Pricing the CBOT T-Bonds Futures[*,†]

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October 6, 2008

[*]Research supported by NSERC Canada and IFM2.

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Abstract

The aim of this paper is to investigate the theoretical and empirical pricing of the Chicago Board of Trade (CBOT) Treasury-bond futures. The difficulty to price it arises from its multiple inter-dependent embedded delivery options, which can be exercised at various times and dates during the delivery month. We consider a continuous-time model with a continuous underlying factor (the interest rate), moving according to a Markov diffusion process consistent with the no-arbitrage principle. We propose a numerical pricing model that can handle all the delivery rules embedded in the CBOT T-bond futures, interpreted here as an American-style interest-rate derivative. Our pricing procedure combines dynamic programming, finite-elements approximation, analytical integration and fixed-point evaluation. Numerical illustrations, provided under the Vasicek (1977) and Cox-Ingesoll-Ross (1985) models, show that the interaction between the quality and timing options in a stochastic environment makes the delivery strategies complex, and not easy to characterize. We also carry out an empirical investigation of the market in order to verify whether short traders in futures contracts are exercising the strategic delivery options skillfully and optimally or if they are under-utilizing them. To do so, we price the futures contract under the Hull-White (1990) model. Empirical results show that futures prices are generally undervalued, which means that the market overvalues the embedded delivery options. According to our findings, observed futures prices are on average 2% lower than theoretical futures prices over the 1990-2008 time period, priced two months prior to the first day of delivery months.

JEL Classification: C61; C63; G12; G13.

Mathematics Subject Classification (2000): 90C39; 49M15; 65D05.

Keywords: Futures; asset pricing; dynamic programming; cheapest-to-deliver; delivery options; interest-rate models.

1 Introduction

A *futures contract* is an agreement between two investors traded on an exchange to sell or to buy an *underlying asset* at some given time in the future, called the *delivery date*, for a given price, called the *futures price*. By convention, at the time the futures is written (the *inception date*), the futures price is known and sets the value for both parties to zero. A futures contract is marked to market once a day to eliminate counterparty risk. Precisely, at the end of each trading day, the futures contract is rewritten at a new *settlement price*, that is, the closing futures price, and the difference with the last settlement futures price is substracted (*resp. added*) from the short (*resp. long*) trader account.

The *Treasury Bond futures* traded on the Chicago Board of Trade (the CBOT T-bond futures in the sequel) is the most actively traded and widely used futures contract in the United States, largely because of its ability to hedge long term interest rate risk. It calls for the delivery of $100,000 of a long-term governmental bond. The *notional* or *reference bond* is a bond with a 6% coupon rate and a maturity of 20 years. Delivery months (DM) are March, June, September and December.

Since the notional bond is a hypothetical bond that is generally not traded in the market place, the short has the option to choose which bond to deliver among a deliverable set fixed by the CBOT. The actual delivery day within the delivery month is also at the option of the short. These two delivery privileges offered to the short trader are known as the *quality option* (or *choosing option*) and the *timing option.*

The quality option allows the delivery of any governmental bond with at least 15 years to maturity or earliest call. To make the delivery fair for both parties, the price received by the short trader is adjusted according to the quality of the T-bond delivered. This adjustment is made via a set of *conversion factors* defined by the CBOT as the prices of the eligible T-bonds at the first delivery date under the assumption that interest rates for all maturities equal 6% par annum, compounded semiannually. The T-bond actually delivered by the short trader is called the *cheapest-to-deliver* (CTD).

The timing option allows the short trader to deliver early within a delivery month according to special features, that is, the *delivery sequence* and the *end-of-month delivery rule.* The delivery sequence consists of three consecutive business days: The *position day*, the *notice day*, and the *delivery day.* During the position day, the short trader can declare his intention to deliver until up to 8:00 p.m., while the CBOT closes at 2:00 p.m. (Central Standard Time). On the notice day, the short trader has until 5:00 p.m. to state which T-bond will be actually delivered. The delivery then takes place before 10:00 a.m. of the delivery day, against a payment based on the settlement price of the position day (adjusted according to the conversion factor). Finally, during the last seven business days before maturity, trading on the T-bond futures contracts stops while delivery, based on the last settlement price, remains possible according to the delivery sequence. The so-called *wild card play* (or *end-of-the day option* or *six hours option*) and the *end-of-month option* refer respectively to the timing option during the three day delivery sequence and to the last seven business days of the delivery month.

The modeling and measurement of the delivery options implicit in T-bond futures contracts has been examined in the literature using different methods and leading to non consensual empirical results. In particular, the issues of the performance of the conversion factor system, the identification of the cheapest-to-deliver bond as well as the valuation of the quality option embedded in Treasury-bond futures contracts have been the subject of a substantial volume of research. A first stream of papers deals with the so-called *conversion factor risk* and its impact on the market, and proposes alternative conversion systems and rules for the identification of the CTD (see for instance Livingston (1984), Kane and Marcus (1984), Arak, Goodman and Ross (1986), Benninga and Wiener (1999) and Oviedo (2006)). A second stream of papers proposes theoretical and empirical valuation approaches for the quality option, which is considered to be the most important, assuming a flat term structure for interest rates. Four main methods are used: the exchange option pricing formula (Gay and Manaster (1984), Boyle (1989), Hemler (1990)), the buy-and-hold approach (Hemler (1990), Kane and Marcus (1986a), Hedge (1990)), the implied option value approach (Hedge (1988, 1990), Hemler (1990)) and the switching option method (Barnhill and Seale (1988a, 1988b), Hedge (1990)). A third stream of research, also concentrating on the quality option, takes into consideration the term structure and stochastic nature of interest rates (Ritchken and Sankarasubramanian (1992, 1995), Bick (1997), Carr and Chen (1997), Chen, Chou and Lin (1999), Lacoste (2002), Ferreira de Oliveira and Vidal-Nunes (2007)). While the quality option is assumed to be the most important, ignoring the other delivery options may lead to mispricing, and fail to suggest optimal delivery strategies. A last stream of research considers the timing option, either separately (Gay and Manaster (1986), Kane and Marcus (1986b)), or in conjunction with the quality option (Arak and Goodman (1987), Boyle (1989), Peck and William (1990), Gay and Manaster (1991), Nielsen and Ronn (1997), Chen and Yeh (2005), Hranaoiva, Jarrow and Tomek (2005)). It is worth mentioning that the papers in this last category use simplifying assumptions on the dynamics of the interest rate or on the strategies. To date, no work has been presented regarding the identification of optimal exercise strategies in the CBOT T-bond futures trading and the pricing of the contract under stochastic interest rates when the interaction of all the delivery options is taken into account. In fact, complexity arising from all the embedded inter-dependent delivery rules makes the contract computationally and analytically difficult to price.

