_k(f) = \underline{\alpha}_{\mathcal{K}}(f)$ if and only if $\underline{\alpha}_k(f) = \bar{\alpha}_{\mathcal{K}}(f)$;*
- (c) *$\mathcal{K}$ has the extension property;*
- (d) *there are no free lunches, i.e. (3.2) holds.*

Proof. $(a)\to(b)$. Let $f \in \mathfrak{B}(\mathcal{F},\mathcal{N})$ be such that $\bar{\alpha}_{\mathcal{K}}(f) \geq 0$ (if not, replace f by $f - \bar{\alpha}_{\mathcal{K}}(f)$) and suppose that $\underline{\alpha}_{k_0}(f) = \bar{\alpha}_{\mathcal{K}}(f)$ for $k_0 \in \mathcal{K}$ (so that $\underline{\alpha}_{\mathcal{K}}(f) = \bar{\alpha}_{\mathcal{K}}(f)$). For each n there is $k_n \in \mathcal{K}$ such that $\underline{\alpha}_{k_0}(f) > \bar{\alpha}_{k_n}(f) - 2^{-n}$. Letting $h_n = (1 + \|k_0 + f\|)^{-1}(k_0 - k_n)$ we conclude that $h_n > -2^{-n}$ outside some $N_n' \in \mathcal{N}$ and $h_n \in \mathcal{K}$. Choose $\langle k_n\rangle_{n\in\mathbb{N}}$ such that $\langle\bar{\alpha}_{k_n}(f)\rangle_{n\in\mathbb{N}}$ is monotonically decreasing to $\bar{\alpha}_{\mathcal{K}}(f)$, let $N_n'' \in \mathcal{N}$ be such that $\bar{\alpha}_{k_n}(f) \geq \sup_{\omega \in N_n''^c}(k_n + f)(\omega) - 2^{-n}$ and set $N_n = N_n' \cup N_n''$. Then, as $N_n \in \mathcal{N}$

$$\begin{aligned}
(k_0 + f) N_n^c &= (k_0 - k_n) N_n^c + (k_n + f) N_n^c \\
&= [(k_0 - k_n) N_n^c + (k_n + f) N_n^c] \wedge \|k_0 + f\| \\
&\leq \left[(k_0 - k_n) N_n^c + \bar{\alpha}_{k_n}(f) + 2^{-(n-1)}\right] \wedge \|k_0 + f\| \\
&\leq [(k_0 - k_n) N_n^c \wedge \|k_0 + f\|] + \bar{\alpha}_{k_n}(f) + 2^{-(n-1)} \\
&\leq (1 + \|k_0 + f\|)(h_n \wedge 1) N_n^c + \bar{\alpha}_{k_n}(f) + 2^{-(n-1)}
\end{aligned}$$

If $\upsilon \in \mathcal{U}$ then $\upsilon(h_n \wedge 1)$ converges to 0 and therefore

$$\begin{aligned}
\upsilon(k_0 + f) &\leq \lim_n \left[(1 + \|k_0 + f\|)\upsilon(h_n \wedge 1) + \bar{\alpha}_{k_n}(f) + 2^{-(n-1)}\right] \\
&= \lim_n \bar{\alpha}_{k_n}(f) \\
&= \bar{\alpha}_{\mathcal{K}}(f)
\end{aligned}$$

By definition of $\mathcal{U}$ then $\bar{\alpha}_{k_0}(f) \leq \sup_{v\in\mathcal{U}} v(k_0+f) \leq \bar{\alpha}_{\mathcal{K}}(f)$, i.e.$\bar{\alpha}_{k_0}(f) = \bar{\alpha}_{\mathcal{K}}(f)$. If, on the other side, $\bar{\alpha}_{k_0}(f) = \underline{\alpha}_{\mathcal{K}}(f)$, then the same argument can be used to show that $\underline{\alpha}_{k_0}(f) = \underline{\alpha}_{\mathcal{K}}(f)$. In other words, (b) holds.

