Theory and Calibration of Swap Market Models

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Abstract

This paper introduces a general framework for market models, named Market Model Approach, through the concept of admissible sets of forward swap rates spanning a given tenor structure. We relate this concept to results in graph theory by showing that a set is admissible if and only if the associated graph is a tree. This connection enables us to enumerate all admissible models for a given tenor structure. Three main classes are identified within this framework, corresponding to the co-terminal, co-initial, and co-sliding model. We prove that the LIBOR market model is the only admissible in the co-sliding class. By focusing on the co-terminal model in a lognormal setting, we develop and compare several approximating analytical formulae for caplets, while swaptions can be priced by a simple Black-type formula. A novel calibration technique is introduced to allow simultaneous consistency to caps and swaptions. Calibration of the co-terminal model is shown to be faster, more robust and more efficient than the same procedure applied to the LIBOR market model. We then argue that the co-terminal approach is the simplest and most convenient market model for pricing and hedging a large variety of exotic interest-rate derivatives.

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1 Introduction

In recent years, market models of interest rate dynamics have attracted much attention among academics, and have become increasingly popular among practitioners. These models look more appealing than classical short-term rate based approaches from both a theoretical and a practical point of view since they are built by directly specifying arbitrage-free dynamics on a set of forward LIBOR or swap rates. Several issues on the implementation side do however exist because of the high dimensionality of the associated Markovian dynamics. Despite this, market models have recently gained an undisputed popularity thanks to the availability of new approximation and model calibration techniques.

Historically, two different (yet mathematically similar) approaches have been introduced. In the LIBOR market model (LMM) developed by Brace, Gatarek and Musiela [7]; Goldys [15]; Miltersen, Sandmann and Sondermann [26]; Musiela and Rutkowski [27]; arbitrage-free dynamics are assigned to a set of non-overlapping forward LIBOR rates while in the swap market model (SMM), first introduced by Jamshidian [20], similar dynamics are applied on a family of forward swap rates. Nonetheless, most of the available literature on the subject has focused on the LMM only. Many authors rely on the simplest assumption that consists in assigning dynamics to forward LIBOR rates driven by a d-dimensional Itô diffusion process with deterministic volatility. In this setting, Glasserman and Zhao [14] study arbitrage-free discrete-time approximations of the LMM, while Jäckel and Rebonato [19] and Hull and White [17] introduce methods to approximate true LIBOR dynamics by high-dimensional lognormal processes. Schoenmakers and Coffey [34] propose parameterized instantaneous correlation structures that simplify model implementation, while model calibration issues for LMM are extensively studied in Rebonato [32], Brigo and Mercurio [8], Schoenmakers and Coffey [35], and d'Aspremont [3]. Andersen and Andreasen [1] introduce a generalized LMM where the instantaneous volatility of the LIBOR rates is allowed to be nonlinearly state-dependent (more precisely, of a CEV type), while stochastic volatility extensions are discussed in Andersen and Brotherton-Ratcliffe [2], Joshi and Rebonato [23], and Wu and Zhang [37]. In this respect, it is also worth mentioning the work of Jamshidian [21] who studies possible extensions of the standard diffusion-based approach to general semimartingale processes, as well as Glasserman and Kou [12], Glasserman and Merener [13], for LMM with jumps. A vast empirical study dedicated to stochastic volatility and jump LMM is given in Jarrow *et al.* [22], while one-factor and two-factor versions of LMM and SMM are compared in De Jong *et al.* [9]. Finally, Hunter *et al.* [18] and Pietersz *et al.* [28] introduce efficient Monte-Carlo simulation methods of a set of correlated LIBOR rates.

Despite the extensive literature available on the LMM, very little has been published on its swap rate counterpart until now. Models based on dynamics assigned to forward swap rates are often considered less tractable than the LMM both in theory and in the applications, although the two approaches are remarkably close in their mathematical construction [32]. On the practical side, many authors seem to prefer the LMM by claiming that LIBOR rates are "more fundamental" financial quantities than swap rates. This view is sometimes justified by noticing that, to a good degree of accuracy, forward swap rates may be thought of as deterministic linear combinations of forward LIBOR rates [8].

In this paper we challenge these claims although our results are, in fact, more general. Our goal is threefold.

First, we consider a general specification of a model that concentrates on evolving market observable rates directly. We assign arbitrage free dynamics to a generic family of forward swap rates, and aim at finding the weakest condition under which this construction yields a unique specification in all equivalent pricing measures. In this respect, the concept of *admissibility* of a set is introduced, and its theoretical and practical implications are discussed. The Market Model Approach is introduced next. It is the most general modelling framework where arbitrage-free dynamics are assigned to an admissible set. The cornerstone of this approach is the direct modeling of forward swap rates (as opposed to discount bond prices or instantaneous forward rates) such that absence of arbitrage and market completeness are both satisfied. Correspondingly, we determine the necessary and sufficient conditions for this to hold. Interestingly, we show that the properties of these admissible sets can be best understood with the use of graph theory. This mapping allows to graphically characterize all admissible sets in a simple and intuitive way and, moreover, to determine the number of distinct market models that are admissible for a given tenor structure. Our study shows that the class of admissible market models is very large, and contains all "standard" market models [7], [20] as special cases. We identify three major sub-classes denominated co-initial, co-sliding and co-terminal according to the nature of the family of forward swap rates. As an important corollary, we prove that the LMM is the only admissible model within the co-sliding sub-class.

Second, we concentrate on the co-terminal SMM (ctSMM, in short), and discuss market situations where the use of that approach is most relevant. In particular, we demonstrate that it enjoys the same degree of mathematical tractability as the LMM. For instance, drift approximations for the pricing of vanilla options work equally well in comparison to a standard LMM framework. Obviously, this contrasts with the current widespread perception on the computational burden of the SMM. Also, pricing of complex derivatives can be efficiently achieved.

Third, we focus on the calibration of the ctSMM, and show how a joint calibration on market prices of caplets and swaptions can be achieved in a faster, more robust and more transparent way than in the LMM. This result suggests that the ctSMM is a fundamental pricing and risk-management tool for a large variety of complex interest-rate (IR) derivatives.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and define the Market Model Approach by assigning arbitrage-free dynamics to a set of admissible (in a sense to be later defined) forward swap rates. We discuss connection with graph theory in Section 3. We show that a set of forward swap rates is admissible if and only if its graph is a tree. By borrowing results of graph theory on labeled trees we enumerate all admissible market models for a given tenor structure exactly. In Section 4 three major subclasses of admissible models are unveiled, and a different use for each class is identified and discussed. From Section 5 on we concentrate on the ctSMM class, and discuss the utility of co-terminal models in the context of exotic IR derivatives. Section 6 is dedicated to the pricing of plain-vanilla IR derivatives such as swaptions and caplets. For caplet pricing, we explore several analytical approximation methods. In Section 7 we concentrate on the calibration of the ctSMM. There, we introduce a new parametric "bootstrap" algorithm that allows to easily calibrate the model to caplets and a set of co-terminal swaptions associated to the same tenor structure. Section 8 contains all numerical results, namely a numerical comparison of the different approximation and calibration schemes developed in Sections 6 and 7. Section 9 gathers some concluding remarks. We refer to the comprehensive study of Galluccio, Huang, Ly and Scaillet [11], hereafter GHLS, for further details and discussions.

2 Market Model Approach

We assume that we are given a pre-specified collection of reset/settlement dates $\mathcal{T} := \{T_1, \cdots, T_M\}$, referred to as the tenor structure, with $T_j < T_k$, $1 \leq j < k \leq M$. Starting time is assumed to be $T_0 = 0$ with $T_0 < T_1$. Let us denote the year fraction between any two consecutive dates by $\delta_j = T_j - T_{j-1}$, for $j = 1, \cdots, M$. Throughout the paper, we will assume that a day-count convention has been assigned, with no loss of generality.

We write $B(t, T_j), j = 1, ..., M$, to denote the price at time t of a discount bond that matures at time $T_j > t$. The forward swap rate $S(t, T_j, T_k)$, for any j and k, and satisfying $1 \leq j < k \leq M$, is defined through, for all $t \in [0, T_j]$,

$$S(t, T_j, T_k) = \frac{B(t, T_j) - B(t, T_k)}{G(t, T_j, T_k)}, \qquad G(t, T_j, T_k) = \sum_{l=j+1}^{k} \delta_l B(t, T_l).$$

For any generic forward swap rate, we will refer to T_j (resp. T_k) as the start (resp. end) date. Here we have introduced the price process $G(t, T_j, T_k)$ of the annuity numéraire, sometimes called the level numéraire.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and W a d-dimensional Wiener process. Throughout this paper, $\mathcal{F}$ will be assumed to be the natural filtration generated by W, i.e. $\{\mathcal{F}_t^W\} = \sigma(W_s, s \leq t)$, so that W is adapted to $\mathcal{F}$. A probability measure $\mathbb{P}^{T_j, T_k}$, equivalent to the historical probability measure $\mathbb{P}$, is said to be the forward swap probability measure associated with the dates T_j and T_k, or simply the forward swap measure, if for every $i = 1, ..., M$, the relative bond price $B(t, T_i)/G(t, T_j, T_k), \forall t \in [0, T_i \wedge T_{j+1}]$, follows a local martingale process under $\mathbb{P}^{T_j, T_k}$. $G(t, T_j, T_k)$ is the price of the numéraire so that the forward swap rate $S(t, T_j, T_k)$ is a $\mathbb{P}^{T_j, T_k}$-martingale. We denote the corresponding Brownian motion under $\mathbb{P}^{T_j, T_k}$ by W^{T_j, T_k}.

We next assume that forward swap rates follow diffusion processes. In particular, $S(t, T_j, T_k)$ is a $\mathbb{P}^{T_j,T_k}$-martingale, so that, under $\mathbb{P}^{T_j,T_k}$:

$$\frac{dS(t,T_j,T_k)}{S(t,T_j,T_k)} = \lambda(t,T_j,T_k)' dW^{T_j,T_k}(t), \qquad \forall t \in [0,T_j], \tag{1}$$

where the vector-valued volatility function $\lambda(t,T_j,T_k)$ is left unspecified. As we show below, however, there is no guarantee in general that these dynamics are well-defined in their respective martingale measure, for a completely generic choice of the set of forward swap rates. For a short while we will then postulate that such a set exists so that dynamics (1) are well-defined [1]. If $\lambda(t,T_j,T_k)$ is deterministic, $S(t,T_j,T_k)$ will be lognormally distributed. In general, however, $\lambda(t,T_j,T_k)$ may be state-variable dependent. In the following we want to derive a modeling approach which is valid for any type of volatility function, and is not constrained by a specific choice of them.

Let us also observe that the forward LIBOR rate $L(t,T_j)$, $j = 1,...,M-1$, defined as

$$L(t,T_j) = \frac{B(t,T_j) - B(t,T_{j+1})}{\delta_{j+1}B(t,T_{j+1})}, \qquad \forall t \in [0,T_j],$$

is itself a forward swap rate $S(t,T_j,T_k)$ corresponding to $k = j+1$. Accordingly, we denote by $\mathbb{P}^{T_j}$ the corresponding forward probability measure associated to the discount bond price $B(t,T_j)$, and by W^{T_j} a Brownian motion under $\mathbb{P}^{T_j}$. Then, for every $i = 1,...,M$, the relative bond price $B(t,T_i)/(\delta_{j+1}B(t,T_{j+1}))$, $\forall t \in [0, T_i \wedge T_{j+1}]$, follows a local martingale under $\mathbb{P}^{T_{j+1}}$. To simplify the exposition, we will often use the following compact notations: $B_j(t) := B(t,T_j)$, $G_{jk}(t) := G(t,T_j,T_k)$, $S_{jk}(t) := S(t,T_j,T_k)$, $\lambda_{jk}(t) := \lambda(t,T_j,T_k)$, $L_j(t) := L(t,T_j)$, $\lambda_j(t) := \lambda(t,T_j)$. Sometimes, we will also omit specifying calendar-time t dependence, if no confusion arises.

A generic arbitrage-free model is then specified by assigning arbitrage-free dynamics to a given set of forward swap rates spanning the tenor structure. Obviously, there is a large number of possible choices one can make regarding which set of forward swap rates has to be chosen. However, only a few are meaningful from a modeling perspective.

To this aim, we introduce the following concept

Definition 1 *Let $\mathcal{T}$ be a generic tenor structure. A collection of reset/settlement dates $\mathcal{C} = \{T_a, \cdots, T_b\}$ is called degenerate tenor structure subset if the following conditions are satisfied*

1. *$\mathcal{C} \subset \mathcal{T}$.*
2. *For any date T_j, $j = a+1, \cdots, b-1$ there exist at least one swap rate starting at T_j and one swap rate ending at T_j.*

[1] This consistency issue could be solved in principle by introducing market models as particular specifications of the Heath-Jarrow-Morton [16] framework as done, for instance, in [7] within the LMM specification. The problems with this indirect approach is that it makes more difficult to determine, in general, the HJM instantaneous forward volatilities that correspond to those of the observable rates one is willing to model.