The aim of this paper is to propose a model and a pricing algorithm that can handle explicitly and simultaneously all the delivery rules embedded in the CBOT T-bond futures in a stochastic interest-rate environment, and then to test it empirically. To do so, we consider a continuous-time model with a continuous underlying factor, the risk-free short-term interest rate. We assume that this rate moves according to a Markov diffusion process that is consistent with the no-arbitrage principle. Our pricing procedure is a backward numerical algorithm combining Dynamic Programming (DP), approximation by finite elements, and fixed-point evaluation. In this context, the DP value function is the value of the contract for the short trader (which is reset to 0 at the settlement dates) and the DP recursion is given by no-arbitrage pricing (Elliot and Kopp (1999)). Under a given assumption about the stochastic evolution of interest rates, the numerical procedure output may be summarized into three results. The first gives the theoretical futures prices at settlement dates. The second gives the delivery position strategy (deliver or not) and the third identifies the CTD on the notice day, given the futures price at the last settlement date. All three results are functions of time and current interest rate.

The paper is organized as follows. Section 2 presents the model and the Dynamic Programming formulation for the value of the contract. Section 3 describes in details the numerical procedure. Section 4 reports on numerical results obtained under both the Vasicek (1977) and the Cox-Ingersoll-Ross (1985) (hereafter CIR) models for the short-rate process. In section 5, we conduct an empirical investigation of our futures pricing model under the Hull-White (1990) model. Section 6 is a conclusion.

2 Model and DP Formulation

2.1 Notation

We consider a frictionless cash and T-bond futures market in which trading takes place continuously. Denote

- $E[\cdot]$ the expectation under the risk neutral measure Q;
- $(c, M) \in \Theta$ an eligible T-bond with a principal of 1 dollar, a continuous coupon rate c, and a maturity M, where the finite set Θ of eligible bonds is known at the date the contract is written;
- $\{r_t\}$ a Markov process for the risk-free short-term interest rate;

- $\rho(r,t,\tau)$ the price at t of a zero-coupon bond maturing at $\tau \geq t$ when $r_t = r$ under the process $\{r_t\}$

$$\rho(r,t,\tau) = E\left[\exp(-\int_t^\tau r_u du) \mid r_t = r\right]; \tag{1}$$

- $p(t,c,M,r)$ the price at t of the eligible T-bond (c,M) when $r_t = r$ under the process $\{r_t\}$

$$p(t,c,M,r) = c\int_t^M \rho(r,t,u)du + \rho(r,t,M); \tag{2}$$

- f_{cM} the CBOT conversion factor corresponding to the T-bond (c,M), where the set $\{f_{cM} : (c,M) \in \Theta\}$ is known at the date the contract is written:

$$f_{cM} = \mathrm{PV}\left(t_{\underline{n}}, c, M, 6\%\right) \tag{3}$$

  where $t_{\underline{n}}$ is the first day of the delivery month and $\mathrm{PV}(t,c,M,r)$ is the price at t of the eligible T-bond (c,M) when its yield to maturity is r

$$\mathrm{PV}(t,c,M,r) = c\int_t^M \exp(-r(u-t))du + \exp\left(-r\left(M-t\right)\right); \tag{4}$$

- $g_t(r)$ the price of the T-bond futures at t when $r_t = r$;

- $g_t^*(r)$ the fair settlement price of the T-bond futures at t when $r_t = r$;

- $\underline{c}$ ($\overline{c}$) the minimum (maximum) bond coupon rate among the deliverable bonds;

- $\underline{M}$ ($\overline{M}$) the minimum (maximum) bond maturity among the deliverable bonds.

To be consistent with the CBOT delivery rules, we consider a sequence of motoring dates t_m^h where the lower index $m = 0, ..., \overline{n}$ is computed in days from the date the contract is written and the upper index $h \in \{2, 5, 8\}$ indicates the time in hours within that day. Assuming that the contract is written at $t_0 = t_0^2$, we denote the marking-to-market dates by t_m^2 for $m = 0, \ldots, n$, where t_n represents the last futures trading date during the delivery month. We denote the delivery position dates by t_m^8 for $m = \underline{n}, \ldots, \overline{n}$, where $t_{\underline{n}}$ and $t_{\overline{n}}$ are respectively the first and the last date of the delivery month, $0 < \underline{n} < n < \overline{n}$. Finally, the delivery notice dates are denoted t_m^5 for $m = \underline{n}+1, \ldots, \overline{n}+1$. Our choice of monitoring dates is justified by the fact that, within the delivery month, it is better for the short trader to wait until 8:00 p.m. each day to decide whether to take or not a delivery position.

Moreover, we assume that the delivery notice date coincides with the actual delivery, since this does not change the (expected) value of the contract.

Our DP model determines the value of the contract for the short trader at each monitoring date, as a function of the interest rate at the current and last settlement dates, assuming that the short trader behaves optimally. We obtain the fair settlement price by making the value of the contract null for both parties at the settlement dates.