$(b)\to(c)$. Let $\pi(f) = \frac{1}{2}\left(\underline{\alpha}_{\mathcal{K}}(f) + \bar{\alpha}_{\mathcal{K}}(f)\right)$ and suppose that there exist $k \in \mathcal{K}$ and $d \in \mathbb{R}$ such that $y = k + d(f - \pi(f)) \in \mathfrak{B}(\mathcal{F},\mathcal{N})_+$ i.e. such that $\{y < -\eta\} \in \mathcal{N}$ for each $\eta > 0$. A shown in the proof of Lemma 9, this implies either $\underline{\alpha}_k(f) = \pi(f)$ (if $d > 0$) or $\bar{\alpha}_k(f) = \pi(f)$ (if $d < 0$): in either case $\pi(f) = \underline{\alpha}_{\mathcal{K}}(f) = \bar{\alpha}_{\mathcal{K}}(f)$. Say $d > 0$. Since $\bar{\alpha}_k(f) = \pi(f)$ by (b), we conclude that for any $\eta > 0$ there must exist $N \in \mathcal{N}$ such that $\sup_{\omega\in N^c} y(\omega) < \eta$ i.e. $\{y > \eta\} \in \mathcal{N}$ from which the implication $y = 0$ follows.

$(c)\to(d)$. Let $f \in \bar{\mathcal{C}} \cap \mathfrak{B}(\mathcal{F},\mathcal{N})_+$. By (c), Lemma 9 and (8.1), $\mathcal{A}(f,\mathcal{K}) = \{0\}$ i.e. $f \in \mathcal{K}(f;0) \cap \mathfrak{B}(\mathcal{F},\mathcal{N})_+ = \{0\}$.

$(d)\to(a)$. Let $\eta,\varepsilon > 0$ and $c \in \bar{\mathcal{C}}$ be such that $c > -\varepsilon$ up to negligibility. If $v \in \mathcal{M}(\mathcal{C})$ then $0 \geq v(c)$ and from this we easily deduce that $v(c \geq \eta) \leq \frac{\varepsilon}{\varepsilon+\eta}$. It follows that every sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\bar{\mathcal{C}}$ converges to 0 in v measure whenever c_n^- converges to 0 in norm. The same property easily extends to the collection $\mathcal{M}(\mathcal{C})^*$ of all finitely additive probabilities absolutely continuous with respect to some $v \in \mathcal{M}(\mathcal{C})$. If (d) holds, then for any set $F \in \mathcal{F}$ not negligible there exists a corresponding $v_F \in \mathcal{M}(\mathcal{C})$ such that $v_F(F) > 0$ while the finitely additive probability $\bar{v}_F = v_F(F)^{-1} F dv_F$ is clearly absolutely continuous with respect to v_F [26, theorem III.2.20, p. 114]. Letting $F = \{h > (1-\eta)\|h\|\}$ for $h \in \mathfrak{B}(\mathcal{F},\mathcal{N})_+$, then we have $\bar{v}_F(h) \geq (1-\eta)\|h\|$. But then the collection $\mathcal{M}(\mathcal{C})^*$ is norm attaining for $\mathfrak{B}(\mathcal{F},\mathcal{N})_+$ and (a) is therefore satisfied. □

The equivalence between (a) and (d) has a direct correspondence in a result of Delbaen and Schachermayer [21, corollary 3.7, p. 477] (see also [40, lemma 2.2, p. 193]) obtained under the assumption $\mathcal{N} = \mathcal{N}_Q$.

Proof of Proposition 3. For each $m \in \mathcal{M}(\mathcal{C})$, denote by T_m the stopping time defined in (6.1) with respect to m and P. Define the quantity

$$\eta = \inf_{m\in\mathcal{M}(\mathcal{C})} P(T_m < \infty) \tag{H.3}$$

If $\langle m_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}(\mathcal{C})$, then $m = \sum_n 2^{-n} m_n \in \mathcal{M}(\mathcal{C})$ and, for each $\tau \in \mathcal{T}$, $m_\tau^e = \sum_n 2^{-n} m_{n,\tau}^e$ (by uniqueness of the decomposition (2.6)). Since $m_\tau^e \gg m_{n,\tau}^e$ it follows that $0 = \sum_n P(T_m < T_{m_n}) \geq P\left(\bigcup_n \{T_m < T_{m_n}\}\right)$ and therefore $P(T_m < \infty) \leq P\left(\bigcap_n \{T_{m_n} < \infty\}\right)$ so that, choosing $m_n \in \mathcal{M}(\mathcal{C})$ such that $P(T_{m_n} < \infty) < \eta + 2^{-n}$, we conclude $P(T_m < \infty) = \eta$.

