Reaching nirvana with a defaultable asset?

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Preliminary and incomplete

Abstract

Before the ongoing crisis, the perception of boundless liquidity and the ability of defaultable assets to offer an 'yield pickup' (a positive excess expected return that persists when volatility falls) drove an extreme appetite for those assets. We parsimoniously show how, in an arbitrage-free model of dynamic asset allocation, such ingredients are indeed conducive to massive long positions in a default-prone security.

1 Introduction

There is more than anedoctal evidence that, at least until the Spring 2007, defaultable securities were in huge demand. They were able to command an yield pickup at a time of surging values for their own asset class and for many other asset classes. A widespread sense of unlimited liquidity supported the picture of deleveraging as a basically painless act. Both the yield pickup and the perception of boundless liquidity

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helped investors to feel comfortable about implementing giant carry trades in those securities. The result were colossal long geared positions in default-prone assets.

Can we parsimoniously rationalize this? We show how, under a thrifty no-arbitrage model of dynamic portfolio choice, oceanic liquidity and the ability of a defaultable asset to provide an 'yield pickup' (a positive risk premium that endures at times of subdued volatility) can lead an aggressive investor to take massive long positions in it.

Our result brings fresh evidence on the importance of perceived liquidity for investment strategies. Longstaff (2001) and Liu, Longstaff, and Pan (2003) show how liquidity risk and event risk (viewed as a form of liquidity risk) makes an investor behave as if she faced borrowing constraints even though none are imposed. They consider non-defaultable investables. We show that, even with a defaultable asset, perceived absence of liquidity risk is conducive to extreme levered positions.

The CEV price process greatly helps interpretation and tractability by empowering straightforward analytical calculations for the chance of the asset's survival as well as for the dynamic optimal portfolios. In particular, optimal portfolios will have the well-known form studied by Kim and Omberg (1996). Kim and Omberg (1996) have provided the workhorse model for any closed-form dynamic portfolio problem with a stochastic opportunity set. Our paper is the first to use their model to explicitly calculate the optimal allocation of wealth to a defaultable security.

The paper is organized as follows. Section 2 details the features of the CEV-type defaultable asset. Section 3 shows how enormous long geared positions in a default-prone security can be the rational outcome of a dynamic portfolio problem with a CEV-type defaultable asset. Section 4 draws preliminary conclusions.

2 The defaultable asset

There are essentially three properties we want the price S of the defaultable asset to have: (i) it must be arbitrage-free; (ii) it must be consistent with easy deleveraging; (iii) it must render default possible and the probability of its occurrence analytic.

Property (i) is meant to rule out the possible emergence of extreme portfolios due to the existence of free lunches. In fact, investors with non-satiated preferences may be unwilling to miss opportunities of adding non-negative bits to their final wealth by means of self-financing strategies with zero initial cost. Property (ii) is related to investors' perception of boundless liquidity. The unwinding of long levered positions in the asset is sensed to be unproblematic. In other words, the risk that sizeable drops in investors' wealth may occur before they have the opportunity to adjust their portfolio is perceived as far-fetched. This is effectively captured by a price process with continuous trajectories. Property (iii) is obviously related to the defaultable nature of the asset of interest. Without loss of generality, we assume zero recovery.

The Constant-Elasticity-of-Variance (CEV) price process[1] with negative elasticity parameter β was introduced by Cox (1975), its parametrization is utterly parsimonious, and it tidily meets the three properties. Its dynamics is

$$\frac{dS_t}{S_t} = (r + X_t\sigma_t)\,dt + \sigma_t dZ_t, \quad \sigma_t = S_t{}^{\beta}, \quad -1 < \beta < 0, \quad S_0 = s \geq 0, \tag{1}$$
$$X_t = \frac{\mu - r}{\sigma_t} \quad \text{(Sharpe ratio)},$$

where the expected return on the asset is the constant μ and the riskfree rate is r, with $\mu > r$ (Z_t is a Wiener process). From the boundary classification, the point 0 is an attainable state only for $\beta < 0$. As soon as the process S falls down to zero, we confine it there so that the level 0 becomes the absorbing state for the asset price process (consistently with zero recovery at default). This is property (iii).