The contract is evaluated by backward recursion in three distinct periods: The end-of-the-month period, where no trading takes place, but delivery is still possible ($m = n, ..., \overline{n}$), the beginning of the delivery month where trading and delivery are both possible ($m = \underline{n}, ..., n$), and the period before the delivery month, where no action is taken by the short trader, but the settlement price is adjusted every day ($m = 0, ..., \underline{n}$).

2.2 End-of-the-month Period

Recall that during the last seven business days before maturity, trading on the T-bond futures contracts stops while delivery remains possible, based on the settlement price at the last settlement date, indexed by m'. If an intention to deliver is issued at the delivery position date t_m^8, for $m = n, ...\overline{n}$, we define the *expected exercise value* $v_m^e(r', r)$ and the *actual exercise value* $v_m^a(r', r)$ for the short trader, as a function of the interest rate at the last settlement date, denoted r', and at the current date, denoted r, as follows:

$$v_m^e(r', r) = E\left[\left(v_m^a\left(r', r_{t_{m+1}^5}\right) e^{-\int_{t_m^8}^{t_{m+1}^5} r_u du}\right) \mid r_{t_m^8} = r\right], \tag{5}$$

$$v_m^a(r', r) = \max_{(c,M)\in\Theta} \left\{ g_{m'}(r') f_{cM} - p\left(t_{m+1}^5, c, M, r\right)\right\}, \tag{6}$$

where $m' = n$ is the last settlement date and $g_{m'}(r')$ is the price of the futures settled at m' when $r_{m'} = r'$.

Otherwise, if the short trader decides not to deliver at t_m^8, for $m = n, ..., \overline{n}$, we define the *holding value* $v_m^h(r', r)$, which is computed by no-arbitrage to be the expected value of the future potentialities of the contract and given by (9) below. The short trader will of course issue an intention to deliver at $\left(t_m^8, r', r\right)$ if and only if

$$v_m^e(r', r) > v_m^h(r', r). \tag{7}$$

The *value function* for the short trader at t_m^8, for $m = n, ..., \overline{n}$, is thus defined recursively by:

$$v_m^8(r', r) = \max\left\{v_m^e(r', r), v_m^h(r', r)\right\} \tag{8}$$

$$v_m^h(r', r) = E\left[v_{m+1}^8\left(r', r_{t_{m+1}^8}\right) e^{-\int_{t_m^8}^{t_{m+1}^8} r_u du} \mid r_{t_m^8} = r\right] \tag{9}$$

$$v_{\overline{n}}^8(r', r) = v_{\overline{n}}^e(r', r), \tag{10}$$

and the *settlement value* for the short trader at $(t_{m'}^2, r_{t_{m'}^2})$ is the discounted expected value at $t_{m'}^8$:

$$v_{m'}^2(r') = E\left[v_{m'}^8\left(r', r_{t_{m'}^8}\right) e^{-\int_{t_{m'}^2}^{t_{m'}^8} r_u du} \mid r_{t_{m'}^2} = r'\right], \tag{11}$$

where $m' = n$. Notice that equations (5) - (11) define a mapping from the space of functions $g_{m'} : \mathbb{R} \to \mathbb{R}$ to the space of functions $v_{m'}^2 : \mathbb{R} \to \mathbb{R}$, but we did not make this dependency on $g_{m'}$ explicit to alleviate the notation. Moreover, the settlement value at m' and r' is defined for any settlement price function $g_{m'}(r')$, constant during the end-of-the-month period, which can be written

$$v_{m'}^2(r') = F_{m'}(g)(r') \tag{12}$$

where $F_{m'}(g) : \mathbb{R} \to \mathbb{R}$ is a function defined by the Dynamic Program (5) - (11) with $g = g_{m'}(r')$ and $m' = n$.

However, the settlement price at $t_{m'}^2$ should be selected so that the value to both parties is 0, taking into account the timing and quality options. Consequently, the fair settlement price at $t_{m'}^2$, denoted $g_{m'}^*(r')$, is a function of the settlement date interest rate such that:

$$v_{m'}^2(r') = F_{m'}(g_{m'}^*)(r') = 0 \text{ for all } r', \tag{13}$$

where $m' = n$.

2.3 Delivery Month

During the delivery month ($m = \underline{n}, ..., n-1$), the value of the contract for the short trader may be evaluated in the same manner as in the previous section, but the holding value must account for the interim payments at the marking-to-market dates. Thus, the exercise value functions $v_m^e(r', r)$ and $v_m^a(r', r)$ at respectively t_m^8 and t_m^5, for $m = \underline{n}, ..., n-1$ are given by (5)-(6), where $m' = m$. The holding value at t_m^8, however, accounts for the interim payment at the next marking-to-market date, that is,

$$\begin{aligned} v_m^h(r', r) &= E\Bigg[\left(g_m(r') - g_{m+1}^*\left(r_{t_{m+1}^2}\right)\right) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du} \\ &\quad + v_{m+1}^2\left(r_{t_{m+1}^2}\right) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du} \mid r_{t_m^8} = r\Bigg] \\ &= E\left[\left(g_m(r') - g_{m+1}^*\left(r_{t_{m+1}^2}\right)\right) e^{-\int_{t_m^8}^{t_{m+1}^2} r_u du} \mid r_{t_m^8} = r\right], \end{aligned} \tag{14}$$

since the settlement value at $m+1$ is the null function for a fair settlement price. The value function at t_m^8 and t_m^2 is then given by (8) and (11), with $m' = m$. Finally, for each marking-to-market date t_m^2, $m = \underline{n}-1, ..., n-1$, the settlement price function $g_{m'}^*(r)$ is such that (13) is verified, with $m' = m$.