3. *There exist at least two distinct swap rates starting at T_a and two distinct swap rates ending at T_b.*

The thinnest configuration that is associated to a degenerate tenor structure subset $\mathcal{C}$ is depicted in Figure 1a. Let $\mathcal{S}_c := \{S(t, T_j, T_k)\}_c$ be the set of rates associated with $\mathcal{C}$. This means that every element of $\mathcal{S}_c$ has both start and end date contained in $\mathcal{C}$ (Figure 1a). The picture shows that the set $\mathcal{S}_c$ yields a closed loop (or cycle) as opposed to a tree. A link between degenerate subsets and graph theory as well as the mathematical definitions of a cycle and a tree will be introduced in the next section. The set $\mathcal{S}_c$ is made of two disjoint subsets $\mathcal{S}_c = \{\mathcal{S}_c^u, \mathcal{S}_c^d\}$ with $\mathcal{S}_c^u := \{S_j^u\}_c$ and $\mathcal{S}_c^d := \{S_j^d\}_c$. Correspondingly, we introduce two collections of dates $\mathcal{H}^u = \{T_1^u, T_2^u, \cdots, T_m^u\}$ and $\mathcal{H}^d = \{T_1^d, T_2^d, \cdots, T_n^d\}$ with $T_1^u = T_1^d = T_a$, and $T_m^u = T_n^d = T_b$. In particular, every date at the tip of the arrows in Figure 1a corresponds to a forward swap ending at that date. As a consequence, the set $\mathcal{S}_c^u$ (resp. $\mathcal{S}_c^d$) is characterized by the following topological property: starting from date T_a, it is possible to "reach" the end date T_b by means of a unique path following arrows between dates belonging to $\mathcal{S}_c^u$ (resp. $\mathcal{S}_c^d$) only. From a financial point of view, this property corresponds to the following investment strategy: if an amount N_0 is invested at time T_a, there exist two distinct sequences of dates such *i)* at any date the notional principal is redeemed and immediately reinvested at the current interest rate and *ii)* the principal N_0 is redeemed at time T_b.

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Figure 1. Degenerate tenor structure and example of a tree

Let $\mathcal{A}$ be the set of all degenerate tenor structure subsets for a given tenor structure, namely $\mathcal{A} := \{\mathcal{C}_1, \cdots, \mathcal{C}_k\}$. The concept of an admissible set of forward swap rates can then be formulated.

Definition 2 *Given a tenor structure $\mathcal{T}$, for any $T_j, T_k \in \mathcal{T}$, a set $\mathcal{S} := \{S(t, T_j, T_k)\}$ of distinct forward swap rates is said to be admissible if*

1. *The cardinality of the set is equal to the number of distinct dates in the tenor structure minus one, i.e. $|\mathcal{S}| = M - 1$.*
2. *Any date T_j, $j = 1, \cdots, M$ in the tenor structure must coincide with a reset/settlement date of at least one forward swap rate in the set $\mathcal{S}$.*
3. *The set $\mathcal{A}$ is empty, i.e. $|\mathcal{A}| = 0$.*

Assume that we are given forward swap rates belonging to an admissible set. In what follows, we implicitly assume that the dynamics of discount bond prices are always associated to Eq.(1). The following result holds:

Proposition 3 *For all $t \in [0, T_j]$, if the set $\mathcal{S} := \{S(t, T_j, T_k)\}$ is admissible, then a set of deflated discount bond prices $\left\{\frac{B(t,T_k)}{B(t,T_i)}\right\}_{k=1,\ldots,M}$ relative to $B(t, T_i)$,*

$i = 1, \dots, M$ exists and is uniquely defined $\mathbb{P}$-a.s. Conversely, if for all $t \in [0, T_j]$ a set of deflated discount bond prices $\left\{\frac{B(t,T_k)}{B(t,T_i)}\right\}_{k=1,\dots,M}$ relative to $B(t, T_i)$, $i = 1, \dots, M$ exists and is uniquely specified $\mathbb{P}$-a.s., then $\mathcal{S}$ is admissible.

Proof. See Appendix A. ■

Proposition 3 states that from an admissible set of forward swap rates it is possible to uniquely determine sets of deflated discount bond prices, relative to a given discount bond, spanning the entire tenor structure. More explicitly, if the set is not admissible, three situations may occur. If $|\mathcal{S}| < M - 1$ then there exist multiple choices of discount bond prices that are all compatible, at any time, with the initial set of forward swap rates. In this situation, dynamics (1) cannot be uniquely specified in all equivalent martingale measures. If $|\mathcal{S}| > M - 1$ it is not possible to determine a set of deflated discount bond prices compatible with $\mathcal{S}$, and dynamics (1) are not defined. Finally if $|\mathcal{S}| = M - 1$ but the set $\mathcal{A}$ is not empty then it is not possible to guarantee the existence of a set of deflated discount bond prices for any choice of $B(t, T_i)$, $i = 1, \dots, M$, as numéraire.

If the set $\mathcal{S}$ is admissible, on the other side, Proposition 2 shows that it is possible to build market models by uniquely assigning dynamics (1) to $\mathcal{S}$ in any martingale measures that is associated to the tenor structure and to rely on the construction of Jamshidian [20] to price LIBOR and swap derivatives by arbitrage [2]. Examples of admissible and non-admissible sets of forward swap rates are given in Figure 2 for $M = 5$.

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Figure 2. Examples of admissible and non-admissible sets of forward swap rates

The following proposition further shows that the deflated discount bond prices are well-behaved and, consequently, changes of numéraire are well defined.

Proposition 4 *If the set $\mathcal{S} := \{S(t, T_j, T_k)\}$ is admissible, all deflated discount bond prices $\left\{\frac{B(t,T_k)}{B(t,T_i)}\right\}_{k=1,\dots,M}$ relative to $B(t, T_i)$, $i = 1, \dots, M$, are non-zero for any $t \in [0, T_j]$, $\mathbb{P}$-a.s..*

Proof. See Appendix B. ■

Combining Propositions 3 and 4, we deduce that the choice of an admissible set of forward swap rates is a necessary and sufficient condition for the associated market model to admit a unique specification in any equivalent LIBOR and/or swap forward probability measure associated to a discrete tenor structure. Correspondingly, we introduce the following concept.

Definition 5 *A Market Model Approach (MMA) is specified by assigning an arbitrage free dynamics of the type (1) to a set of admissible forward swap rates.*

[2] We point out that a different situation arises when $T_0 = T_1$. In this case, $B(T_0, T_1) = 1$ and, as a consequence, we need to choose $B(T_0, T_1)$ itself as the reference discount bond in order to ensure existence and unicity of solutions.

The Market Model Approach is the most general modelling framework for market models of interest rate dynamics. For practical purposes, however, the class associated to admissible sets of forward swap rates is still too large. In Section 4 we will select, among all admissible models, those being the most relevant from a financial point of view.

3 Connection with graph theory

The above considerations suggest an interesting way to simplify the concept of admissibility of a set of forward swap rates through the mathematical theory of graphs. Let us first recall some useful concepts related to that theory. We follow the standard text by Diestel [5].

Let $Gr := (V, E)$ be a graph made of a given set V of vertices (or points) and a set E of edges (or lines). A generic edge e is a line connecting two adjacent vertices. If x and y are adjacent vertices, the edge connecting them will be indicated by the set $e = \{x, y\} := xy$. Pair-wise, non-adjacent vertices are called independent. A vertex v is incident with and edge f if $v \in f$, e.g. x and y are incident with e. Let $Gr' \cap Gr := (V \cap V', E \cap E')$ the intersection between two graphs. If $Gr' \cap Gr = \emptyset$, then Gr and Gr' are disjoint graphs. Two disjoint graphs are graphically unconnected, i.e. there are no edges linking a vertex of Gr with one of Gr'. If $Gr' \subseteq Gr$ is a spanning subgraph of Gr if V' spans all vertices of Gr, that is if $V = V'$. The degree $d(v)$ of a vertex v is the number $|E(v)|$ of edges at v. A path is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \cdots, x_k\}$, $V = \{x_0x_1, x_1x_2, \cdots, x_{k-1}x_k\}$. For ease of notation, a path is usually defined by the sequence of its vertices only. If P is a path with $k \geq 3$, then the graph $C = P + x_{k-1}x_0$ is called a cycle. From a graphical point of view, a cycle is a sequence of lines joining adjacent vertices where every vertex is simultaneously the start and the end of the path. A non-empty graph is called connected if any two of its vertices are linked by a path. A connected graph with no cycles (or acyclic graph) is called a tree (see Figure 1b for an example). Spanning subgraphs that are also trees are called spanning trees. A labeled tree is a tree with its nodes labeled.

A simple link exists between sets of forward swap rates associated to a tenor structure and a graph. To this aim, we observe that if $\mathcal{T} := V$ and $\mathcal{S} := E$ then the set $Gr = \{\mathcal{T}, \mathcal{S}\}$ is a graph. A degenerate tenor structure subset is equivalent to the presence of a cycle in Gr. More generally, it is possible to recast Definition 2 by means of graph theory as follows.

Definition 6 *A set $\mathcal{S}$ is said to be admissible if*

1. *The graph Gr has M vertices and $M - 1$ edges, i.e. $|\mathcal{S}| = |\mathcal{T}| - 1 = M - 1$,*
2. *The graph Gr is connected.*

The following result states the equivalence between Definitions 2 and 6.

Proposition 7 *Definition 2 and Definition 6 are equivalent.*

Proof. See Appendix C. ■

Definition 6 allows to translate the concept of admissibility in terms of graph properties. A further step in this direction can be achieved by recalling a fundamental result in graph theory.

Theorem 8 *A connected graph with* n *vertices is a tree if and only if it has* $n-1$ *edges.*

Proof. See [5]. ■

As a consequence, the simplest characterization of admissibility of a set of forward swap rates reads as follows.

Proposition 9 *A set* $\mathcal{S}$ *is admissible if and only if* Gr *is a tree.*

Proposition 9 allows to determine, by simple graphical inspection, whether a set of forward swap rates is admissible or not. In addition, the mapping to graph theory paves the way to the solution of the interesting problem of the exact enumeration of all admissible market models for a given tenor structure.

Proposition 10 *For a given tenor structure* $\mathcal{T} := \{T_1, \cdots, T_M\}$ *there are* M^{M-2} *admissible sets.*

Proof. See Appendix D. ■

These M^{M-2} admissible sets can be characterized via the Prüfer code (Prüfer [30]). This well-known algorithm is an encoding which provides a bijection between the M^{M-2} labeled trees on M nodes and strings of $M-2$ integers chosen from an alphabet of the numbers 1 to M. Symbolic calculus packages such as Mathematica provide routines which can convert a Prüfer code to a labeled tree, and vice-versa.

4 Model selection

As shown by Proposition 10, the number of admissible models becomes rapidly extremely large at increasing M. The first few values are in fact $1, 1, 3, 16, 125, 1296, \ldots$. In a typical market configuration with annual reset/settlement dates and a 10-year maturity, we can define 10^8 different admissible models. Thus, the choice of the set of forward swap rates that underlie model dynamics must be driven by practical considerations. In ultimate analysis, dynamic IR models are used to price and hedge exotic IR derivatives for which no direct market information exists. To ensure meaningful hedging, a model should be made consistent with the term structure of interest rates as well as the volatility information provided by a set of quoted plain-vanilla derivatives. These are typically caps and swaptions. Depending on the type of exotic option to be priced, however, some of these "market" instruments may be more relevant than others. As an example, we consider a Bermudan swaption. If the option gives the holder the right to enter at times $T_1, T_2, \cdots, T_{M-1}$, into a plain-vanilla swap maturing at T_M, the only relevant European swaptions from a pricing and hedging perspective will be those whose expiry dates are $T_1, T_2, \cdots, T_{M-1}$ and with underlying swap maturity T_M. In this case, it seems quite natural to introduce a MMA where the relevant set coincides with the co-terminal forward swap rates. The spanning set underlying the ctSMM is shown in Figure 3a. In this case, all forward swap rates share the same terminal date T_M but have a variable starting date. In summary, we can introduce the following model.

Definition 11 *A co-terminal swap market model (ctSMM) is built from*

1. *An admissible set* $\{S_{jM}\}$, $j = 1, \cdots, M-1$, *of forward swap rates with different start dates and equal maturity date* T_M.
2. *A collection of mutually equivalent probability measures* $\mathbb{P}^{T_j, T_M}$, $j = 1, \cdots, M-1$.
3. *A family* W^{T_j, T_M} *of processes such that: (i) for any* $j = 1, \cdots, M-1$, W^{T_j, T_M} *follows a d-dimensional Brownian motion under the forward swap probability measure* $\mathbb{P}^{T_j, T_M}$, *(ii) for any* $j = 1, \cdots, M-1$ *the forward swap rate satisfies the SDE, for all* $t \in [0, T_j]$:

$$dS_{jM}(t) = S_{jM}(t)\lambda_{jM}{}'(t)dW^{T_j, T_M}(t), \quad S_{jM}(0) = \frac{B_j(0) - B_M(0)}{\sum_{l=j+1}^{M} \delta_l B_l(0)}.$$

European-style derivatives give the holder the right to exercise an option at a single future date T. Therefore, the option payoff, no matter how complex, is by definition $\mathcal{F}_T$-measurable. Qualitatively speaking, a set of admissible forward swap rates sharing the same initial date T contains all the information needed to evaluate the payoff, the latter being in fact a function of a set of admissible "co-initial" forward swap rates at time T. Hence one may expect that a MMA based on a set of co-initial forward swap rates will provide a powerful tool to price and hedge a large variety of European style derivatives including forward-start, amortizing and zero-coupon swaptions as well as bond options consistently with the swaption initial volatility surface. These issues are further discussed in Galluccio and Hunter [10]. As opposed to the ctSMM, all forward swap rates in the co-initial SMM share the same initial date T_1 but have a variable end date (see Figure 3b). We therefore arrive at the following description.