Interestingly, default becomes possible only if asset returns' local volatility inflates as the asset price deteriorates ($\beta < 0$). In fact, this can be realistically interpreted as the impact of price-discovery activity that accompanies falling prices. Even more interestingly, what makes default possible engenders also what we refer to as the asset's 'yield pickup' (the return spread $\mu - r$ is positive and constant no matter how small the local volatility S^{β} is).

[1]The geometric Brownian motion asset price process of Black and Scholes (1973) is obtained with the special case $\beta = 0$ and the square-root process of Cox and Ross (1976) is obtained with $\beta = -\frac{1}{2}$.

The CEV price process patently enjoys property (ii). Property (i) is embodied by the following proposition.

Proposition 1 *The CEV price process with $\beta \in (-1, 0)$ complies with the no-arbitrage assumption.*

Proof. See Theorem 2.3 in Delbaen and Shirakawa (2002). ■

The CEV price process nicely substantiates property (iii) with an analytical expression for the probability of default.

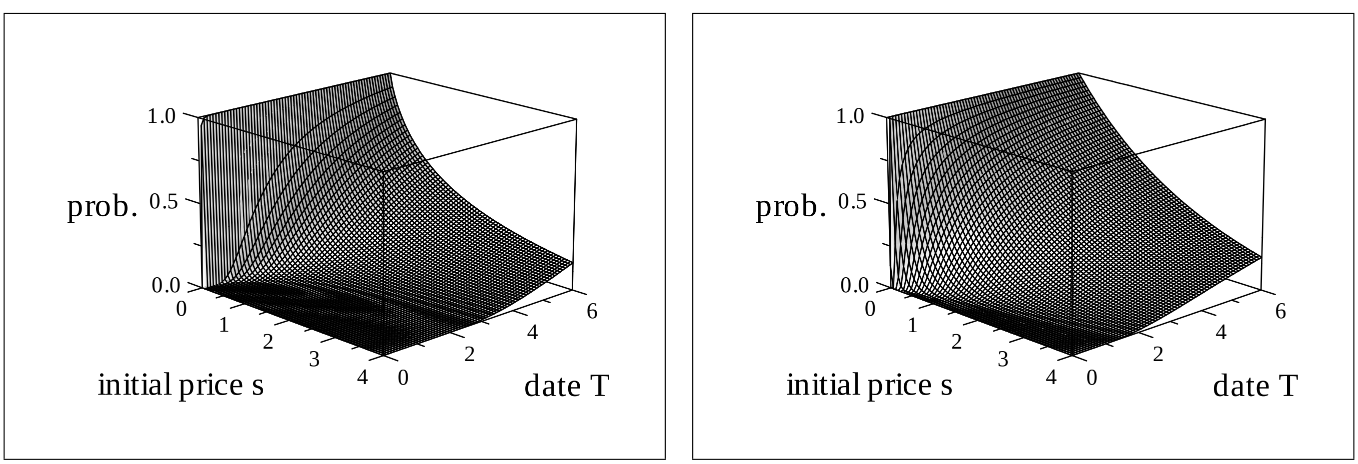
Proposition 2 *If the asset price process follows a CEV price process with $\beta \in (-1, 0)$, then the objective probability of the asset defaulting within the date $T > 0$ admits a closed form (see the Appendix) and is bounded from above as follows:*

$$\mathbb{P}\left[S_h = 0,\ 0 \leq h \leq T \mid S_0 = s\right] \leq \frac{1 - e^{2\mu\beta T}}{2\mu s^{-2\beta}} .$$

Proof. See the Appendix, which reports Cox's (1975) derivation of the probability of default and Randal's (1998) result for the upper bound. ■

Given $\mu = 7\%$, Figure 1 plots the probability of default against the current price s and the date $T > 0$. As s approaches 0, the probability evidently converges to 1 for any T.