2.4 Initial period

Within the time period $\left[t_0, t_{\underline{n}-1}^2\right]$, delivery is not possible, so that the value of the contract for the short trader only involves taking into account the interim payments in the marking-to-market account. The value function at t_m^2, for $m = 0, ..., \underline{n}-1$, is thus given by

$$v_m^2(r) = E\left[\left(g_m^*(r) - g_{m+1}^*\left(r_{t_{m+1}^2}\right)\right) e^{-\int_{t_m^2}^{t_{m+1}^2} r_u du} \mid r_{t_m^2} = r\right] = 0, \tag{15}$$

where g_m^* is such that (13) is satisfied for $m' = m$, $m = 0, ..., \underline{n}-1$. Therefore, the successive settlement prices can be obtained by the recursive relation

$$g_m^*(r) = \frac{E\left[g_{m+1}^*\left(r_{t_{m+1}^2}\right) e^{-\int_{t_m^2}^{t_{m+1}^2} r_u du} \mid r_{t_m^2} = r\right]}{\rho(r, t_m^2, t_{m+1}^2)} \text{ for all } r,\ m = 0, ..., \underline{n}-1. \tag{16}$$

3 Dynamic Programming Procedure

Equations (5)-(16) define a dynamic program which can be used to find the fair settlement prices and the optimal timing and choosing strategies for the short trader by backward induction. This dynamic program is defined on the state space $\{(r', r) : r' \geq 0,\ r \geq 0\}$ and does not admit a closed-form solution, even for the most simple case where the interest rate is assumed to be constant. In this section, we describe a numerical procedure for the solution of this dynamic program. Three specific numerical problems must be addressed:

- the optimization in (6) which involves the price of the eligible bonds according to the underlying interest-rate model,
- the computation of the expectations in (5), (9), (11), (14) and (16) of functions which are analytically intractable,
- the determination of the root of (13).

The numerical procedures consists in

- finding the CTD by an appropriate search over the eligible set according to the properties of the bond prices,
- approximating the functions $v_m^8(g,\cdot)$ and $g_m^*(\cdot)$ by expectations of linear finite elements interpolation functions,
- finding the fair settlement price as a fixed point of a contraction mapping.

3.1 Optimization Procedure

Finding the CTD at m, r', r consists in solving the following:

$$v_m^a\left(r', r\right) = \max_{(c,M)\in\Theta}\left\{g_{m'}(r')f_{cM} - \left(c\int_{t_{m+1}^5}^{M}\rho\left(r, t_{m+1}^5, u\right)du + \rho\left(r, t_{m+1}^5, M\right)\right)\right\} \tag{17}$$

where the finite set of eligible bonds and their conversion factors are fixed at the signature of the contract and $g_{m'}(r')$ is given. The function to be maximized is linear in c, so that the optimal coupon is extremal and given by either $\underline{c} \equiv \min c$ or $\overline{c} \equiv \max c$ in the set of eligible bonds. If an analytic expression for $\rho\left(r, t, \tau\right)$ is known, it is straightforward to check the properties of the projections of the function to optimize on $c = \overline{c}$ and $c = \underline{c}$. If these are convex, simple inspection of the extremal maturities will yield the CTD. If either one is concave, a line search for fixed $\overline{c}$ and/or $\underline{c}$ can be performed. Otherwise, since the number of eligible bonds is fixed, an enumeration of all eligible bonds with extremal coupons will yield the CTD and the value of $v_m^a\left(r', r\right)$. Notice that, while the value of $v_m^a\left(r', r\right)$ can be obtained with as much precision as the price of a given bond for any r', r and $g_{m'}$, it cannot be expressed in closed-form.

3.2 Interpolation Procedure

The interpolation procedure consists in approximating an analytically intractable function, the value of which is known at a finite number of points, by a piecewise linear continuous function.

Let $\mathcal{G} = \{a_1, \ldots, a_q\}$ be a grid defined on the set of interest rates, with the convention that $a_0 = -\infty$ and $a_{q+1} = +\infty$. Given a function $h : \mathcal{G} \to \mathbb{R}$, the interpolation function $\widehat{h} : \mathbb{R} \to \mathbb{R}$ is given by:

$$\widehat{h}\,(r) = \sum_{i=0}^{q} (\alpha_i + \beta_i r)\, I\,(a_i \leq r < a_{i+1})\,, \text{ for all } r \in \mathbb{R}, \tag{18}$$

where I is the indicator function and the coefficients α_i and β_i are obtained by matching $\widehat{h}$ and h on $\mathcal{G}$ and extrapolating outside of $\mathcal{G}$, that is

$$\alpha_i = \frac{a_{i+1} h(a_i) - a_i h(a_{i+1})}{a_{i+1} - a_i}, \tag{19}$$

$$\beta_i = \frac{h(a_{i+1}) - h(a_i)}{a_{i+1} - a_i}, i = 1, ...q-1, \tag{20}$$

and $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\alpha_q = \alpha_{q-1}$, $\beta_q = \beta_{q-1}$.

3.3 Expectations of Interpolation Functions

Define the *transition parameters*

$$A_{k,i}^{t,\tau} \equiv E\left[I\,(a_i \leq r_\tau < a_{i+1})\, e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \tag{21}$$

and

$$B_{k,i}^{t,\tau} \equiv E\left[r_\tau I\,(a_i \leq r_\tau < a_{i+1})\, e^{-\int_t^\tau r_u du} \mid r_t = a_k\right], \tag{22}$$

where $t_0 \leq t \leq \tau$, $k = 1, \ldots, q$, and $i = 0, \ldots, q$.

We assume that these transition parameters and the discount factor $\rho(r, t, \tau)$ can be obtained with precision from the dynamics of $\{r_t,\, t \geq t_0\}$. Notice that for several dynamics of the interest rates, closed-form solutions exist for the transition parameters and discount factor, as discussed in Ben-Ameur *et al.* (2007). Examples include Vasicek (1977), CIR (1985), and Hull and White (1990). Closed-form formulas for the transition parameters and discount factor in the Vasicek, the CIR and the Hull-White models are recalled in the Appendix.