Definition 12 *A co-initial swap market model (ciSMM) is built from*

1. *An admissible set* $\{S_{1j}\}$, $j = 2, \cdots, M$, *of forward swap rates with equal start date* T_1 *and variable end dates.*
2. *A collection of mutually equivalent probability measures* $\mathbb{P}^{T_1, T_j}$, $j = 2, \cdots, M$.
3. *A family* W^{T_1, T_j} *of processes such that: (i) for any* $j = 2, \cdots, M$, W^{T_1, T_j} *follows a d-dimensional Brownian motion under the forward swap probability measure* $\mathbb{P}^{T_1, T_j}$, *(ii) for any* $j = 2, \cdots, M$, *the forward swap rate*

satisfies the SDE, for all $t \in [0, T_1]$*:*

$$dS_{1j}(t) = S_{1j}(t)\lambda_{1j}{}'(t)dW^{T_1,T_j}(t), \quad S_{1j}(0) = \frac{B_1(0) - B_j(0)}{\sum_{l=2}^{j} \delta_l B_l(0)}.$$

PLEASE INSERT

Figure 3. Sets of co-terminal, co-initial and co-sliding forward swap rates

A third common class of products is that of IR derivatives that depend on a set of fixed-maturity instruments, such as constant maturity swaps (CMS), at any coupon date. In this case, an efficient choice consists of introducing a family of admissible forward swap rates having variable start and end date, but sharing the same period interval between tenor dates or time to maturity. We call this class "co-sliding" (see Figure 3c).

Definition 13 *A co-sliding swap market model (csSMM) is built from*

1. *An admissible set* $\{S_{j,j+1}\}$*,* $j = 1, \cdots, M-1$ *of forward swap rates.*
2. *A collection of mutually equivalent probability measures* $\mathbb{P}^{T_j,T_{j+1}}$*,* $j = 1, \cdots, M-1$*.*
3. *A family* $W^{T_j,T_{j+1}}$ *of processes such that: (i) for any* $j = 1, \cdots, M-1$*,* $W^{T_j,T_{j+1}}$ *follows a d-dimensional Brownian motion under the forward swap probability measure* $\mathbb{P}^{T_j,T_{j+1}}$*, (ii) for any* $j = 1, \cdots, M-1$ *the forward swap rate satisfies the SDE, for all* $t \in [0, T_1]$*:*

$$dS_{j,j+1}(t) = S_{j,j+1}(t)\lambda'_{j,j+1}(t)dW^{T_j,T_{j+1}}(t), S_{j,j+1}(0) = \frac{B_j(0) - B_{j+1}(0)}{\sum_{l=j+1}^{j+1} \delta_l B_l(0)}. \tag{2}$$

Definition 13 introduces the LMM, since non-overlapping forward swap rates of the form $\{S_{j,j+1}\}$, $j = 1, \cdots, M-1$ are in fact forward LIBOR rates.

A direct consequence is the following

Corollary 14 *The LMM is the only admissible co-sliding swap market model.*

In fact, all co-sliding models that are obtained by generalizing (2) to arbitrary families of overlapping and non-overlapping forward swap rates, like those in the form $\{S_{j,j+n}\}$, $j = 1, 2, \ldots, M-n$, with $n = 2, \ldots, M-1$, or $\{S_{j,j+m}\}$, $j = 1, m+1, \ldots, (km+1) \wedge (M-m)$, with $m = 2, \ldots, M-1$ and $k \in \mathbb{N}$ are not admissible after Propositions 3 and 4.

Beside the three major classes described above, it is possible to introduce "mixed" models by assigning dynamics on admissible sets of forward swap rates where some swaps are co-sliding and some others are co-terminal. This interesting application has been recently studied in [29].

In conclusion, the Market Model Approach (MMA) constitutes the most general framework to price and hedge IR derivatives where the starting point is the modelling of financial observable quantities like forward swap rates. Note, finally, that Definitions 11 to 13 can be easily extended to far more general classes of processes, see [21].

5 The co-terminal SMM

The co-terminal class, first introduced by Jamshidian [20], is the one we concentrate from now on. In our opinion, its relevance and importance for the pricing of complex IR derivatives has been largely overlooked in the literature.

To clarify this point, we start by stressing that the underlying of many Over-The-Counter (OTC) interest rate derivatives are indeed swap (as opposed to LIBOR) rates. The simplest and most important example is the Bermudan swaption. Apart from plain-vanilla caps and swaptions, Bermudan swaptions are by far the most liquid IR derivatives. The owner of such a contract at a generic time t has the right to enter into a plain-vanilla swap maturing at a date T_M at any date among those in the set $\{T_1, \cdots, T_{M-1}\}$. The optimal exercise boundary can be found by using a standard optimal control approach and the pricing formula can be formally decomposed into a hierarchical sequence of European swaptions [20]. At any time $T_i\,(i < M-1)$ the exercise decision will depend on the assessment of the volatility that will drive future swap rate dynamics, prevailing at that particular time. To price (and hedge) the Bermudan option at any previous time t we need an arbitrage-free model for the evolution of the underlying rates. This simple example shows that the optimal pricing methodology must concentrate on evolving swap rates, as they constitute the natural underlying of the option.

The usual way around this obstacle, within the LMM, consists of trying to force the LIBOR-based dynamics to be consistent with the market information provided by the price of European swaptions. However, a joint calibration of the LMM on caps and swaptions is difficult to achieve without introducing strong assumptions on the input swaption volatilities [8] so that global minimization algorithms are often the preferred choice. In this situation, only a selection of options is considered for calibration, see for instance [32]. In general, achieving a simultaneous calibration to market caps and swaptions within their bid/ask volatility spread by means of a fast and robust algorithm that is capable of preserving a meaningful shape of the (calibrated) instantaneous forward volatility is problematic [3]. If, on the other hand, one forces the LMM to be (artificially) consistent with the swaption market only, then the resulting errors between market and model-implied Black volatilities of caplets are typically of the order of 2%, much beyond the bid-ask spread [32].

Below we show that a ctSMM can be easily calibrated to a careful selection

[3] As many authors have emphasized, caplet and swaption markets show some degree of "inconsistency" when one tries to fit a LMM to both.

of caplets and "diagonal" swaptions [4]. The volatility risk associated to a large fraction of exotic IR derivatives is almost entirely captured by these two sets only. In this respect, apart from the aforementioned Bermudan swaptions, we mention Callable Cap and Reverse Floaters [35], Ratchet Cap Floaters and Knock-in/out swaps [32]. As immediate consequence, we deduce that a ctSMM is the most appropriate approach for these complex financial derivatives. On the other side of the spectrum, products that are written on CMS rates, like callable CMS, are better handled within a co-sliding model.

Finally, it is worth mentioning that LIBOR rates are not directly quoted by the market, while swap rates are. Typically, forward LIBOR rates enter the valuation of forward rate agreements (FRAs). These simple instruments allow at some time t to lock-in the discretely compounded interest rate that will be paid between two future dates. FRAs are OTC derivatives, and only LIBOR futures and plain-vanilla swaps are quoted on a daily basis in the market. As a consequence, the forward LIBOR rates must be derived or "stripped out" of quoted instruments, i.e. LIBOR futures and swaps. In Section 6 we will see that a forward LIBOR rate can be formally written as the sum of two consecutive forward swap rates with given coefficients, pretty much the same as a forward swap rate can be formally interpreted as a weighted sum of forward LIBOR rates. In conclusion, the claim that swap rates are derived from LIBOR rates, the latter being a "more fundamental" financial quantity than the former, is not justified by evidence.

6 European option pricing

Model calibration is a reverse engineering procedure aimed at identifying the relevant model parameters from a set of liquid instruments quoted in the market. In the IR derivatives context, these instruments are plain-vanilla options written on forward swap and LIBOR rates, i.e. swaptions and caplets respectively. To achieve a fast and stable model calibration and avoid slow numerical procedures one should be able to express plain-vanilla option prices in closed or quasi-closed form. In this section, we address the problem of the pricing of European options within the ctSMM. When closed-form solutions are not available, namely for caplet prices, we introduce several approximation procedures. Speed and accuracy of these different approaches are analyzed in Section 8.

For simplicity, we consider dynamics driven by deterministic volatility structures as in e.g. Rutkowski [33] [5]. The extension to more general classes of processes and the implications relating to the pricing and risk management of IR derivatives is left to future research. It is however worth mentioning that it is possible to easily extend the diffusion-based approach to more general classes of

[4] The set of diagonal swaptions is composed of those options that are written on co-terminal swap rates, see below.

[5] In GHLS we derive the dynamics under any forward probability swap measure for both the ctSMM and ciSMM. The co-sliding case (or LMM) can be handled as well, see the list of references.

processes within the co-initial model framework and simultaneously ensure that the model dynamics is consistent with any given initial volatility surface [10].

To get compact notations we will make extensive use of the following auxiliary processes as in [20]: $\nu_{ij}(t) := \nu_{ij,M}(t) := \sum_{k=j}^{M-1} \delta_{k+1} \prod_{l=i+1}^{k} (1+\delta_l S_{lM}(t))$, $\nu_i(t) := \nu_{ii}(t)$. Note also that the following relations hold: (i) $G_{jM}(t)/B_M(t) = \nu_j(t)$, (ii) $B_j(t)/B_M(t) = 1 + \nu_j(t) S_{jM}(t)$.

6.1 Swaption pricing

Under a deterministic volatility structure, the co-terminal forward swap rates are lognormally distributed, so that the corresponding swaption can be priced via a Black formula [6]. The price of a European swaption, giving the right to enter at time T_j into a swap maturing at T_M, is given at time t by

$$\mathbf{Swn}\left(t, T_j, K\right) = G_{jM}\left(t\right)\left[S_{jM}\left(t\right) N\left(d_1\right) - K N\left(d_2\right)\right], \tag{3}$$

where, as usual, $d_1 = (\ln(S_{jM}\left(t\right)/K) + \frac{1}{2}\sigma_{jM}^2\left(T_j - t\right))/(\sigma_{jM}\sqrt{T_j - t})$, $d_2 = d_1 - \sigma_{jM}\sqrt{T_j - t}$, with $\sigma_{jM}^2 = \frac{1}{T_j - t}\int_t^{T_j} \lambda_{jM}'(s)\lambda_{jM}(s)ds$.

We use σ to indicate Black implied volatilities. Similarly to the notation used for instantaneous volatilities, σ_{jM} is the Black implied volatility of the swaption written on S_{jM}, and σ_j is the Black implied volatility of the caplet written on L_j.

6.2 Caplet pricing

We consider the price of a caplet at time t giving the right to buy a forward LIBOR rate between T_j and T_{j+1}:

$$\mathbf{Cpl}\left(t, T_{j+1}, K\right) = \delta_{j+1} B\left(t, T_{j+1}\right) \mathbb{E}_t^{T_{j+1}}\left[\left(L_j\left(T_j\right) - K\right)_+\right],$$

where the expectation is taken with respect to the forward measure $\mathbb{P}^{T_{j+1}}$ such that the forward LIBOR rate $L_j\left(t\right)$ follows a martingale process: $dL_j\left(t\right) = L_j\left(t\right)\lambda_j'\left(t\right) dW^{T_{j+1}}(t)$.

In the ctSMM forward LIBOR rates are not lognormally distributed, and we cannot price caplets using the Black formula directly. In the LMM setting a similar situation exists: caplets can be priced in closed-form using Black formula while swaptions cannot. Quick and accurate approximation techniques to price swaptions in the LMM have been proposed by Rebonato [31], Hull and White [17], and Brace, Gatarek and Musiela [7]. Their construction and accuracy are reviewed in detail in Brigo and Mercurio [8]. Here we will parallel these suggestions in the context of the ctSMM, and provide similar approximated formulas for caplet prices. In addition, we also suggest a new approximation based on a spread option approach, which leads to a Margrabe-type formula [25].

6.2.1 Rebonato approach

This method is similar to the one first advocated by Rebonato [31] for the LMM. The starting point consists in observing that a forward LIBOR rate can be written as a weighted sum of two consecutive co-terminal forward swap rates:

$$L_j(u) = w_{jj}(u)S_{jM}(u) + w_{j,j+1}(u)S_{j+1,M}(u), \tag{4}$$

where $w_{jj}(u) = \nu_j(u)/(\nu_j(u) - \nu_{j+1}(u))$, $w_{j,j+1}(u) = -\nu_{j+1}(u)/(\nu_j(u) - \nu_{j+1}(u))$, with $w_{jj}(u) + w_{j,j+1}(u) = 1$.

The two weights can also be expressed as $w_{jj}(u) = G_{jM}(u)/(\delta_{j+1}B_{j+1}(u))$ and $w_{j,j+1}(u) = -G_{j+1,M}(u)/(\delta_{j+1}B_{j+1}(u))$. Note that they sum to one but $w_{jj}(u)$ is positive whereas $w_{j,j+1}(u)$ is negative. This is at difference with the LMM where all weights are positive in interpreting a forward swap rate as a weighted sum of forward LIBOR rates.