In the attempt to derive a contradiction, assume that $\eta > 0$ and fix $\epsilon > 0$ and $m \in \mathcal{M}(\mathcal{C})$. Consider the set

$$F_\epsilon(m) = \left\{f \in \mathfrak{B}(\mathcal{F},\mathcal{N})_+ : f \leq 1,\ P(f) \geq \eta(1-\epsilon),\ m(f) < \epsilon\right\}$$

and let F_ϵ be the set-valued map $m \to F_\epsilon(m)$ defined on $\mathcal{M}(\mathcal{C})$. F_ϵ is convex valued and non empty (in fact $m(T_m < \infty) = m^p(T_m < \infty)$ and m^p and P are orthogonal). Denote by $\mathfrak{X}$ the space $\mathcal{M}(\mathcal{C})$ endowed with the weak* topology of $ba(\mathcal{F},\mathcal{N})$ and put $\mathfrak{Y} = \mathfrak{B}(\mathcal{F},\mathcal{N})$: $\mathfrak{X}$ is Hausdorff and compact and $\mathfrak{Y}$ a Banach space. Consider an open set $\mathcal{U}_{f_0} \subset \mathfrak{Y}$ containing $f_0 \in F_\epsilon(m_0)$. It is clear that $\mathcal{V}_f = \{m \in \mathcal{M}(\mathcal{C}) : m(f) < \epsilon\}$ is open and that

$$F_\epsilon^{-1}(\mathcal{U}_{f_0}) = \{m \in \mathcal{M}(\mathcal{C}) : F_\epsilon(m) \cap \mathcal{U}_{f_0} \neq \varnothing\} = \bigcup_{\left\{f\in\mathcal{U}_{f_0}:\ P(f)\geq\eta(1-\epsilon)\right\}} \mathcal{V}_f$$

In other words, the lower inverse F_ϵ^{-1} of F_ϵ maps open sets into open sets, i.e. F_ϵ is lower hemicontinuous. By virtue of Michael selection theorem [46, footnote 7, p. 364.], there is a continuous function $\phi_\epsilon : \mathfrak{X} \to \mathfrak{Y}$

such that $\phi_\epsilon(m) \in \overline{F_\epsilon(m)}$ for each $m \in \mathcal{M}(\mathcal{C})$ so that (i) $0 \le \phi_\epsilon(m) \le 1$, (ii) $P(\phi_\epsilon(m)) \ge \eta(1-\epsilon)$ and (iii) $m(\phi_\epsilon(m)) \le \epsilon$.

Consider now the set valued map $M : \mathfrak{Y} \to \mathfrak{X}$ defined as

$$M(f) = \{m \in \mathcal{M}(\mathcal{C}) : m(f) = \bar{\alpha}_{\mathcal{K}}(f)\}$$

$M(f)$ is clearly a non empty, compact and convex subset of $\mathfrak{X}$. Let $\mathcal{V}$ be a closed subset of $\mathfrak{X}$ and $f_0 \in \overline{M^{-1}(\mathcal{V})}$: for each δ there exists then $f_\delta \in M^{-1}(\mathcal{V})$ such that $\|f_\delta - f_0\| < \delta$. By definition this implies that for some $m_\delta \in \mathcal{V}$, $m_\delta(f_\delta) = \bar{\alpha}_{\mathcal{K}}(f_\delta)$ so that

$$m_\delta(f_0) \ge m_\delta(f_\delta) - \delta = \bar{\alpha}_{\mathcal{K}}(f_\delta) - \delta \ge \bar{\alpha}_{\mathcal{K}}(f_0) - 2\delta$$