Given an interpolation function $\widehat{h} : \mathbb{R} \to \mathbb{R}$, the expected value at t and $r_t = a_k$ of a future payoff

$\widehat{h}(\cdot)$ at τ is given by:

$$\begin{aligned}
\widetilde{h}(t,\tau,a_k) &\equiv E\left[\widehat{h}(r_\tau)\, e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \\
&= E\left[\sum_{i=0}^{q} (\alpha_i + \beta_i r_\tau)\, I(a_i \le r_\tau < a_{i+1})\, e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \\
&= \sum_{i=0}^{q} \alpha_i A_{k,i}^{t,\tau} + \beta_i B_{k,i}^{t,\tau} \text{ for all } a_k \in \mathcal{G} \text{ and } 0 \le t \le \tau. \qquad (23)
\end{aligned}$$

3.4 Root Finding Procedure

At a given r' and m', the root finding procedure consists in finding a constant g such that

$$v_{m'}^2(r') = F_{m'}(g)(r') = 0$$

where $F_{m'}(g)$ is defined by the Dynamic Program (5) - (11), (14) with $g = g_{m'}(r')$. Consider two settlement prices g_1 and g_2 such that $g_1 < g_2$. Since

$$\max_{(c,M)\in\Theta} \left\{ g_1 f_{cM} - p\left(t_{m+1}^5, c, M, r\right) \right\} < \max_{(c,M)\in\Theta} \left\{ g_2 f_{cM} - p\left(t_{m+1}^5, c, M, r\right) \right\} \qquad (24)$$

for all r and m, it is easy to show that $F_{m'}$ is strictly monotone in g. Moreover,

$$\lim_{g\to-\infty} F_{m'}(g)(r') = -\infty \qquad (25)$$

$$\lim_{g\to+\infty} F_{m'}(g)(r') = +\infty. \qquad (26)$$

Therefore, $F_{m'}(\cdot)(r')$ admits a unique root.

Consider the following successive approximation scheme:

$$g^1 = g^0 - F_{m'}\left(g^0\right)(r') \qquad (27)$$

$$g^{k+1} = g^k - F_{m'}\left(g^k\right)(r') \frac{g^k - g^{k-1}}{F_{m'}\left(g^k\right)(r') - F_{m'}\left(g^{k-1}\right)(r')}, \qquad k > 1, \qquad (28)$$

where g^0 is given.

Now, since $F_{m'}(\cdot)(r')$ is strictly increasing, this Quasi-Newton successive approximation scheme will converge to the unique root from any starting point; for example, a good starting point is

$$g^0_{t_{m'}}(r) = \min_{(c,M)\in\Theta}\left\{\frac{c\int_T^M \rho(r,t_{m'},u)du + \rho(r,t_{m'},M)}{\rho(r,t_{m'},T)f_{cM}}\right\} \tag{29}$$

which is the price at $t_{m'}$ of a forward contract maturing at an appropriately chosen T (either $T = 1$ day during the delivery month or $T = 7$ days during the end-of-the-month) with a choosing option on the same basket Θ. Moreover, since the number of exercise strategies is finite, it can be shown that there exists a neighborhood of the root where $F_{m'}(\cdot)(r')$ is linear in g, so that this approximation scheme will converge in a finite number of steps.

3.5 Algorithm

The algorithm consists in solving the dynamic program (5)-(16) by backward induction from the last delivery position date $t^8_{\overline{n}}$ on the grid $\mathcal{G}$.

In each of the three periods spanning the contract, the main loop of the algorithm consists in iteratively finding, from an initial guess, the fair settlement price at the settlement dates, as a function of the current spot interest rate, considering all the delivery options. This is realized by applying, at a given marking-to-market date, the root finding procedure on all points of $\mathcal{G}$, and then applying the interpolation procedure to obtain the settlement price as a continuous function of the interest rate.

The inner loop of the algorithm consists in obtaining, for a given settlement price function, the value of the contract for the short trader at a given position date, considering all the delivery options, as a function of the interest rate at the last settlement date and the current interest rate. This is realized by applying, on all points of $\mathcal{G} \times \mathcal{G}$, the optimization procedure to find the CTD and the actual exercise value on the grid. The interpolation procedure is then applied to obtain a continuous function, and the expectation procedure is applied on the time interval between the position and the notice dates, yielding the exercise value at the position date. This is compared with the holding value, which is known on the grid points. The optimal value function at the position date is then interpolated and the expectation procedure is applied between either two successive position dates (during the end-of-the-month period) or the last settlement and current position dates. This yields the value for the short seller at the settlement date as a function of the interest rate, which is null if the settlement price is fair.

The detailed algorithm is provided in the Appendix.

4 Numerical Illustration

In our numerical experiments, the finite set of deliverable bonds contains 62 bonds with maturity ranging from $\underline{M} = 15$ years to $\overline{M} = 30$ years in steps of 6 months. Since only the bonds with extremal coupon rates are optimal to deliver, we consider only two coupon rates corresponding to the highest and lowest coupon rates in the current CBOT set of deliverable bonds, namely $\overline{c} = 7.625\%$ and $\underline{c} = 4.5\%$. The inception date is chosen to be three months prior to the first day of the delivery month. These properties of the set of eligible bonds are summarized in Table 1.

Table 1: Properties of the deliverable set

	Min	Max	Step	Notional
Coupon (%)	4.5	7.625	-	6
Maturity (years)	15	30	0.5	20

In our numerical illustrations, we use both the Vasicek and CIR interest-rate process models given respectively in (37) and (44), using the closed-form formulas (40)-(42) or (49)-(51) for the discount factor and transition parameters.