The two steps underlying the Rebonato approach are a) "freezing" the weights at their initial value (at time t) in Equation (4), and then differentiate both sides, b) "freezing" the remaining random forward LIBOR and swap rates in the volatility function. These two steps provide a simple lognormal approximation (see GHLS) of the LIBOR rate. The approximation is effective if the variability of the weights is much smaller than the variability of the forward swap rates. This can be tested both historically and through Monte Carlo simulations (see the results in Section 8.2). The small variability can be intuitively explained by the fact that the weights are ratios of linear combinations of discount bond prices. Hence the volatility of the weights is small by construction since weight dynamics derive from ratios of highly correlated processes.

6.2.2 Hull and White approach

A slightly more sophisticated version of the above procedure follows the path of Hull and White (HW) [17]. It consists in a) differentiating Equation (4) without an initial freezing of the weights, b) freezing the remaining random forward LIBOR and swap rates in the volatility function. We get then the lognormal dynamics:

$$\frac{dL_j(u)}{L_j(u)} \approx \sum_{l=j}^{M-1} \widehat{w}_{jl}\,(t)\,\lambda_{lM}(u)'dW^{T_{j+1}}(u),$$

with $\widehat{w}_{jl}\,(t) = \overline{w}_{jl}\,(t)\,S_{lM}(t)/L_j(t)$, and $\overline{w}_{jl}(t)$ is equal to

$$\begin{cases} \nu_j(t)/(\nu_j(t) - \nu_{j+1}(t)), & l = j, \\ -\delta_{j+1}\nu_{j+1}(t)\left(1 + \nu_{j+1}(t)S_{jM}(t)\right)/\left(\nu_j(t) - \nu_{j+1}(t)\right)^2, & l = j+1, \\ \delta_l\delta_{j+1}\nu_{j+1,l}(t)\left(S_{jM}(t) - S_{j+1,M}(t)\right)/[\left(\nu_j(t) - \nu_{j+1}(t)\right)^2\left(1 + \delta_l S_{lM}(t)\right)], & \\ & j+2 \le l \le M-1, \\ 0, & \text{otherwise.} \end{cases}$$

The volatility parameter to plug into the Black caplet price is given by the square root of

$$(T_j - t)\sigma_j^2 = \sum_{l=j}^{M-1} \sum_{k=j}^{M-1} \widehat{w}_{jl}(t)\, \widehat{w}_{jk}(t) \int_t^{T_j} \lambda'_{lM}(u)\lambda_{kM}(u)du.$$

We recall that the above approximation is only valid if the weights $\widehat{w}$ do not vary too much. Numerical experiments (see below) show that the variability of the weights is small when compared to the variability of the forward swap and LIBOR rates. Besides, it is worth noticing that the first two weights are much larger than the others by a factor of about 1000 in absolute terms. Hence the price or implied volatility of a caplet should only change marginally if other weights apart from the two first ones are neglected (see Section 8.1). This further approximation results in a "truncated" HW approach, which will be instrumental in the bootstrap calibration approach of Section 7.

6.2.3 Rank-one approach

The rank-one approach has been suggested by Brace, Gatarek and Musiela [7]. In the LMM it starts from recognizing that a swaption can be viewed as a sum of caplets whose exercise regions depend on the forward swap rate instead of the forward LIBOR rate. The approximation then relies on freezing the drift in the forward LIBOR rate dynamics and on a rank-one approximation of the covariance matrix of the forward LIBOR rates. A similar approach can be adopted in an SMM framework since a caplet can be viewed as a sum of swaptions whose exercise regions depend on the forward LIBOR rate instead of the forward swap rate (see GHLS).

6.2.4 Spread-option approach

Recall that the forward LIBOR rate can be written as a basket of two consecutive co-terminal forward swap rates (cf. Equation (4)). Hence once the weight factors have been frozen at their initial values at t, the caplet can be viewed as an option on a spread between two consecutive forward swap rates. Here we use the same freezing technique as in the Rebonato approach but we do not rely on the approximation of the basket of two lognormally distributed variables by means of a single lognormal variable (which is necessary to get a Black formula). The caplet price is then akin to the formula given in Margrabe [25] (see GHLS).

7 Model calibration

The problem of the calibration of the LMM has attracted much interest recently, see for instance [8], [32], [35] and [36]. Yet, no similar results are currently available in the ctSMM, although the mathematical similarities existing between the two approaches.

When dealing with calibration, it is convenient to use the following scalar specification of the co-terminal forward swap rates S_{jM} for $j = 1, ..., M-1$, under their appropriate forward swap measures:

$$\frac{dS_{jM}(t)}{S_{jM}(t)} = \Lambda_{jM}(t) d\overline{W}^{T_j,T_M}(t),$$

where $\Lambda_{jM}(t) := \|\lambda_{jM}(t)\|$ is a scalar time-inhomogeneous function equal to the Euclidean norm of the corresponding instantaneous volatility vector $\lambda_{jM}(t)$ and $d\overline{W}^{T_j,T_M}(t) = \lambda'_{jM}(t) dW^{T_j,T_M}(t)/\|\lambda_{jM}(t)\|$ corresponds to a one-dimensional Brownian motion under the forward measure $\mathbb{P}^{T_j,T_M}$ associated with the numéraire G_{jM}. Since forward swap rate S_{jM} is undefined at times $t > T_j$, we can equally extend all dynamics up to T_M by requiring that $\Lambda_{jM}(t) \neq 0$ at $t \in [0, T_{j-1})$, and $\Lambda_{jM}(t) = 0$ at $t > T_j$. Note that there exists a simple transformation between the two specifications in terms of λ's and Λ's, i.e. $\lambda'_{iM}(t)\lambda_{jM}(t) = \rho_{ij}(t)\Lambda_{iM}(t)\Lambda_{jM}(t)$, where $\rho_{ij}(t)$ is the instantaneous correlation between the Brownian motions.

The ctSMM is completely specified once the correlation matrix $\rho_{ij}(t)$ and the scalar instantaneous volatility functions $\Lambda_{jM}(t)$ have been assigned. To simplify model calibration and avoid data overfitting, it is good and standard practice to assume that either $\rho_{ij}(t)$ or $\Lambda_{jM}(t)$ are time-homogeneous functions. This choice is not too restrictive and actually avoids redundancy in specifying the time dependency of the covariance matrix. Thus, we will throughout assume that the correlation matrix ρ_{ij} is time independent and leave alone the functions $\Lambda_{jM}(t)$ to completely specify the time dependency in the covariance matrix.

The choice of the model calibration instruments must be driven by practical (in particular hedging) considerations. If a model is calibrated to a finite set $\mathcal{A}$ of plain-vanilla options then it will show no risk sensitivity against any other set $\mathcal{B}$ disjoint from the set $\mathcal{A}$. Thus, the choice of $\mathcal{A}$ must be associated to that portion of the volatility matrix that we consider the most appropriate in capturing the volatility risk. Below we show that our ctSMM can be easily and efficiently calibrated to a set $\mathcal{A}$ made of all caplets and co-terminal swaptions associated to the tenor structure. This set is indeed optimal for a large fraction of all exotic IR derivatives as above mentioned. This is a crucial property of the ctSMM since, in comparison, a LMM must in general be calibrated to a larger set even in situations where some calibration instruments are redundant as far as volatility risk is concerned. Finally, we recall the importance of a joint calibration on caps and co-terminal swaptions. As noticed by many authors (see for instance [32]), it allows to infer (given today's information) relevant information on the volatility that will drive the future evolution of forward swap rates, once the correlation matrix has been attributed and a particular parameterization of the instantaneous volatility functions has been chosen.

The main idea to calibrate the ctSMM consists in using a "parametric bootstrap procedure" [6]. This choice has three main advantages. First, it ensures that the model is simultaneously consistent with a set of co-terminal swaptions and caplets spanning the same tenor structure. Second, it provides smooth calibrated instantaneous volatility functions and avoids the overfitting problem.

[6] The terminology "bootstrap" is standard among practitioners and refer to a "step-by-step recursive" procedure. This is not related to the bootstrap method of the statistical literature.

Third, the algorithm is extremely fast and avoids the instability problems associated to global minimization techniques (see GHLS for further discussion about the advantage of this method over alternative nonparametric and parametric approaches).

The choice of the parametric form for the instantaneous volatility of the forward swap rates we adopt is similar in spirit to that of the LMM, see for instance [32]. We thus look for shapes that replicate the initial term structure of Black implied swaption volatilities and are (as much as possible) time stationary. The last constraint is driven by the observation that, to a large extent, the global shape of the term structure of at-the-money (ATM) volatilities tends to be preserved through time. We thus introduce the following generic form of the scalar instantaneous volatility:

$$\Lambda_{jM}(t) := \phi_j(t)\psi_j(T_j - t), \quad j = 1, \cdots, M-1. \tag{5}$$

The stationary factor $\psi_j(T_j - t)$ is meant to reproduce the well-known "hump" of the implied volatility of the swaption with underlying S_{jM}, while the calendar-time dependent function $\phi_j(t)$ represents a perturbation mode around the stationary solution. A simple and effective parameterization of $\psi_j(T_j - t)$ is provided by

$$\psi_j(T_j - t; \{\theta_j\}) := (a_j(T_j - t) + b_j)e^{-c_j(T_j - t)} + d_j, \quad j = 1, \cdots, M-1, \tag{6}$$

where $\{\theta_j := (a_j, b_j, c_j, d_j), j = 1, \cdots, M-1\}$ is a set of positive parameters. This specification of $\psi_j(T_j - t)$ is usually suggested in the LMM literature (see [8] and [31]). The factor $e^{-c_j(T_j-t)}$ is used to model the decreasing shape of the term structure at the long end while $a_j(T_j - t) + b_j$ models the upward shape at the short end. Put together, they reproduce the observed hump as a function of time-to-maturity. Finally, parameter d_j sets a global level. Note that θ_j is indexed by j due to the fact that different co-terminal swaptions have in general different shapes of implied volatility term structure.

Ideally, if $\phi_j(t)$ were unitary functions for any j, model dynamics would be perfectly stationary. Unfortunately, the constraint of perfect consistency with the initial term structure of swaption volatilities is usually too strong to allow for the model being simultaneously consistent with the caplet market as well. Therefore a perturbation around the stationary solution $\psi_j(T_j - t)$ is needed, and the goal will be to keep perturbation $\phi_j(t)$ as close to unity as possible in the calibration process.

The calibration is performed in two sequential and independent steps. First, we fit $\psi_j(T_j - t)$ to the humped shape of market implied volatility of co-terminal swaptions indexed by $j = 1, \cdots, M-1$, by adjusting the set $\{\theta_j\}$. Second, for a given instantaneous correlation matrix ρ_{ij}, we calibrate $\phi_j(t)$ on both caplet and swaption market volatilities via a bootstrap algorithm. The correlation matrix can be statistically estimated from historical data or further modelled according to suitable parametric forms, as we discuss below. If the correlation structure is taken as an input, the only freedom left is the first component of the instantaneous volatility of forward swap rates. However, as it will be shown below, forcing ρ_{ij} to match its historical value is in general too restrictive to achieve a sound result. Below, we suggest an alternative method.

Assume that $\psi_j(T_j - t; \{\theta_j\})$ has already been identified. We may then proceed as follows in order to make our ctSMM consistent with the caplet volatilities. Consider the next objects: the forward swap rates $S_{jM}(t), S_{j+1,M}(t)$, the forward LIBOR rate $L_j(t)$ and their associated Black volatilities $\sigma_{j,M}, \sigma_{j+1,M}, \sigma_j$. In the scalar representation of the dynamics it is straightforward to show that

$$(T_j - t)\sigma_{j,M}^2 = \int_t^{T_j} \Lambda_{j,M}(s)^2 ds, \quad (T_{j+1} - t)\sigma_{j+1,M}^2 = \int_t^{T_{j+1}} \Lambda_{j+1,M}(s)^2 ds, \quad (7)$$

$$\begin{aligned}(T_j - t)\sigma_j^2 &= \widehat{w}_j(t)^2 \int_t^{T_j} \Lambda_{j,M}(s)^2 ds + \widehat{w}_{j+1}(t)^2 \int_t^{T_j} \Lambda_{j+1,M}(s)^2 ds \\ &\quad + 2\widehat{w}_j(t)\,\widehat{w}_{j+1}(t)\,\rho_{j,j+1}(t) \int_t^{T_j} \Lambda_{j,M}(s)\Lambda_{j+1,M}(s) ds, \quad (8)\end{aligned}$$

where the coefficients $\widehat{w}_j(t), \widehat{w}_{j+1}(t)$ can be those provided by either the Rebonato or the truncated HW approximation of Section 6.2. Factor $\phi_j(t)$ is then introduced as a smooth perturbation around the stationary solution. In particular, it will be defined as follows:

$$\phi_j(t) = \phi_j f_j(t) = \begin{cases} \phi_j^a/(1+\alpha_j t), & t \in [0, T_{j-1}), \\ \phi_j^b/(1+\alpha_j t), & t \in [T_{j-1}, T_j], \end{cases} \quad (9)$$

where α_j, ϕ_j^a and ϕ_j^b are positive constants. This choice for $\phi_j(t)$ is only one of the possible parametric forms, but it yields very satisfactory results since (in contrast with exponentially decaying function of the form $e^{-\alpha_j t}$) it does not introduce dramatic alterations of the initial shape associated to $\psi_j(T_j - t; \{\theta_j\})$. Equation (8), in terms of $\phi_j(t)$ and the Black volatility σ_{jM}, reads

$$\begin{aligned}(T_j - t)\sigma_j^2 &= \widehat{w}_j(t)^2 \sigma_{jM}^2 (T_j - t) + \widehat{w}_{j+1}(t)^2 \left(\phi_{j+1}^a\right)^2 \int_t^{T_j} f_{j+1}^2(s)\psi_{j+1}^2 ds \\ &\quad + 2\widehat{w}_j(t)\,\widehat{w}_{j+1}(t)\,\rho_{j,j+1}(t)\phi_{j+1}^a \left[\phi_j^a \int_t^{T_{j-1}} f_j(s)\psi_j f_{j+1}(s)\psi_{j+1} ds \right. \\ &\quad \left. + \phi_j^b \int_{T_{j-1}}^{T_j} f_j(s)\psi_j f_{j+1}(s)\psi_{j+1} ds \right]. \quad (10)\end{aligned}$$