Figure 1: Probability of default within date T (years)



$\beta = -0.2$ $\beta = -.5$

A crucial implication can be drawn by combining Proposition 1 with the following Proposition 3. Under no arbitrage, default risk balances the presence of bullish price trajectories along which the Sharpe ratio X_t bloats. If no free lunches are to emerge, default risk must counteract the 'yield pickup' on the defaultable asset: The ability of the asset to create enticing circumstances in which subdued volatility is associated with a positive excess expected return $\mu - r$ must be 'offset' by the chance of default.

Proposition 3 *Given* $\beta \in (-1, 0)$*, for the positive price process of the CEV model, i.e. :* $\{S_t; 0 \leq t \leq T\}$ *under the conditional objective probability measure* $\mathbb{P}[\ \cdot \mid S_T > 0]$*, there always exist arbitrage opportunities.*

Proof. Theorem 4.2 of Delbaen and Shirakawa (2002) holds. ■

The next section shows that the allure of the CEV price trajectories along which the Sharpe ratio X_t swells can become supreme for sufficiently aggressive investors. Their full confidence in painless deleveraging can make them 'forget' about the balancing force represented by default risk.

3 Dynamic asset allocation and nirvana

The choice of the CEV price process for the defaultable asset value shows all its potential by enabling an uncomplicated and closed-form dynamic portfolio analysis. We closely follow the setup in Kim and Omberg (1996). The investor trades two assets, a risk-free money-market account and the risky defaultable asset. There is no consumption or labor income during the investment horizon. The investor has Constant-Relative-Risk-Aversion (CRRA) utility from final wealth:

$$U(W) = (W)^{1-\frac{1}{\gamma}} \qquad \text{with} \qquad \gamma \in (0,\infty),$$

$$\frac{-WU_{WW}}{U_W} = \frac{1}{\gamma} \in (0,\infty) \quad \text{(constant relative risk aversion)},$$

$$\gamma \to +\infty \qquad \text{implies risk neutrality},$$

$$\gamma \to 1 \qquad \text{implies log utility}.$$

The defaultable asset price S_t has the CEV dynamics specified in (1) and the point 0 is its cemetery state. By construction, the Sharpe ratio X_t is always non-negative, also admits a cemetery state at 0, and the correlation between its innovations and risky-asset price innovations is 1. The current Sharpe ratio X supplies all currently available information on current and future investment opportunities. Indeed, Nielsen and Vassalou (2006) show that, in typical continuous-time portfolio problems, the only time-variation that matters for portfolio choice is the time-variation in the slope (the Sharpe ratio) and the intercept (the riskfree rate) of the instantaneous capital market line.

Let τ be the investor's current horizon and $y^*(W,X,\tau)$ be the optimal monetary investment in the defaultable asset. The following proposition qualifies dynamic optimal portfolios[2].

Proposition 4 *The optimal fraction of wealth allocated to the defaultable asset is*

$$\frac{y^*(W,X,\tau)}{W} = \underbrace{\gamma\frac{X^2}{\mu - r}}_{myopic\ demand} + \underbrace{\gamma(-\beta)\,C(\tau)\,X^2}_{hedging\ demand},$$

$$C(\tau) = \frac{2a\left(1-\exp\left(-\sqrt{q}\tau\right)\right)}{2\sqrt{q} - \left(b+\sqrt{q}\right)\left(1-\exp\left(-\sqrt{q}\tau\right)\right)},$$

[2] Gao (2009) studies the investment problem for a defined-contribution pension plan that can allocate wealth to a CEV stock. Gao (2009) does not use the asset-allocation analysis developed by Kim and Omberg (1996) so that nirvana solutions and a detailed Sharpe-ratio-based discussion of the hedging demands' sign are absent in his work. Absent is also the consideration of CEV-stock default's impact on optimal allocation.

$$
\begin{aligned}
q &= 4\beta^2\left(r^2+\gamma\left(\mu^2-r^2\right)\right), \\
a &= \gamma-1, \\
b &= 2\left(\left(1-\gamma\right)\beta\left(\mu-r\right)-\beta\mu\right), \\
c &= \gamma\beta^2\left(\mu-r\right)^2.
\end{aligned}
$$