The interest-rates grid points $a_1, ..., a_q$ are selected to be equally spaced with $a_0 = -\infty$, $a_1 = \overline{r} - 8\sigma d^V(T)$, $a_q = \overline{r} + 8\sigma d^V(T)$ for the Vasicek model, while $a_1 = \max(0; \overline{r} - 8\sigma d^C(T))$ and $a_q = \overline{r} + 8\sigma d^C(T)$ for the CIR model, where

$$d^V(T) = \sqrt{\frac{1}{2\kappa}(1 - \exp(-2T\kappa))}, \tag{30}$$

$$d^C(T) = \sqrt{\frac{\overline{r}}{\kappa}(\exp(-T\kappa) - \exp(-2T\kappa)) + \frac{\overline{r}}{2\kappa}(1 - \exp(-T\kappa))^2} \tag{31}$$

and $\overline{r}$ is the long-term mean reversion level, κ is the mean reversion speed, σ is the volatility of the short-term interest rate, q is the number of grid points and T is the horizon in years (for an inception date of three months before the first day of the delivery month, $T = 1/3$).

We disentangle the individual effects of each implicit option by pricing four futures contracts embedding different combinations of these options, namely

- F1 : the *straight futures contract* offering no options at all and corresponding to the case where the short trader declares his intention to deliver on the first position day of the delivery month and delivers the notional bond,
- F2 : the contract offering the quality option alone, where the short trader chooses on day $\underline{n}$+1

the bond to be delivered among the deliverable basket,

- F3 : the contract offering only the timing option, allowing the short trader to deliver the notional bond anytime during the delivery month according to the delivery sequence, and
- F4 : the full contract offering all the embedded delivery options to the short trader.

The computation of these four prices allows us to price each option alone as well as in the presence of the other option. For instance, we compute the following differences

- F1-F4 is the value of all the embedded options,
- F3-F4 gives the value of the quality option in the presence of the timing option,
- F1-F2 is the value of the quality option without timing,
- F1-F3 is the value of the timing option without quality and
- F2-F4 is the value of the timing option when the quality option is offered to the short trader.

Notice that definitions of implicit delivery options are not uniform throughout the literature, and one must be cautious in comparing results across studies. According to our definition, the *timing option* gives the short trader the right to deliver late on any day during the delivery month. Some papers define the timing option as the option to deliver early in the delivery month. The small value they obtain can be explained by the fact that delaying delivery is often optimal.

4.1 Convergence

We first examine the convergence properties of the DP procedure. To do so, we record the relative error in the price of the futures contract F4 at the inception date as the number of grid points is doubled from 10 to 1280. We report on the error with respect to the price obtained for the best precision level ($q = 1280$).

Table 2 gives an example of futures prices, under the Vasicek model, corresponding to the interest rate $r_0 = 4\%$ when $\overline{r} = 0.05$, $\kappa = 0.2$ and $\sigma = 0.05$. Figure 1 below represents the log of the relative error as a function of the log of the distance between grid points in the Vasicek model for various combinations of the input parameters. The average rate of convergence is 1.2. Notice that in many cases good precision levels of futures prices can be reached for a relatively small number of grid points.

q	Futures price	Relative error	CPU (sec.)
10	1.393001762	0.052609564	1
20	1.353692561	0.025098768	2
40	1.331981229	0.009207850	3
80	1.322748743	0.002292345	7
160	1.320638214	0.000697896	23
320	1.319952069	0.000178433	125
640	1.319771148	4.1372e-05	603
1280	1.319716546	-	5081

Table 2: Convergence of the DP futures prices (Vasicek)

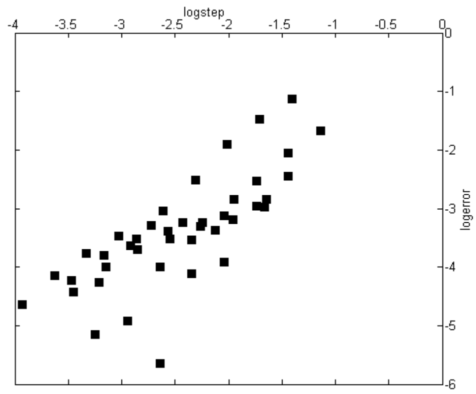

Figure 1: Convergence of the DP futures prices (Vasicek)

4.2 Options values

We now report on the values of the quality and timing options at the inception date for levels of interest rates ranging from 3.5% to 8.5%. In this numerical experiment, we use the (risk neutral) parameter values of Cases 1 and 2 given in Table 3 below, $q = 600$ and $r_0 = 6\%$. These parameters are those of Shoji and Osaki (1996) who estimate these models using the 1-month U.S. Treasury-Bill rate over the period 1964-1992.

Table 3: Input data

	Case 1	Case 2	Case 3
Model	Vasicek	CIR	Vasicek
$\overline{r}$	0.062098	0.061677	0.059
κ	0.565888	0.545788	0.5
σ	0.025416	0.091471	0.02

It is worthwhile mentioning that the aim of this section is to give a numerical illustration of the results obtained by the DP procedure for a chosen set of parameters. Therefore, the reported results are not representative and one could obtain different values and shapes for the embedded delivery options as a function of the interest rate if some other combinations of the input parameters or different properties of the deliverable set are considered. The sensitivity of the implicit delivery options with respect to the input parameters is addressed in section 4.4.

Figure 2 compares, at the inception date, the values of the embedded quality option (with and without the timing option) as a function of the current interest rate for the Vasicek and CIR dynamics. For this example, we find that without the timing option, the quality option is worth an average 0.33 percentage points of par (ppp) for the Vasicek model while this value is 0.26 ppp for the CIR model. When the timing option is embedded in the contract, we notice a slight increase in the value of the quality option which is then worth an average 0.36 ppp (Vasicek) or 0.28 ppp (CIR). These values are consistent with the literature about the valuation of the quality option. The fact that the quality option is more valuable in the presence of the timing option is due to the interaction and the interdependence that exist between these two options. In fact, if the short trader enjoys both the quality and the timing options, then at each decision time during the delivery month, he has the opportunity to choose the cheapest-to-deliver among the deliverable basket and benefit from bond price movements as well as switches in the CTD that can occur during the delivery month. So, in the presence of timing, the short is offered the quality option repeatedly, and can choose the best time to exercise it. The quality option has more value when combined to the timing option to reflect the price that the short should pay to benefit more than once from having the privilege of choosing the CTD among the deliverable set.