On the other hand, we have from Equation (7):

$$\begin{aligned}(T_{j+1} - t)\sigma_{j+1,M}^2 &= \left(\phi_{j+1}^a\right)^2 \int_t^{T_j} f_{j+1}^2(s)\psi_{j+1}^2 ds \\ &\quad + \left(\phi_{j+1}^b\right)^2 \int_{T_j}^{T_{j+1}} f_{j+1}^2(s)\psi_{j+1}^2 ds. \quad (11)\end{aligned}$$

Note that, if the scalar instantaneous volatility function $\Lambda_{j,M}(s), s \in [t, T_j]$ of the j^{th} forward swap rate were known and if we used the above functional form of the volatility function $\Lambda_{j+1,M}(s)$, $s \in [t, T_{j+1}]$, of the $(j+1)^{\text{th}}$ forward swap rate then it would be possible to determine uniquely $\Lambda_{j+1,M}(s)$, $s \in [t, T_{j+1}]$ from the knowledge of: *i)* the three Black volatilities $\sigma_{j,M}, \sigma_{j+1,M}, \sigma_j$ and, *ii)* the correlation $\rho_{j,j+1}(t)$. Mathematically, this stems from the observation that the only unknown in Equation (10) is ϕ^a_{j+1}. Assume now that ϕ^a_{j+1} has been determined from Equation (10), then Equation (11) can be solved for ϕ^b_{j+1} and the volatility $\Lambda_{j+1,M}(s)$ is then uniquely identified. This procedure can thus be repeated by bootstrap, i.e. step-by-step, until the last co-terminal swaption and caplet. In detail, the procedure goes as follows for $j = 0, ..., M-2$:

1. Select α_j, for $j = 1, ..., M-1$. We need to initialize α_j to large values in order for the solutions to exist.

2. Set $\phi^a_1 = 0$, and solve Equation (11) when $j = 0$ for ϕ^b_1.

3. After $\left(\phi^a_j, \phi^b_j\right)$ is known, the only remaining unknown in Equation (10) is ϕ^a_{j+1}. ϕ^b_{j+1} can then be solved from Equation (11). Repeat this procedure from $j = 1$ to $j = M-2$. Notice that Equation (10) is a quadratic algebraic equation in $x := \phi^a_{j+1}$, namely $ax^2 + bx + c = 0$, with $a := \widehat{w}^2_{j+1}(t) \int_t^{T_j} f^2_{j+1}(s)\psi^2_{j+1} dt$, $b := 2\widehat{w}_j(t)\, \widehat{w}_{j+1}(t)\, \rho_{j,j+1}\phi^a_j \left(\int_t^{T_{j-1}} f_j(s)\psi_j f_{j+1}(s)\psi_{j+1} ds + \phi^b_j \int_{T_{j-1}}^{T_j} f_j(s)\psi_j f_{j+1}(s)\psi_{j+1} ds\right)$, and $c := \left[\widehat{w}^2_j(t)\, \sigma^2_{jM} - \sigma^2_j\right](T_{j+1} - t)$. The real-valued solutions are given by

$$\begin{aligned} \phi^a_{j+1} &= \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \\ \phi^b_{j+1} &= \left(\frac{\sigma^2_{j+1,M}(T_{j+1} - t) - \left(\phi^a_{j+1}\right)^2 \int_t^{T_j} f^2_{j+1}(s)\psi^2_{j+1} ds}{\int_{T_j}^{T_{j+1}} f^2_{j+1}(s)\psi^2_{j+1} ds}\right)^{1/2} . \end{aligned} \quad (12)$$

   Notice that $a > 0, b < 0$ and $c > 0$ so there are two positive roots of Equation (10). We choose the smaller one as it is more likely that a real-valued solution ϕ^b_{j+1} exists in this case.

4. If we go through steps 1 to 3 successfully, repeat steps 1 to 3 with smaller α_j.

5. Stop when we have the set of smallest α_j for which the solution of a set of $\left(\phi^a_j, \phi^b_j\right)$ exists.

Finally let us remark that Equations (8) and (7) hold for caplets written on the 12-month (12M) forward LIBOR rate. If only caplets written on the 3M or 6M forward LIBOR rate are available in the market, one can easily modify (see GHLS) the approach introduced in [8] for the LMM.

In practical applications it is also sometimes preferable to reduce the number of factors driving the dynamics of the set of co-terminal forward swap rates. All above considerations apply with little modification to the case where $W^{T_j,T_M}(t)$ is a d-dimensional Brownian motion with $d < M - 1$ (see GHLS).

8 Numerical results

The numerical tests below are conducted in the EUR market using a family of co-terminal annual swaptions over a 10-year tenor structure and a family of 1-year LIBOR caplets ($T_0 = 0 < T_1 = 1 < \cdots < T_M = 10$). The market forward swap rates and Black implied volatilities correspond to data in the EUR market observed on March, 17^{th} 2003 and provided by BNPParibas proprietary systems. Empirical tests performed in USD and GBP markets, not reported here, provided qualitatively similar results to those in EUR. A different Brownian motion is used for each forward swap rate and the instantaneous correlation between the forward swap rates S_{jM} and S_{kM} is (initially) set to an estimate of the historical correlation.

8.1 Caplet pricing approximations

In order to evaluate the efficiency of the caplet pricing approximations introduced in Section 6.2, we run several Monte Carlo simulations using 100,000 paths and 16 steps per period to compute the benchmark caplet prices. These benchmark results serve as a reference against which we can compare the Rebonato, the HW, the truncated HW, the spread-option, and the rank-one analytical approaches. Our pricing engine by Monte Carlo follows the Glasserman and Zhao [14] methodology which gives very accurate results in terms of swaption prices and distribution of forward swap rates. In our tests, we use a simple correlation parameterization of the type $\rho_{jk} = e^{-\xi|j-k|}$. More complex and interesting parameterization give similar results to the ones presented here. We set $\xi = 0.01, \phi_j = 1$, $\alpha_j = 0$ and $(a_j, b_j, c_j, d_j) = (a, b, c, d)$ and use four different volatility shapes that are meant to represent real market scenarios. These shapes are plotted in Figure 4.

PLEASE INSERT

Figure 4. Instantaneous volatility term structures used in the simulation

In Table 1 we report the Mean Absolute Relative Errors (MARE) of a 2Y maturity against 12M LIBOR caplet implied volatilities and prices. The average is computed on the nine maturities for the four different shapes. Results indicate that all approaches, except the rank-one, give rather good approximations with a maximum MARE of 2.23%. The HW method seems to be the one to be preferred in practice since it outperforms all three others in most cases. As previously anticipated, using only the first two weights in the HW approximation has a marginal impact on the accuracy of the method but has the major advantage of downsizing the expression of a caplet to a simple spread of two forward swap rates. This will enable us to exploit the full facility of a fast and accurate bootstrap calibration algorithm.

Table 1: Accuracy of the different caplet pricing approximations for different volatility term structures

Whereas the rank-one approach performs reasonably well in the LMM [8], it gives unsatisfactory results in the ctSMM, especially for short maturity caplets. Indeed, whereas the weights of the swaption approximation in the LMM are always positive and much smaller than 1 for short maturity swaptions, the weights involved in the LIBOR representation as a spread of two forward swap rates have different signs and are much larger than 1 for short maturity caplets. This phenomenon leads to intrinsic numerical problems that can be best illustrated by a simple example. To fix the ideas, assume that the volatilities $\lambda_j(t) = \lambda$ are constant so that the j^{th} caplet variance reads

$$\begin{aligned} \sigma_j^2 &\approx (\widehat{w}_j|\lambda| + \widehat{w}_{j+1}|\lambda|)^2 - 2\widehat{w}_j\widehat{w}_{j+1}|\lambda||\lambda|(1-\rho_{j,j+1}) \\ &\approx \lambda^2 - 2\widehat{w}_j\widehat{w}_{j+1}\lambda^2(1-\rho_{j,j+1}), \end{aligned} \tag{13}$$

with $\widehat{w}_j > 0$, $\widehat{w}_{j+1} < 0$ and $\widehat{w}_j + \widehat{w}_{j+1} \approx 1$. The first term in Equation (13) is recognized as the result of the rank-one approximation whereas the second term is interpreted as a perturbation around it. This second term cannot be neglected for short maturity caplets since it has the same order of magnitude of the first one unless all forward swap rates are perfectly correlated. A consequence of the above equation is that the rank-one approach systematically overprices the caplet volatility.

8.2 Weight stability and distributional characteristics

In the next lines we provide some evidence concerning the stability of the weights involved in the caplet pricing approximations through numerical simulations. We use an instantaneous volatility of the form $\Lambda_{jM}(t) = \phi_j f_j(t)\psi_j(T_j - t)$, where $(a_j, b_j, c_j, d_j) = (0.08, 0.10, 0.45, 0.06)$, and a correlation structure given by $\rho_{jk} = e^{-0.01|j-k|}$, for simplicity. However, different and more consistent correlation matrix parametrizations are indeed possible [35], but the qualitative picture is not altered by this choice. The behavior of averages and standard deviations of relative changes $(w_{jj}(t) - w_{jj}(0))/w_{jj}(0)$ are plotted in Figures 5 and 6. Results indicate that the weights present very stable averages across paths. This is similar to what is observed in the LMM. In terms of probability distribution, we noticed that long term forward LIBOR rates are very close to lognormal densities whereas short-term ones do not result in such a good fit. This does not come as a surprise since all pricing approximations are much more accurate for long term caplets than from short-term ones. We have measured a positive probability for the LIBOR rate to be negative but this probability is usually very small and negligible for practical purposes. In addition, empirical results based on historical data instead of simulations confirm these findings [7].

PLEASE INSERT

[7] All results not explicitly reported here are available from the authors upon request

Figures 5 and 6. Mean and standard deviation of relative changes of weights. Results are obtained from 400,000 Monte Carlo simulations using March, 13^{th}, 2003 as the starting date and volatility term structures as defined in the text.

Lengthy but straightforward computations lead to the following characterization of the instantaneous variance of the weight $w_{jj}(t)$:

$$d\langle w_{jj}, w_{jj}\rangle(t) =$$

$$w_{jj}(t)^2 \sum_{k=j}^{M-1} \sum_{l=j}^{M-1} C_{k+1,j}(t) C_{l+1,j}(t) \lambda'_{kM}(t) \lambda_{lM}(t) S_{k,M}(t) S_{l,M}(t) dt,$$

where $C_{k+1,j}(t) = A_{k+1,j}(t) - D_{k+1,j}(t)/(\nu_j(t) - \nu_{j+1}(t))$, with $A_{k+1,j}(t) = \sum_{l=k+1}^{M-1} \delta_{l+1} \prod_{i=j+1, i\neq k+1}^{l} (1 + \delta_i S_{iM}(t))$, $D_{k+1,j}(t) = A_{k+1,j}(t) - A_{k+1,j+1}(t)$, for $k \neq j$, and $D_{j+1,j}(t) = A_{j+1,j}(t)$. Since this expression can be in principle computed along simulated paths, one may numerically compare its values to the ones taken by the instantaneous variance of the forward swap rate along the same paths $d\langle S_{jM}, S_{jM}\rangle(t) = \lambda'_{jM}(t)\lambda_{jM}(t) S_{j,M}(t)^2 dt$.

8.3 Calibration

We calibrate the ctSMM to swaption and caplet ATM volatilities by using a truncated HW method for caplet formulae. Recall that the generic form of the scalar instantaneous volatility for each forward swap rate S_{jM} is (5), with $\psi_j(T_j - t; \{\theta_j\})$ and $\phi_j(t)$ parameterized as in Equations (6) and (9). We anticipate that these functional choices, in conjunction with the aforementioned bootstrap algorithm, guarantee that the following objectives can be achieved. *i)* The calibration algorithm is extremely fast and stable between any two consecutive dates. *ii)* Caplet and co-terminal swaption volatilities are matched within their market bid-ask spread. *iii)* The resulting instantaneous forward volatilities $\Lambda_{j,M}(s), s \in [t, T_j]$ are smooth in t for a given T_j. *iv)* The perturbation functions $\phi_j(t)$ do not dramatically alter the stationary solutions associated to $\psi_j(T_j - t; \{\theta_j\})$. To the best of our knowledge, it is virtually impossible to match all above targets within a single calibration procedure in the LMM.