The condition for nirvana normal solutions and the critical horizon τ_c are

$$
2\sqrt{q}<b+\sqrt{q}\Longleftrightarrow\sqrt{q}<b, \qquad \tau\in[0,\tau_c), \qquad \tau_c=\frac{1}{\sqrt{q}}\ln\left(\frac{b+\sqrt{q}}{b-\sqrt{q}}\right).
$$

A nirvana solution is one for which expected utility $J\left(W,X,\tau\right)$ increases so rapidly with the investment horizon that expected utility approaches infinity ($\gamma>1$ investors) or its upper bound ($\gamma<1$ investors) at a finite critical horizon τ_c, which depends on the solution's parameters. Nirvana solutions[3] also have investment functions approaching infinity at the critical horizon. Given $\beta<0$, $\gamma>0$, and $\mu>r$, nirvana can be achieved only by investors who are more aggressive than the log-utility agent[4]:

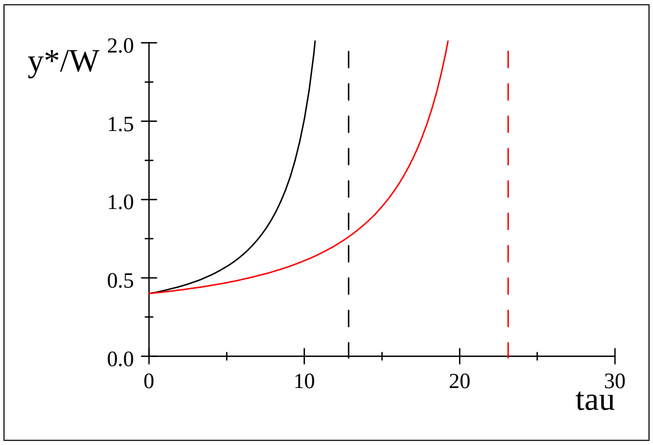
$$
\sqrt{4\beta^2\left(r^2+\gamma\left(\mu^2-r^2\right)\right)}<-2\beta\left(r+\gamma\left(\mu-r\right)\right)\Longleftrightarrow\gamma\in\left(1,\infty\right).
$$

The left-hand side of Figure 1 pictures nirvana-type investment functions. It is based on the following parameter values: $r=2\%$, $\mu=7\%$, $k=1$. For $X=10\%$, it shows how the critical horizon τ_c at which nirvana is attained is shortened by a stronger leverage effect.

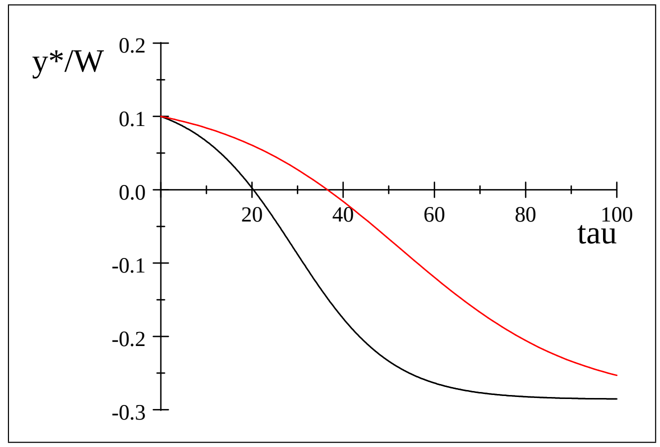
[3] As Kim and Omberg (1996) point out, the nirvana solutions are not defined for longer horizons than the critical horizon. Investors with a longer horizon can pursue any policy up to the critical horizon and attain nirvana by commencing the optimal policy at that time.

[4] Notice that nirvana will never be attained with a positive elasticity ($\beta>0$). Hence, the stock price bubble implied by a CEV stock price with positive elasticity (see Heston, Loewenstein, and Willard (2007)) is not matched by nirvana solutions.