Quality option at inception (Vasicek) | Quality option at inception (CIR)

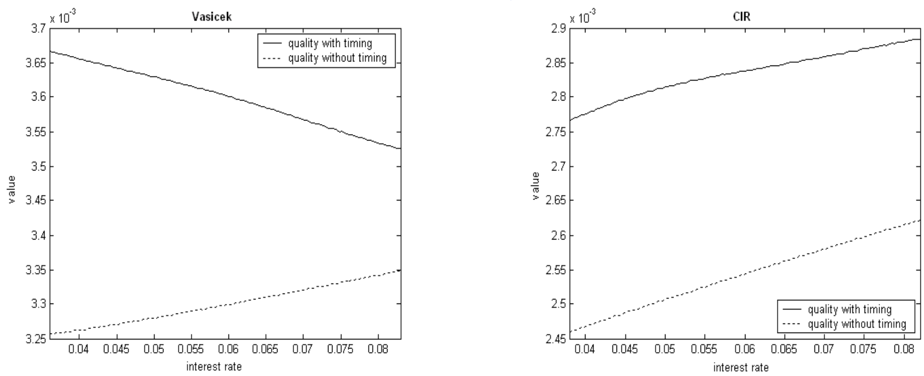

Figure 2: Quality option values vs. interest rates at inception (CFS)

In Figure 3, we report on the evolution, during the delivery month, of the quality option values (with timing) as a function of the interest rate at the settlement time under both the Vasicek and the CIR dynamics. We observe that the quality option does not exhibit a specific behavior with respect to time to maturity or interest rates. For example, we can see for the Vasicek dynamics that the relation of the quality option with elapsed time is nearly flat for low levels of interest rates while this relation becomes negative for higher levels. Also, we can see for the CIR dynamics that the shape of the quality option with respect to interest rates differ significantly on day 1 and day 15 of the delivery month, especially when interest rates are high.

Evolution of the quality option during the DM (Vasicek)

Evolution of the quality option during the DM (CIR)

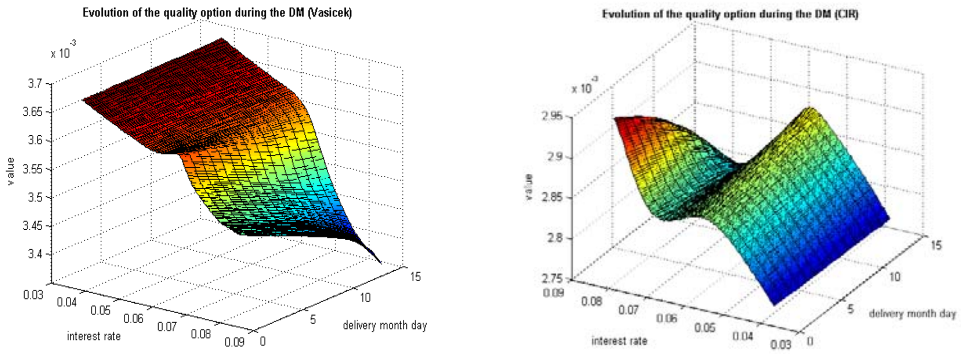

Figure 3: Evolution of the quality option during the delivery month (CFS)

Figure 4 plots the value of the timing option at the inception date (with and without the quality option) for both dynamics considered here.

Without the quality option, the timing option is worth an average 0.072 ppp (Vasicek) and 0.12 ppp (CIR) while when the quality option is offered to the seller, the timing option is more valuable and is worth an average 0.1 ppp (Vasicek) and 0.15 ppp (CIR). Again, because of the interaction between the options, the timing option is more valuable in the presence of the quality option. In fact, the short can benefit, not only from changes in the price of the CTD (always the reference bond without the quality option), but also from switches in the CTD. It is interesting to notice here that even if the quality option is on average more valuable, the value of the timing option can exceed the value of the quality option, especially for low interest rates. The timing option is observed to be always negatively related to interest rates and this can be easily explained by the fact that, since we are valuing the option to deliver late, when interest rates increase, the short trader can invest the proceeds of early exercise at higher rates. Also, when interest rates are lower than the long-term mean, one would expect them to go up and consequently lower the cash bond prices which make late delivery optimal. The opposite effect will lead the seller to exercise the timing option in order to avoid a general increase in the cash price of the cheapest-to-deliver.

Timing option at inception (Vasicek) — Timing option at inception (CIR)

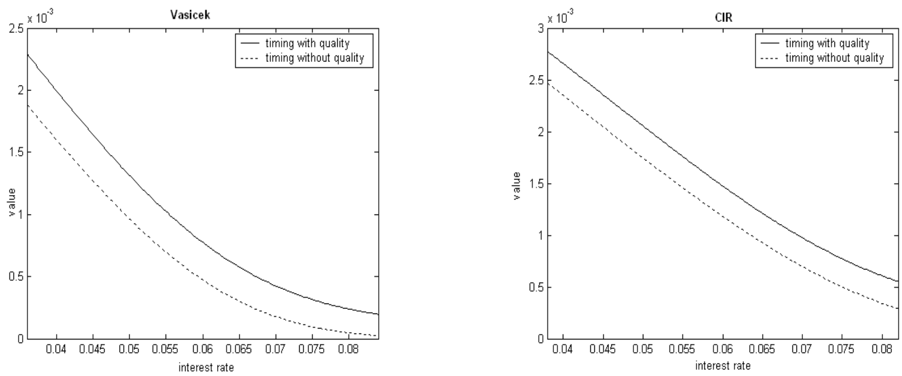

Figure 4: Timing option values vs. interest rates at inception (CFS)