The first step of the calibration is to keep $\phi_j(t) = 1$, and to find a set of parameters $\{\theta_j\} := (a_j, b_j, c_j, d_j)$ to achieve a "best fit" of the initial term structure of swaption volatilities. This technique implies $M - 1$ independent least square minimizations, and is therefore fast and straightforward to achieve. We then adjust proportionally a_j, b_j and d_j (keeping c_j constant) so that each cumulative instantaneous variance exactly matches the market implied Black volatility of the swaption written on S_{jM}. We end up with a scalar instantaneous volatility $\Lambda_{jM}(t)$ which exhibits a shape consistent with the market hump and also matches the ATM volatilities of the set of co-terminal swaps, see Figure 7a. We also found that that the calibrated set $\{\theta_j\}$ is rather stable with time so that the trader does not need to re-adjust it too often.

The next logical step is to input a correlation structure ρ_{ij} in Equations (10) and (11), and to calculate function $\phi_j(t) = \phi_j f_j(t)$ using the bootstrap algorithm to match the Black implied volatility of caplets and swaptions, as explained in Section 7. At this stage, a few considerations are due. As many authors have observed (see for instance [8] and [32]) the LMM seems to be inconsistent with the market quotes of caplet and swaption volatilities in the sense that it is impossible to achieve exact calibration of both markets once an input correlation matrix has been assigned. It is often claimed that this feature is due to a "misalignment" between different swaption volatilities due to liquidity reasons. Besides, at least on the two most liquid markets, i.e. EUR and USD, typical bid/ask spreads range between 0.25% and 0.75% in lognormal units, and are therefore relatively tight. A model that is unable to reprice vanilla options within the market bid-ask volatility spread is in principle liable to generate arbitrage in hedging exotic derivatives. Usually, the LMM is calibrated on a *selection* of swaption volatilities in the ATM matrix that are meant to capture, for the problem at hand, a large portion of the actual volatility risk. In this way simultaneous calibration to caplets is possible while still keeping meaningful (i.e. smooth) instantaneous volatility functions. However, since these approaches are usually based on non-linear global minimization algorithms [32],[36], they are relatively slow and may suffer from numerical stability problems. In addition, there is no guarantee, in general, that all selected instruments can be matched within their bid/ask spread. On the other hand full nonparametric approaches aimed at calibrating the whole set of caplet and swaption volatilities, like the one proposed in [8], do not allow to achieve convergence unless the input swaption volatilities are artificially shocked from their mid-market values, often beyond their bid/ask spread. Even when convergence can be achieved, the resulting instantaneous volatility term structures are usually too rough to be used for practical purposes. These misalignment problems do not indicate that caplet/swaption arbitrage opportunities may be detected through the LMM, as already pointed out in [32]. Instead, they show that the LMM has a rigid mathematical construction and is not entirely consistent with the ATM volatility market.

In the context of the ctSMM we are faced with the same issue. In this case, however, these misalignment problems are less dramatic, and a simultaneous calibration within bid/ask spread is more easily achievable. To understand the origin of this asymmetry between the LMM and the ctSMM, we recall that in the context of the former a forward swap rate can be formally decomposed as a linear combination of $M-1$ forward LIBOR rates, while in the latter a forward LIBOR can be formally written as a weighted difference between two consecutive forward swap rates. As a consequence, while the price of a swaption within the LMM depends on a correlation matrix among LIBOR rates of dimension $M-1$, the price of a caplet within the ctSMM depends on a single correlation factor [8]. Therefore, calibrating the LMM to a set of caplets and swaptions is more problematic since any algorithm has to face the issue of coupling simultaneously $\frac{M(M-1)}{2}$ correlations and volatilities [8]. On the opposite, our parametric bootstrap algorithm for the ctSMM is essentially unidimensional since for any new maturity only one new correlation is needed and thus it only involves $M-1$

[8] In the last argument we have implicitly assumed a freezing of the weights to their initial values as before.

steps.

Table 2 summarizes the values of the historical instantaneous correlation matrix computed with three years of data from March, 17$^{\text{th}}$ 2000 to March, 17$^{\text{th}}$ 2003. We now describe one possible approach to calibration. Three other sensible approaches are discussed in GHLS but they work less well in practice.

This approach is based on a small perturbation of the correlation matrix around its historical estimate and a shift of the volatilities around their mid-market values by keeping them within their bid-ask spread. This calibration solution provides excellent results, see Table 3 and Figure 8. The calibrated α_j are very small and the perturbation $\phi_j(t)$ modifies the stationary trend $\psi_j(T_j - t; \{\theta_j\})$ by a small amount only. The choice of bumping the input matrix from the historical level can be easily justified by observing that a generic d-dimensional diffusion is fully specified by its covariance structure. In our case, we need to calibrate the model to the market-implied volatility. Thus, once the volatility structure is implied from caplets and swaptions quotes, the instantaneous correlation matrix needs not be consistent with its historical estimate. By lightly shocking the input correlation from historical levels, we are indirectly inferring some information from the market on the correlation itself, although a direct implied calibration of the whole correlation matrix is impossible. On this last point, we refer to [32].

PLEASE INSERT

Table 2. Historical correlation matrix

Table 3. Results for calibration methodology 4) . See text for explanation.

Figure 7. Instantaneous volatility curves before calibration to caplets

Figure 8. Instantaneous volatility curves after calibration to caplets

In Table 3a we compare the market Black volatilities of caplets and swaptions against the volatilities used in the calibration. We see that differences in volatility are always less than 0.27%. Thus all errors are within the bid/ask spread in the EUR volatility market. Table 3b shows the differences between the correlation matrix used in the calibration and its historical estimate [9]. Finally, the values of the different parameters that define the instantaneous volatility functions after calibration are summarized in Table 3c. Notice, in particular, that calibration can be achieved with values of α_j smaller than 0.15.

Figure 7 shows the shape of the instantaneous volatility functions for the different swap rates as a function of t with fixed maturity T_j after calibration to the initial term structure of swaption Black volatilities. In Figure 8 we plot the same functions after having performed a joint calibration on caplets and swaptions. It is worth noticing that the initial shapes (consistent with a full stationary model) are still preserved to a large extent after calibration and they are, in addition, very smooth.

[9] When shocking the instantaneous correlation matrix from its historical values we have checked that its positive definite nature is preserved. Alternatively, one could use a parametric correlation matrix.

9 Concluding remarks

In this paper we have studied a general approach suitable to price IR derivatives. This "Market Model Approach" gives birth to three major classes: the co-terminal, co-initial and co-sliding SMM. The Market Model Approach is based on the concept of admissibility of a set of forward swap rates. We presented and analyzed the link between these concepts and graph theory. In particular, we proved that the LMM is the only admissible co-sliding SMM.

By further developing the important example of the co-terminal SMM, we have shown that accurate and fast approximations are available in that setting. Besides user-friendly calibration algorithms work efficiently in terms of speed and stability properties. They further lead to smooth and meaningful shapes for the instantaneous forward volatility of forward swap rates, while delivering an almost perfect match of both swaption and caplet implied Black volatilities. An application to Bermudan swaptions reported in GHLS also reveals the interest of the approach when pricing exotic IR instruments. Some important theoretical extensions (and their respective calibration) related to the inclusion of stochastic volatility or the generalization to multicurrency underlyings are left to future research.

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A Appendix : Proof of Proposition 3

Consider the tenor structure depicted in Section 2, namely $\mathcal{T} := \{T_1, \cdots, T_M\}$. Assume that a generic family $\mathcal{S} := \{S(t)\}$ of distinct forward swap rates is given. In general, the set $\mathcal{S}$ comprises N elements. We use Greek letters to indicate start/end dates of each forward swap rate belonging to the set, i.e.

$$S_{\alpha_1\beta_1}(t) := \frac{B(t,T_{\alpha_1}) - B(t,T_{\beta_1})}{G_{\alpha_1\beta_1}(t)}, \cdots, S_{\alpha_N\beta_N}(t) := \frac{B(t,T_{\alpha_N}) - B(t,T_{\beta_N})}{G_{\alpha_N\beta_N}(t)}.$$

Let us start with the proof of the sufficient part. Definition 2 implies that $N = M - 1$. Therefore, the following linear homogeneous system

$$\begin{aligned} B(t,T_{\alpha_1}) - B(t,T_{\beta_1}) &= S_{\alpha_1\beta_1}(t) \sum_{k=\alpha_1+1}^{\beta_1} \delta_k B(t,T_k), \quad (14) \\ &\cdots \\ B(t,T_{\alpha_{M-1}}) - B(t,T_{\beta_{M-1}}) &= S_{\alpha_{M-1}\beta_{M-1}}(t) \sum_{k=\alpha_{M-1}+1}^{\beta_{M-1}} \delta_k B(t,T_k), \end{aligned}$$

comprises $M-1$ equations in the n unknowns $B(t,T_{\alpha_1}), \cdots, B(t,T_{\beta_1})$; $B(t,T_{\alpha_2}), \cdots, B(t,T_{\beta_2})$; $\cdots$; $B(t,T_{\alpha_{M-1}}), \cdots, B(t,T_{\beta_{M-1}})$. To simplify the notation, we introduce the set of dates $\mathcal{U} := \left\{ \left(T_{\alpha_1}, \cdots, T_{\beta_1}\right), \cdots, \left(T_{\alpha_{M-1}}, \cdots, T_{\beta_{M-1}}\right) \right\}$.

The number n of independent unknowns is fixed by Definition 2, and the fact that all dates must be in the tenor structure. If any date in the tenor structure coincides with at least one reset/settlement date of a swap rate in $\mathcal{S}$, then $\mathcal{U} \supseteq \mathcal{T}$. At the same time, since by construction the set $\mathcal{S}$ is defined in relation to the above tenor structure, the inclusion $\mathcal{U} \subseteq \mathcal{T}$ must also hold. Therefore, we have that $\mathcal{T} = \mathcal{U}$ identically. Since the cardinality $|\mathcal{U}| = |\mathcal{T}| = M$, we deduce that the above linear system $\mathcal{L}(M-1, M)$ comprises $M-1$ equations and M independent unknowns. Let C be the rectangular $(M-1) \times M$ matrix associated to (14). In a more compact notation,

$$CB = 0, \;\; B := (B(t,T_1), \cdots, B(t,T_M))'. \quad (15)$$

If we consider $B(t,T_{\alpha_1}) - B(t,T_{\beta_1}) = S_{\alpha_1\beta_1}(t) \sum_{k=\alpha_1+1}^{\beta_1} \delta_k B(t,T_k)$, for example, the corresponding row of C reads as

$$\left(\cdots 0, \quad 1, \quad -\delta_{\alpha_{1+1}} S_{\alpha_1\beta_1}(t), \cdots - \delta_{\beta_{1-1}} S_{\alpha_1\beta_1}(t), \quad -(1 + \delta_{\beta_1} S_{\alpha_1\beta_1}(t)), \quad 0 \cdots \right),$$

the 1 entry being in column α_1.

Consider the set of deflated discount bonds relative to $B(t,T_i)$, that is $\widetilde{B}_i(t,\cdot) := B(t,\cdot)/B(t,T_i)$. Since by construction $T_i \in \mathcal{T}$, by dividing both sides of all equations in (15) by $B(t,T_i)$, we obtain a new set of $M-1$ linear equations in the $M-1$ deflated discount bond prices $\widetilde{B}_i(t,T_1), \cdots, \widetilde{B}_i(t,T_{i-1}), 1, \widetilde{B}_i(t,T_{i+1}), \cdots, \widetilde{B}_i(t,T_M)$. For any $(M-1) \times M$

homogeneous system (15) it exists an associated $(M-1) \times (M-1)$ non-homogeneous system on the corresponding deflated discount bond prices, i.e.

$$\begin{aligned} D\widetilde{B} &= \Psi, \\ \widetilde{B} &:= \left(\widetilde{B}_i(t,T_1), \cdots, \widetilde{B}_i(t,T_{i-1}), \widetilde{B}_i(t,T_{i+1}), \cdots, \widetilde{B}_i(t,T_M)\right)', \end{aligned} \tag{16}$$

where vector Ψ entries are either zero or one, with at least one non-vanishing entry.

A necessary and sufficient condition for this system to possess a unique solution is that matrix C has full rank. In fact, if $rank(C) = M-1$, then $rank(D) = M-1$ as well, for any choice of $B(t,T_i)$, $i = 1, \ldots, M$, as numéraire. In this case, the existence of a unique set of discount bond prices is guaranteed by elementary theorems of linear algebra. On the opposite, if two or more rows of C are linearly dependent, then $rank(C) < M-1$. In this case, depending on the choice of the numeraire, i.e., $B(t,T_i)$ $i = 1, \ldots, M$, matrix D might not have full rank and therefore there is no guarantee that a solution exists for any choice of the numéraire. In other words, the existence and unicity of solutions is reduced to the study of the linear dependence among the rows of C.