Figure 2. Allocation to the defaultable asset versus the investment horizon τ



$\gamma = 2 \quad (\beta = -0.9 \;\text{ black; } \beta = -0.5 \;\text{ red})$



$\gamma = 0.5 \quad (\beta = -0.9 \;\text{ black; } \beta = -0.5 \;\text{ red})$

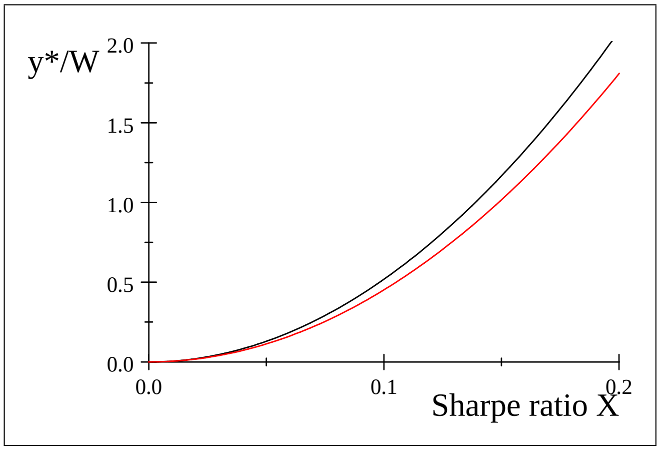
The absence of intermediate consumption hampers the use of arguments based on the substitution/income effects on consumption to gauge the sign of the hedging demand for the stock. However, since an unexpected drop in the stock price implies a deterioration in the investment opportunities offered by the stock (the Sharpe ratio X drops), one expects that a non-log-utility and sufficiently risk-averse investor will hedge such an adverse effect by shorting the stock to profit from the unexpected drop in the stock price. Indeed, an inspection of $C(\tau)$'s sign confirms that investors who are less aggressive than the log-utility agent ($\gamma < 1$) always have negative hedging demands, which are visualized in the right-hand side of Figure 1.

The zero level is a cemetery state for the stock price (the stock is worth zero at the issuing corporate's default). A natural conjecture is that the investor will gradually move out of the stock as its price plunges, that is, as the defaultable security becomes more likely to disappear soon. The conjecture is patently verified by the optimal investment function (3):

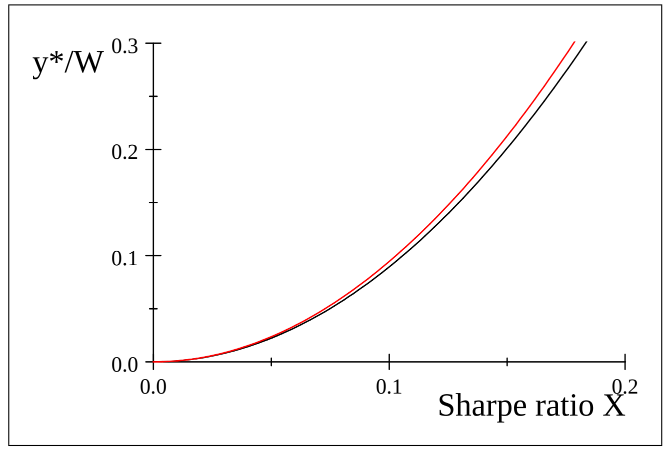
$$\lim_{X \to 0^+} \frac{y^*(W, X, \tau)}{W} = 0.$$

Given the same parameter levels employed in Figure 1 and an investment horizon equal to $\tau = 4$, Figure 2 shows how quickly the optimal percentage allocation to stocks is quashed as the Sharpe ratio approaches the zero level.

Figure 2. Allocation to the defaultable asset versus the Sharpe ratio X



$\gamma = 2 \quad (\beta = -0.9 \;\text{ black}; \;\; \beta = -0.5 \;\text{ red})$



$\gamma = 0.5 \quad (\beta = -0.9 \;\text{ black}; \;\; \beta = -0.5 \;\text{ red})$

4 Preliminary conclusions

We show how dynamic portfolio theory can parsimoniously explain the vast carry trades in defaultable securities observed before the crisis. Our paper is the first to employ the Kim and Omberg (1996) model to explicitly calculate the optimal allocation of wealth to a default-prone security. We emphasize how a frugal no-arbitrage model of dynamic portfolio choice rationalizes huge geared bets in a defaultable asset that is able to grant an 'yield pickup' at times of oceanic liquidity.