Put it differently, for each $\delta > 0$ the set $\mathcal{V}_{f_0,\delta} = \{m \in \mathcal{V} : m_\delta(f_0) \ge \bar{\alpha}_{\mathcal{K}}(f_0) - 2\delta\}$ is non empty. It then ensues from the finite intersection property that $\bigcap_{\delta>0} \mathcal{V}_{f_0,\delta} = \{m \in \mathcal{V} : m_\delta(f_0) = \bar{\alpha}_{\mathcal{K}}(f_0)\}$ is also non empty or, in other words, that $f_0 \in M^{-1}(\mathcal{V})$ and therefore $M^{-1}(\mathcal{V})$ is closed. We conclude that M is upper hemicontinuous and that so is the composite map $\Phi_\epsilon = M \circ \phi_\epsilon : \mathfrak{X} \to \mathfrak{X}$; further, Φ_ϵ is convex and compact valued. Φ_ϵ has therefore closed graph and, $\mathfrak{X}$ being a Hausdorff, locally convex topological vector space, it admits a fixed point m^* as a result of a well known theorem of Glicksberg [31, p. 171]. Letting $f^* = \phi_\epsilon(m^*)$ we have that

$$\epsilon \ge m^*(f^*) = \bar{\alpha}_{\mathcal{K}}(f^*)$$

while $P(f^*) \ge \eta(1-\epsilon)$. This can be considered as an orthogonality condition between P and $\bar{\alpha}_{\mathcal{K}}$.

Given that ϵ was entirely arbitrary, we can establish the same conclusion replacing ϵ with $\epsilon_n = 2^{-n-1}$: let m_n be the fixed point of Φ_{ϵ_n} and $f_n = \phi_{\epsilon_n}(m_n)$ so that $P(f_n) \ge \eta(1-\epsilon_n)$ and $\epsilon_n \ge \bar{\alpha}_{\mathcal{K}}(f_n)$. Both inequalities remain valid if we replace f_n by $g'_n = \sum_{i \ge n} a_{i,n} f_i$ where the positive sequence $\langle a_{i,n} \rangle_{i\in\mathbb{N}}$ contains finitely many non null elements and $\sum_{i\ge n} a_{i,n} = 1$. In fact

$$\bar{\alpha}_{\mathcal{K}}(g'_n) \le \sum_{i\ge n} a_{i,n} \bar{\alpha}_{\mathcal{K}}(f_i) \le \sum_{i\ge n} a_{i,n} 2^{-i-1} \le 2^{-n-1}$$

Choosing weights conveniently we obtain, by Komlòs lemma [21, lemma A1.1, p. 515], that the sequence $\langle g'_n \rangle_{n\in\mathbb{N}}$ converges P a.s. while by Egoroff theorem there is a set $F \in \mathcal{F}$ such that $P(F^c) < \eta\epsilon$ and that, letting $g_n = g'_n F$, the sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ converges uniformly to some $g \ge 0$. As $0 \le g'_n \le 1$

$$P(g) = \lim_n P(g_n) \ge \lim_n P(g'_n) - P(F^c) \ge \lim_n \eta(1 - \epsilon_n - \epsilon) = \eta(1-\epsilon)$$

so that $g \ne 0$. However

$$\bar{\alpha}_{\mathcal{K}}(g) = \lim_n \bar{\alpha}_{\mathcal{K}}(g_n) \le \lim_n \bar{\alpha}_{\mathcal{K}}(g'_n) = 0$$

By the second claim in Lemma 3, the last inequality implies $g \in \bar{\mathcal{C}}$ and the property (3.2) is therefore violated, a contradiction. The last claim easily follows from Theorem 5(i). □

Appendix I. Proofs from Section 9.

Define the functional $\pi_{\mathcal{K}} : \mathfrak{B}(\mathcal{F}, \mathcal{N}) \to \mathbb{R}$ implicitly as $\pi_{\mathcal{K}}(f) = -\underline{\alpha}_{\mathcal{K}}(-f)$. The following is a fairly trivial lemma.