We therefore analyse the structure of C when the set of forward swap rates is admissible. By construction, D is not block-diagonal, and none of the rows of C contains only zero elements [10]. Furthermore the first column of C, namely the column made of the coefficients associated to the shortest discount bond $B(t,T_1)$, contains either a single 1 entry (corresponding to a single swap starting at T_1) or multiple 1 entries (corresponding to many swaps starting at T_1). In the first case the remaining $M-2$ elements of the first column entries are all zeros. Hence, it is impossible to find a linear combination of the rows reducing to a zero vector. If there are multiple 1 entries then, once again, the remaining entries of the column must fill in with zeros. Unlike before, it is possible to find linear combinations of the rows which will annihilate the first entry of the resulting vector. However, once such a combination is held fixed, it cannot annihilate the other entries and yield a zero vector for any realization of the diffusion processes $S_{\alpha_1\beta_1}(t), \cdots, S_{\alpha_{M-1}\beta_{M-1}}(t)$. Note, however, that this argument does not exclude the possibility that a zero vector is indeed obtained by a linear combination of the rows for one particular realisation of the set of swap rates. This event occurs with probability 0 with respect to $\mathbb{P}$. In summary, if the set of forward swap rates is admissible, the system (16) admits a unique solution in terms of deflated discount bond prices, $\mathbb{P}$-a.s. and this holds for any t in the specified time interval. This ends the proof of the sufficient part.

We now present the proof of the necessary part. If a system of $M-1$ deflated discount bond prices admits a unique solution as a function of a set of distinct forward swap rates, then necessarily it must be a linear non-homogeneous system $\mathcal{L}(M-1, M-1)$. This in turn implies that $\mathcal{T} = \mathcal{U}$ and that $|\mathcal{S}| = M-1$. Condition 1 in Definition 2 is then satisfied. Then, inclusions $\mathcal{U} \supseteq \mathcal{T}$ and $\mathcal{U} \subseteq \mathcal{T}$

[10] If the matrix were block-diagonal, then the graph associated to the tenor structure would be made of two separate subgraphs. Proposition 4 shows that this is incompatible with the notion of admissibility.

must simultaneously hold. The former constraint is equivalent to Condition 2 in Definition 2.

To prove that Condition 3 is also satisfied, we proceed by contradition. Assume that set $\mathcal{S}$ is not admissible. Given that Conditions 1 and 2 are already satisfied, this is equivalent to assume that Condition 3 is not, i.e. there exists at least one degenerate tenor structure subset $\mathcal{C}$ in $\mathcal{T}$. Now recall the definitions of $\mathcal{H}^u$ and $\mathcal{H}^d$ given in the main text. For every pair of consecutive dates T^u_{i-1}, T^u_i in $\mathcal{H}^u$ the following equation holds by absence of arbitrage opportunities $S^u_{i-1,i}(t)G^u_{i-1,i}(t) = B(t, T^u_{i-1}) - B(t, T^u_i)$ with obvious notations. A similar result holds for all dates in the set $\mathcal{H}^d$. The following two identities:

$$\begin{aligned}
\sum_{i=2}^{m} \left[B(t, T^u_{i-1}) - B(t, T^u_i)\right] &= B(t, T^u_1) - B(t, T^u_m), \\
\sum_{i=2}^{n} \left[B(t, T^d_{i-1}) - B(t, T^d_i)\right] &= B(t, T^d_1) - B(t, T^d_n),
\end{aligned}$$

imply that $\sum_{i=2}^{m} \left[B(t, T^u_{i-1}) - B(t, T^u_i)\right] = \sum_{i=2}^{n} \left[B(t, T^d_{i-1}) - B(t, T^d_i)\right]$, since $B(t, T^u_1) = B(t, T^d_1) = B(t, T_a)$ and $B(t, T^u_m) = B(t, T^d_n) = B(t, T_b)$. By absence of arbitrage opportunities, we deduce that

$$\sum_{i=2}^{m} S^u_{i-1,i}(t)G^u_{i-1,i}(t) = \sum_{i=2}^{n} S^d_{i-1,i}(t)G^d_{i-1,i}(t).$$

This identity implies that one equation in the system (15), among those associated to $\mathcal{H}^u$ and $\mathcal{H}^d$, is redundant. Therefore, the $M-1$ equations in the system do not form a linearly independent set, and $rank(C) < M-1$. In that case there is no guarantee that a solution exists for a generic choice of $B(t, T_i)$, $i = 1, \ldots, M$, as numéraire. This contradicts the hypothesis.

B Appendix : Proof of Proposition 4

If the set $\mathcal{S}$ is admissible then, after Proposition 3, there exists a unique set of deflated discount bond prices $\widetilde{B}_i(t, T_1)$, $\cdots$, $\widetilde{B}_i(t, T_{i-1}), 1, \widetilde{B}_i(t, T_{i+1})$, $\cdots$, $\widetilde{B}_i(t, T_M)$ relative to $B(t, T_i)$. This set is the unique solution of a non-homogeneous linear system

$$\begin{aligned}
D\widetilde{B} &= \Psi, \qquad (17)\\
\widetilde{B} &:= \left(\widetilde{B}_i(t, T_1), \cdots, \widetilde{B}_i(t, T_{i-1}), \widetilde{B}_i(t, T_{i+1}), \cdots, \widetilde{B}_i(t, T_M)\right)',
\end{aligned}$$

and vector Ψ entries are either zero or one, with at least one non-vanishing entry. The solution can be easily found by means of the Cramer's rule: the j-th. solution $\widetilde{B}_i(j)$ to (17) can be symbolically expressed as

$$\widetilde{B}_i(j) = \frac{\det[D^{(j)}_i]}{\det[D]}; \quad j = 1, \cdots, M-1,$$

where $D^{(j)}$ is the square matrix obtained from D by replacing its j-th. column with vector Ψ. A sufficient condition for $\widetilde{B}_i(j)$ to be non zero is that $\det[D_i^{(j)}] \neq 0$. A necessary and sufficient condition for a determinant to be zero is that its rows are linearly independent. By assumption, matrix C (associated to the homogeneous system (15)) has rank $r = M - 1$. This implies that its rows are linearly independent. Also, we notice that $-1 \times \Psi$ coincides with one of the columns of C. Thus, the columns of $D_i^{(j)}$ coincide with the $M - 1$ columns of C, apart from an irrelevant sign in one of the columns. From a general property of linear spaces, we deduce that the rows of matrix $D_i^{(j)}$ must necessarily be linearly independent, too. For the same reason as above, it might happen that for a given realization of the set of forward swap rates, two (or more) rows of $D^{(j)}$ are linearly dependent. This event, however, occurs with probability 0 with respect to $\mathbb{P}$. This ends the proof.

C Appendix : Proof of Proposition 7

We need to prove that Conditions 1, 2 and 3 in Definition 2 are necessary and sufficient for Gr to be connected and have M vertices linked by $M - 1$ edges.

We start by proving the necessary part. We observe that Condition 2 in Definition 2 means that the graph Gr has no isolated vertices. In particular all vertices of Gr have a minimum degree of 1. Since a connected graph has a minimum degree of 1 by definition, Condition 2 of Definition 6 implies Condition 2 of Definition 2. The following result can be found in [5]: If Gr is a connected graph, then Gr has a spanning tree subgraph Gr'. Thus Gr' has M vertices and $M-1$ edges. By Condition 1 of Definition 6 this implies that Gr is itself a tree. Since a tree is acyclic, this implies that Condition 3 of Definition 2 is satisfied too. Conditions 1 of the two definitions are trivially the same. This ends the first part of the proof.

To prove the sufficient part we assume, by contradiction, that Gr is not connected. In this case, let $Gr = \{Gr_1, Gr_2, \cdots, Gr_l\}$ be its decomposition in disjoint subgraphs $Gr_1, Gr_2, \cdots, Gr_l$. From Condition 2 of Definition 2, none of them has an isolated vertex. From Condition 1 of Definition 2, we deduce then that one among $Gr_1, Gr_2, \cdots, Gr_l$ is a graph with a number of edges equal to the number of vertices minus 1. All other subgraphs, on the other side, must be restricted to have an equal number of edges and vertices. This is the only partition that is compatible with the total number of edges, $M-1$, and vertices, M. We assume, with no loss in generality, that graph $Gr_i = \{V_i, E_i\}$ has $n_i - 1$ edges and n_i vertices. Since it is connected then, from the same argument as before, Gr_i is a tree. Thus, it is acyclic. We next consider one among the remaining $l - 1$ graphs $Gr_j = \{V_j, E_j\}$ with n_j edges and vertices. Since it is connected, let Gr'_j be its spanning tree subgraph having $n_j - 1$ edges. The following result can be found in [5]: a tree H is maximally acyclic. This means that a tree H contains no cycle but $H + xy$ does, for any two non-adjacent vertices x and $y \in H$. Since Gr_j is obtained by the tree Gr'_j by adding one edge between non-adjacent vertices [11], it must contain a cycle which is in contrast with Condition 3 in Definition 2. The contradiction comes from having assumed that Gr is not connected. This ends the second part of the proof.

D Appendix : Proof of Proposition 10

From Proposition 9 we know that a set is admissible if and only if its graph is a tree. Since a tenor structure is made of ordered dates, we must attach date labels $\{T_1, \cdots, T_M\}$ to all nodes of the tree. In this way, isomorphic trees are considered distinct for the purpose of enumeration. Such graph is known in graph theory as "labeled" tree. The number of labeled trees on M nodes is known to be M^{M-2} after Cayley [4], which yields the stated result.

[11] This is a direct consequence of the hypothesis on the set S comprising distinct swap rates. In fact, if all swap rates are distinct, two vertices cannot be both incident on two different edges.