APPENDIX A

Cox (1975) shows that, for a CEV price process with $-1 < \beta < 0$, the probability density function of S_T, the asset price at T, conditional on the current price s, under the objective probability measure is

$$
\begin{aligned}
f\left(S_T = m, T \mid S_t = s, t = 0\right) &= 2\left(1-p\right) k^{\frac{1}{2-p}} \left(xw^{1-2p}\right)^{\frac{1}{2(2-p)}} e^{-x-w} I_{\frac{1}{2-p}}\left(2\sqrt{xw}\right), \qquad m > 0,\ s > 0, \\
p &= 2\left(\beta+1\right), \\
k &= \frac{2\mu}{2\left(1-p\right)\left(e^{\mu(2-p)T}-1\right)}, \\
x &= ks^{2-p}e^{\mu(2-p)T}, \\
w &= km^{2-p},
\end{aligned}
$$

where $I_q\left(\cdot\right)$ is the modified Bessel function of the first kind of order q. Hence,

$$
\begin{aligned}
\mathbb{P}\left[S_h = 0,\ 0 \leq h \leq T \mid S_0 = s\right] &= 1 - \int_0^{+\infty} f\left(S_T = m, T \mid S_t = s, t = 0\right) dm \\
&= \frac{\Gamma\left(-\frac{\mu s^{-2\beta}}{\beta\left[1-e^{2\mu\beta T}\right]}, -\frac{1}{2\beta}\right)}{\Gamma\left(-\frac{1}{2\beta}\right)}, \\
\Gamma\left(z, l\right) &= \int_z^{+\infty} u^{l-1}e^{-u}du, \quad z \geq 0, \quad \text{(Incomplete Gamma function)} \\
\Gamma\left(l\right) &= \int_0^{+\infty} u^{l-1}e^{-u}du \qquad \text{(Gamma function).}
\end{aligned}
$$

Randal (1998) shows that, letting Y be a gamma random variable, with shape parameter $-\frac{1}{2\beta}$ and unit scale parameter, Markov's Inequality gives

$$
\begin{aligned}
\frac{\Gamma\left(-\frac{\mu s^{-2\beta}}{\beta\left[1-e^{2\mu\beta T}\right]}, -\frac{1}{2\beta}\right)}{\Gamma\left(-\frac{1}{2\beta}\right)} &= \Pr\left[Y \geq -\frac{\mu s^{-2\beta}}{\beta\left[1-e^{2\mu\beta T}\right]}\right] \\
&\leq \frac{E\left[Y\right]}{-\frac{\mu s^{-2\beta}}{\beta\left[1-e^{2\mu\beta T}\right]}} \\
&= \frac{-\frac{1}{2\beta}}{-\frac{\mu s^{-2\beta}}{\beta\left[1-e^{2\mu\beta T}\right]}} \\
&= \frac{1-e^{2\mu\beta T}}{2\mu s^{-2\beta}}.
\end{aligned}
$$

APPENDIX B

Given (1), the Sharpe ratio X_t evolves as follows:

$$dX_t = -\beta \left(X_t \mu - \frac{1}{2} (\beta + 1) \frac{(\mu - r)^2}{X_t} \right) dt - \beta (\mu - r) dZ_t.$$

Let $J(W, X, \tau)$ be the optimal expected utility. The standard dynamic programming conditions result in the optimal investment function

$$y^* (W, X, \tau) = \underbrace{\left(\frac{J_W}{-J_{WW}} \right) \frac{X}{\sigma}}_{\text{myopic demand}} + \underbrace{\left(\frac{J_{WX}}{-J_{WW}} \right) \left(-\frac{\beta (\mu - r)}{\sigma} \right)}_{\text{hedging demand}}, \qquad (1)$$

the partial differential equation

$$-J_\tau + J_W r W - \frac{1}{2} J_{WW} (y^*)^2 \sigma^2 - J_X \beta \left(X\mu - \frac{1}{2} (\beta + 1) \frac{(\mu - r)^2}{X} \right) + \frac{1}{2} J_{XX} \beta^2 (\mu - r)^2 = 0, \qquad (2)$$

the horizon-end condition,

$$J(W, X, 0) = U(W),$$

and the second-order condition for a maximum $J_{WW} < 0$.