Lemma 10. *Let Assumption 6 hold.*

1. *The functional $\pi_{\mathcal{K}}$ is positive, sub additive, positively homogeneous and such that $\pi_{\mathcal{K}}(1) \le 1$;*
2. *if (3.1) holds, then $\pi_{\mathcal{K}}(1) = 1$ and $\pi_{\mathcal{K}}(k) \le 0$ when $k \in \mathcal{K}$.*

Proof. Let $f, g \in \mathfrak{B}(\mathcal{F}, \mathcal{N})$. By definition (3.4), $\underline{\alpha}_{\mathcal{K}}(f) \geq \underline{\alpha}_0(f) \geq \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} f(\omega)$ so that $\underline{\alpha}_{\mathcal{K}}$ is positive and $\pi_{\mathcal{K}}(1) \leq 1$. As $k \in \mathcal{K}$ if and only if $k = k_1 + k_2$ with $k_1, k_2 \in \mathcal{K}$

$$\begin{aligned}
\underline{\alpha}_{\mathcal{K}}(f+g) &= \sup_{k \in \mathcal{K}} \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} (k + f + g)(\omega) \\
&= \sup_{k_1, k_2 \in \mathcal{K}} \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} (k_1 + k_2 + f + g)(\omega) \\
&\geq \sup_{k_1 \in \mathcal{K}} \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} (k_1 + f)(\omega) + \sup_{k_2 \in \mathcal{K}} \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} (k_1 + f)(\omega) \\
&= \underline{\alpha}_{\mathcal{K}}(f) + \underline{\alpha}_{\mathcal{K}}(g)
\end{aligned}$$

Since $\lambda^{-1} k \in \mathcal{K}$ whenever $\lambda > 0$, then

$$\begin{aligned}
\underline{\alpha}_{\mathcal{K}}(\lambda f) &= \lambda \sup_{k \in \mathcal{K}} \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} \left(\lambda^{-1} k + f\right)(\omega) \\
&= \lambda \sup_{k' \in \mathcal{K}} \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} (k' + f)(\omega) \\
&= \lambda \underline{\alpha}_{\mathcal{K}}(f)
\end{aligned}$$

from which $\pi_{\mathcal{K}}(0) = 0$ also follows.

If $\underline{\alpha}_k(-1) > -1$ then there exists $N \in \mathcal{N}$ such that $k > 0$ on N^c, a contradiction of (3.1). If $k_0 \in \mathcal{K}$ then $\pi_{\mathcal{K}}(k_0) = \inf_{k \in \mathcal{K}} \inf_{N \in \mathcal{N}} \sup_{\omega \in N^c} (-k + k_0)(\omega) \leq \underline{\alpha}_{k_0}(-k_0) = 0$. $\pi_{\mathcal{K}}[\mathcal{K}] \leq 0$ follows from positivity of $\pi_{\mathcal{K}}$. □

Proof of Theorem 9. Consider the functional $\pi_{\mathcal{K}}$ on $\mathfrak{B}(\mathcal{F}, \mathcal{N})$ and, appealing to Hahn Banach theorem, construct a functional ϕ on $\mathfrak{B}(\mathcal{F}, \mathcal{N})$ such that $\phi(\Omega) = \pi_{\mathcal{K}}(\Omega)$ and $\phi \leq \pi_{\mathcal{K}}$. By Lemmas 4 and 10 we may thus represent ϕ via a finitely additive probability m vanishing on $\mathcal{N}$ and such that $m[\mathcal{C}] \leq 0$. If $k \in \mathcal{K}$, then

$$m(k > 2^n) \leq \pi_{\mathcal{K}}(k > 2^n) \leq \pi_{\mathcal{K}}\left(\frac{k + \|k^-\|}{2^n}\right) \leq 2^{-n} \left\|k^-\right\|$$

Then k is m integrable and $m(k) = \lim_n m(k \wedge n) \leq \pi_{\mathcal{K}}(k) = 0$. □

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