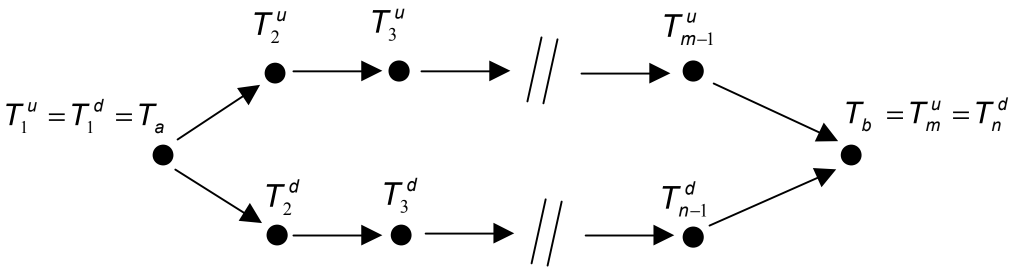


Figure 1.a: Degenerate tenor structure

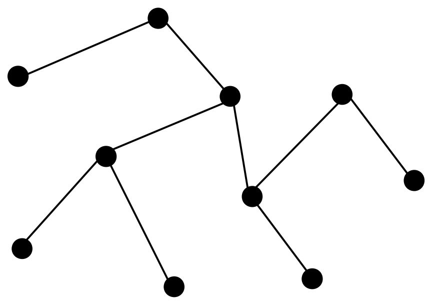


Figure 1.b: Example of a tree

Figure 1. Degenerate tenor structure and example of a tree

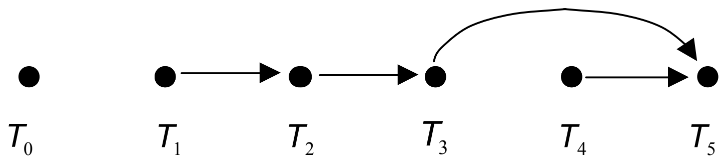


Figure 2.a: Example of an admissible set

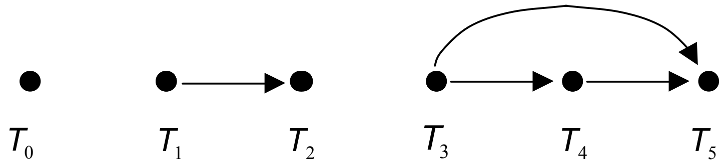


Figure 2.b: Example of a non-admissible set

Figure 2. Examples of admissible and non-admissible sets of forward swap rates

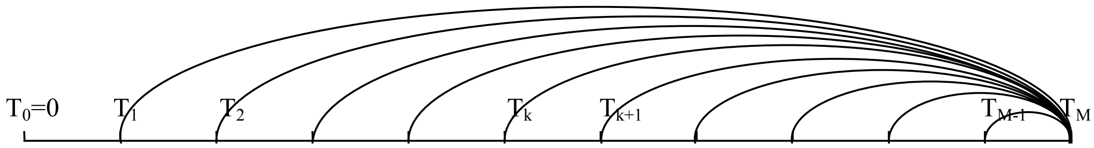


Figure 3.a: Co-terminal forward swap rates

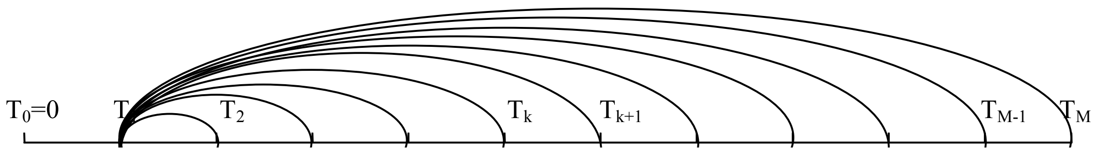


Figure 3.b: Co-initial forward swap rates

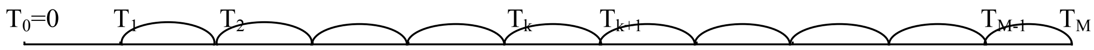


Figure 3.c: Co-sliding forward swap rates

Figure 3. Sets of co-terminal, co-initial and co-sliding forward swap rates

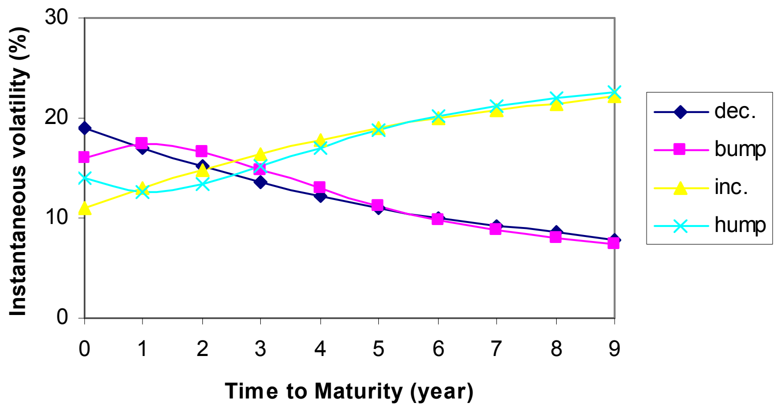
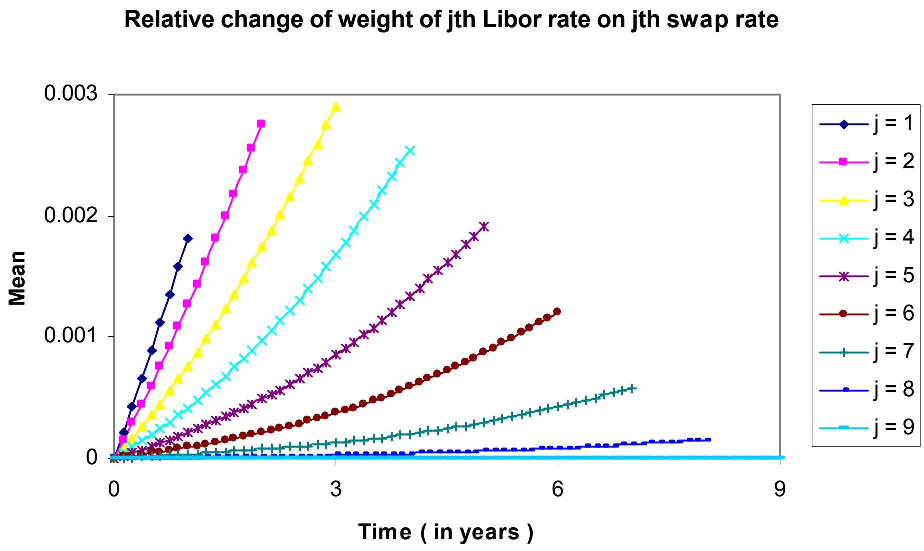
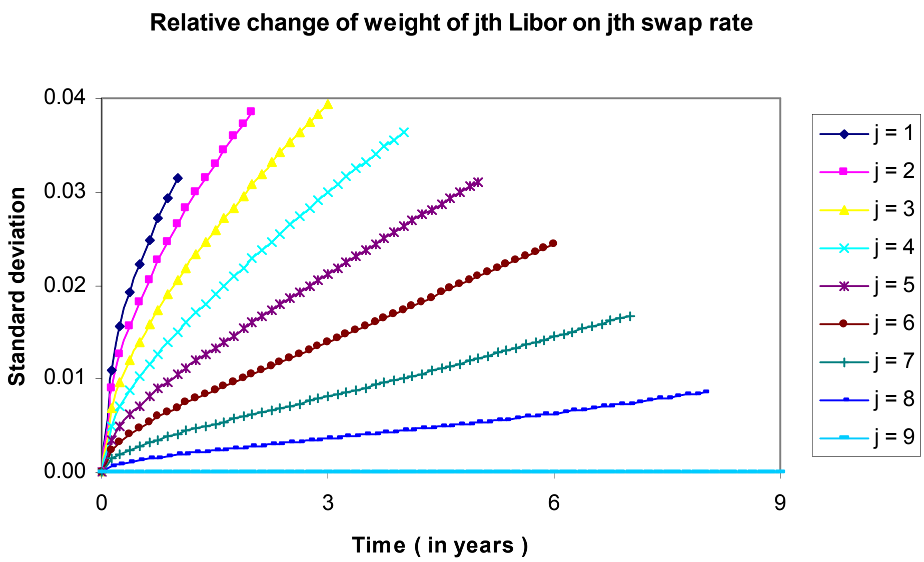


Figure 4. Instantaneous volatility term structures used in the simulation

Implied Vol	HW	Truncated HW	Rebonato	Spread Option	Rank One
Dec.	0.27%	0.43%	1.15%	0.74%	9.25%
Bump	0.64%	0.35%	1.59%	1.03%	17.02%
Inc.	0.79%	0.90%	1.31%	2.23%	38.73%
Hump	0.79%	1.25%	0.67%	1.40%	33.50%
Prices	**HW**	**Truncated HW**	**Rebonato**	**Spread Option**	**Rank One**
Dec.	0.26%	0.42%	1.12%	0.73%	9.09%
Bump	0.63%	0.35%	1.56%	1.02%	16.84%
Inc.	0.79%	0.89%	1.30%	2.21%	38.64%
Hump	0.79%	1.24%	0.66%	1.39%	33.40%

Table 1. Accuracy of the different caplet pricing approximations for different volatility term structures





Figures 5 and 6. Means and standard deviations of relative changes of weights. Results are obtained from 400,000 Monte Carlo simulations using March, 13th, 2003 as the starting date and volatility term structures as defined in the text

Historical correlation with percentage format:

Rho(j,k)	1	2	3	4	5	6	7	8	9
1	100%	99.09%	96.71%	94.17%	91.14%	88.71%	87.29%	86.45%	85.42%
2	99.09%	100%	99.07%	97.32%	95.00%	93.02%	91.76%	90.93%	89.74%
3	96.71%	99.07%	100%	98.73%	97.41%	95.91%	94.78%	93.99%	92.65%
4	94.17%	97.32%	98.73%	100%	99.42%	98.55%	97.80%	97.04%	95.81%
5	91.14%	95.00%	97.41%	99.42%	100%	99.70%	99.19%	98.53%	97.19%
6	88.71%	93.02%	95.91%	98.55%	99.70%	100%	99.79%	99.30%	98.07%
7	87.29%	91.76%	94.78%	97.80%	99.19%	99.79%	100%	99.69%	98.58%
8	86.45%	90.93%	93.99%	97.04%	98.53%	99.30%	99.69%	100%	99.22%
9	85.42%	89.74%	92.65%	95.81%	97.19%	98.07%	98.58%	99.22%	100%

Table 2. Historical correlation matrix

Calibration methodology 4)

Swaption_j	Market volatility	Adjusted volatility
1	0.1750	0.1723
2	0.1598	0.1574
3	0.1508	0.1487
4	0.1443	0.1425
5	0.1386	0.1371
6	0.1345	0.1333
7	0.1314	0.1305
8	0.1301	0.1295
9	0.1289	0.1286

Caplet_j	Market volatility	Adjusted volatility
1	0.2871	0.2898
2	0.2449	0.2473
3	0.2012	0.2033
4	0.1801	0.1819
5	0.1635	0.1650
6	0.1513	0.1525
7	0.1396	0.1405
8	0.1339	0.1345
9	0.1289	0.1292

Table 3.a : Market and calibration-adjusted prices of swaption and caplets

Rho(j,k)	1	2	3	4	5	6	7	8	9
1	0.00%	0.45%	1.64%	2.91%	4.43%	5.64%	6.35%	6.77%	7.29%
2	0.45%	0.00%	0.46%	1.34%	2.50%	3.49%	4.12%	4.53%	5.13%
3	1.64%	0.46%	0.00%	0.63%	1.29%	2.04%	2.61%	3.00%	3.67%
4	2.91%	1.34%	0.63%	0.00%	0.29%	0.72%	1.10%	1.48%	2.09%
5	4.43%	2.50%	1.29%	0.29%	0.00%	0.15%	0.40%	0.73%	1.40%
6	5.64%	3.49%	2.04%	0.72%	0.15%	0.00%	0.10%	0.35%	0.96%
7	6.35%	4.12%	2.61%	1.10%	0.40%	0.10%	0.00%	0.15%	0.71%
8	6.77%	4.53%	3.00%	1.48%	0.73%	0.35%	0.15%	0.00%	0.39%
9	7.29%	5.13%	3.67%	2.09%	1.40%	0.96%	0.71%	0.39%	0.00%

Table 3.b : Differences between the input and the historical correlation matrix

Forward swap j	a_j	b_j	c_j	d_j	phi1	phi2	Alpha
1	0.0000	0.0844	0.3320	0.1030	0.0000	1.0592	0.1500
2	0.0000	0.0869	0.3526	0.0970	1.2364	1.0283	0.1500
3	0.0000	0.0930	0.3649	0.0932	1.2987	1.0464	0.1500
4	0.0000	0.1008	0.3626	0.0894	1.3962	1.0003	0.1500
5	0.0000	0.1103	0.3757	0.0864	1.4650	0.9753	0.1400
6	0.0134	0.1141	0.4450	0.0833	1.4479	0.9594	0.1100
7	0.0267	0.1189	0.5058	0.0820	1.4326	1.0091	0.0900
8	0.0524	0.1221	0.6096	0.0835	1.4325	1.1208	0.0800
9	0.1041	0.1049	0.7152	0.0849	1.3703	1.1292	0.0600

Table 3.c : Calibrated volatility term structures using adjusted swaptions and caplets

Table 3. Results for calibration methodology 4). See text for explanation

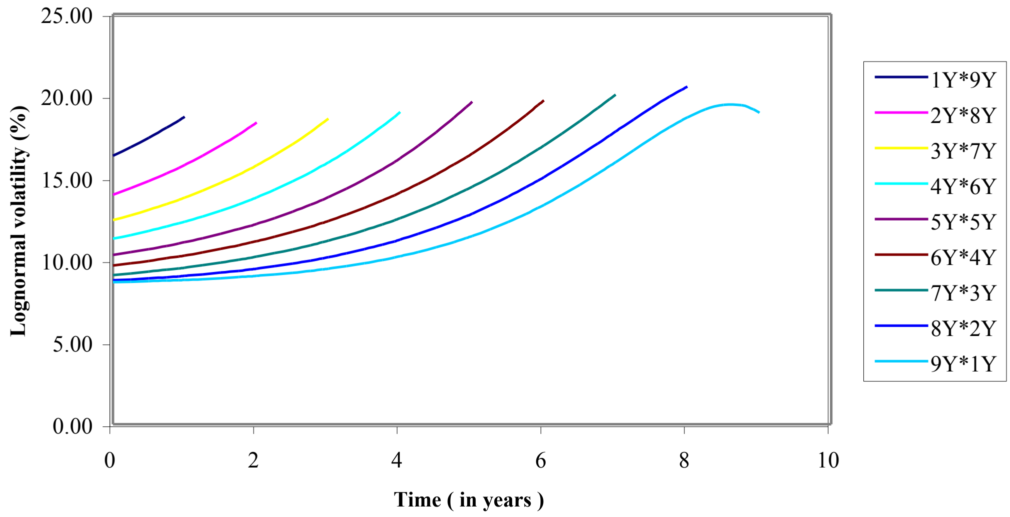


Figure 7. Instantaneous volatility curves before calibration to caplets

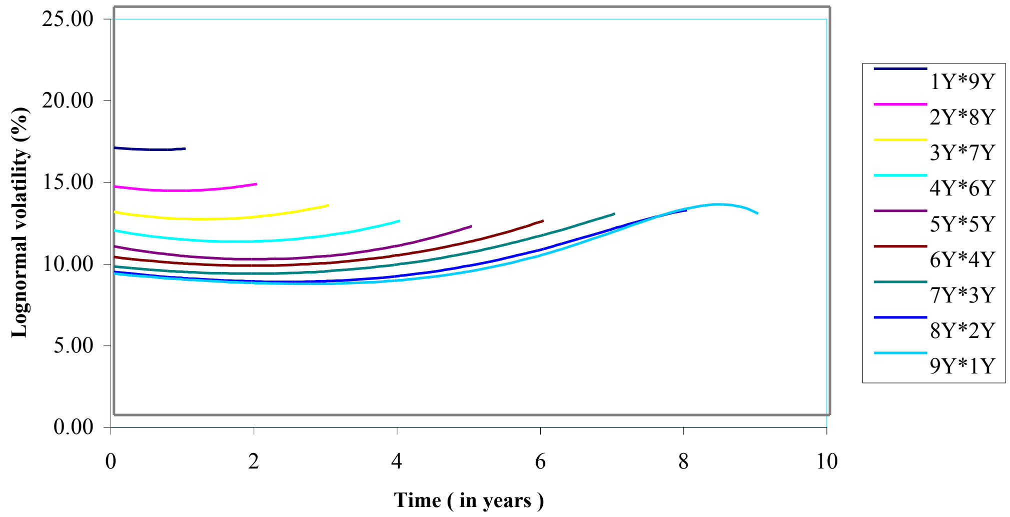


Figure 8. Instantaneous volatility curves after calibration to caplets