The optimal investment function for the logarithmic case is the well-known myopic investment function. For the nonlogarithmic cases, assume trial solutions of the form

$$J(W, X, \tau) = U(W) e^{A(\tau) + C(\tau)\frac{X^2}{2}},$$

$$A(0) = C(0) = 0.$$

Solutions of this form automatically satisfy both the horizon-end condition and the second-order condition for a maximum. We have

$$J_\tau = J\left(A' + C' \tfrac{X^2}{2} \right), \qquad J_X = J(CX), \qquad J_W = \tfrac{U_W}{U} J,$$

$$J_{XX} = J(C) + J(CX)^2, \qquad J_{WW} = \tfrac{U_{WW}}{U} J,$$

$$J_{WX} = \tfrac{U_W}{U} J(CX),$$

and

$$\frac{U_W}{U}W = \frac{\gamma - 1}{\gamma}, \qquad \frac{U_{WW}}{U}W^2 = -\frac{\gamma - 1}{\gamma^2}.$$

Substitution of the trial solutions into Equation (1) yields the investment function

$$y^* (W, X, \tau) = \gamma W \frac{X}{\sigma} - \gamma W C(\tau) X \frac{\beta(\mu - r)}{\sigma}, \qquad (3)$$

Further substitution of the trial solutions into Equation (2) produces a quadratic equation for X; its three coefficients must be zero, resulting in this system of first-order nonlinear ordinary differential equations:

$$C' = C^2 c + C b + a,$$

$$A' = C f + g,$$

$$A(0) = C(0) = 0,$$

$$\begin{aligned} a &= \gamma - 1, \\ b &= 2\left((1-\gamma)\beta(\mu - r) - \beta\mu\right), \\ c &= \gamma\beta^2 (\mu - r)^2, \end{aligned}$$

$$\begin{aligned} f &= \frac{1}{2}\beta\left((\beta + 1)(\mu - r)^2 + \beta(\mu - r)^2\right), \\ g &= \frac{\gamma - 1}{\gamma} r. \end{aligned}$$

The condition for normal solutions is

$$q = b^2 - 4ac > 0.$$

The condition for well-behaved normal solutions is

$$2\sqrt{q} > b + \sqrt{q} \Longleftrightarrow \sqrt{q} > b.$$

Given the assumptions that $\gamma > 0$ and $\mu > r$, the quantity q,

$$\begin{aligned} q &= b^2 - 4ac \\ &= 4\beta^2 \left(r^2 + \gamma \left(\mu^2 - r^2\right)\right), \end{aligned}$$

is always positive. Hence, the solutions have this normal form:

$$C(\tau) = \frac{2a\left(1 - \exp\left(-\sqrt{q}\tau\right)\right)}{2\sqrt{q} - \left(b + \sqrt{q}\right)\left(1 - \exp\left(-\sqrt{q}\tau\right)\right)},$$

$$A(\tau) = f \int_0^\tau \frac{2a\left(1 - \exp\left(-\sqrt{q}u\right)\right)}{2\sqrt{q} - \left(b + \sqrt{q}\right)\left(1 - \exp\left(-\sqrt{q}u\right)\right)} du + g\tau.$$

The condition for nirvana normal solutions and the critical horizon τ_c are

$$2\sqrt{q} < b + \sqrt{q} \Longleftrightarrow \sqrt{q} < b, \qquad \tau \in [0, \tau_c), \qquad \tau_c = \frac{1}{\sqrt{q}} \ln\left(\frac{b + \sqrt{q}}{b - \sqrt{q}}\right).$$

Given $\beta < 0$, $\gamma > 0$, and $\mu > r$, nirvana can be achieved only by investors who are more aggressive than the log-utility agent:

$$\sqrt{4\beta^2 \left(r^2 + \gamma \left(\mu^2 - r^2\right)\right)} < -2\beta \left(r + \gamma \left(\mu - r\right)\right) \Longleftrightarrow \gamma \in (1, \infty).$$

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