4.3 Optimal delivery strategy

We present here some examples of the optimal delivery strategy and the associated change in the CTD during the delivery month under the Vasicek model. We use two sets of parameters for this illustration, namely those corresponding to Case 1 and Case 3 given in Table 3 in order to compare the delivery strategies as the position of $\overline{r}$ with respect to the reference coupon rate changes. We choose to report on the optimal strategy on days 16, 19 and 22 of the delivery month (we assume 22 business days in the delivery month). Notice that the optimal decision depends not only on the level of interest rates at the time the decision is made (8:00 p.m.), but also on the last settlement futures price, which is directly related to the level of interest rates at the settlement date. We report on the optimal strategy for various combinations of the two state variables r^2 and r^8, corresponding respectively to the levels of interest rates at the last settlement date (2:00 p.m.) and the current position date (8:00 p.m.). Figure 5 presents the optimal strategy on day 16 of the delivery month (the decision to deliver or not is made on day 15 (the last futures trading day)) for possible values of interest rates according to Case 1, assuming a rate of 6% at inception. Notice that only a small area around the diagonal of Figure 5 is likely to be observed, corresponding to possible variations of the rate in the 6 hours period separating settlement and position times. This area is presented in Figure 6 for Cases 1 and 3. We notice from this figure that early exercise can be optimal during the delivery month and that the CTD is either the longest (Case 1) or shortest duration bond (Case 3). Early exercise during the delivery month is driven essentially by the differential in interest rates during the wild card period. If during this six hours period, a large move in interest rates occurs, such a move could make it worth the short's while to deliver early, even at the cost of giving up the valuable remaining strategic delivery options. In Case 1, early delivery is optimal when interest rates decrease substantially during the six hours period, and the CTD is the longest duration bond $\left(\underline{c}; \overline{M}\right)$ corresponding to a conversion factor less than one. This is consistent with the rule for exercising early during the delivery month proposed in the literature, requiring an increase in the issue's price for bonds with conversion factors less than one to make the wild card profitable (see for example Kane and Marcus (1986b) and Gay and Manaster (1986)). Notice that, in Case 3, an increase in interest rates is required to make the early delivery of the shortest duration bond $(\overline{c}; \underline{M})$ optimal.

In that context, it is interesting to compute the value of the so-called wild card option. To do so, we price a futures contract where the short is hypothetically forced to make the decision to deliver or not at 2:00 p.m. instead of 8:00 p.m. on each futures trading day during the delivery month. The value of the wild card option is computed as the difference between the prices of the full contract and the contract without it, both in the presence of the quality option. We find that the wild card option is worth an average 0.007 ppp when the short enjoys the quality option. This result is consistent with the very small value reported in the literature for this option. We also verify that, under Case 1, the optimal decision at 2:00 p.m. if the short is not allowed to play the wild card is found on the diagonal in Figure 5.

Figures 7 and 8 present the optimal delivery strategy for both sets of the input parameters considered in this section on days 19 and 22 of the delivery month, respectively (end-of-the-month). Is is worthwhile observing from Figure 8 that the CTD is not necessarily the bond with the longest or shortest duration ($\left(\left(\underline{c}; \overline{M}\right)\right.$ or $(\overline{c}; \underline{M})$), as is often suggested in the literature, and bonds $(\underline{c}; \underline{M})$ or $\left(\overline{c}; \overline{M}\right)$ can be optimal to deliver. Finally, we price the so-called end-of-month option by computing the difference between the full contract and the hypothetical contract where the last possible delivery day and the last futures trading day coincide. We find that this option is worth an average 0.061 ppp.

Similar results about the delivery strategy are obtained under the CIR dynamics.

Optimal delivery strategy on day 16
of the DM (Case 1)

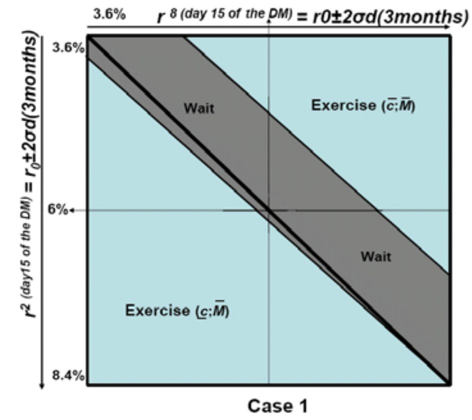

Figure 5: Optimal delivery strategy on day 16 of the DM (Vasicek)

Optimal delivery strategy on day 16 of the DM (Case 1)

Optimal delivery strategy on day 16 of the DM (Case 3)

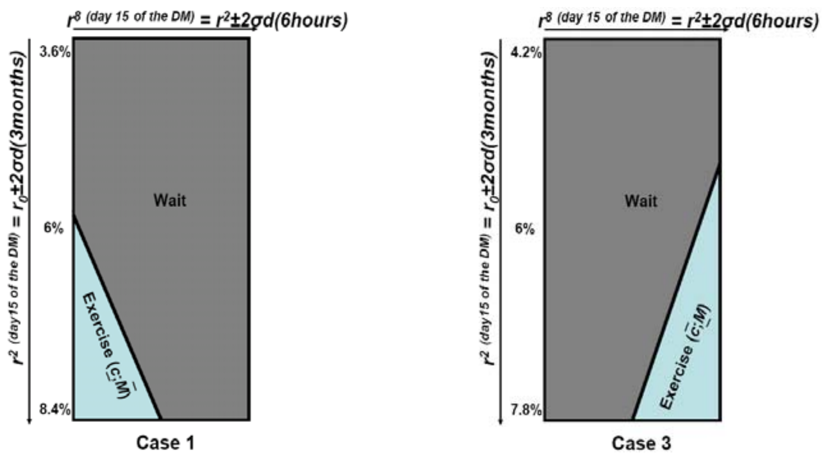

Figure 6: Optimal delivery strategy on day 16 of the DM for possible variations of interest rates (Vasicek)

Optimal delivery strategy on day 19 of the DM (Case 1)

Optimal delivery strategy on day 19 of the DM (Case 3)

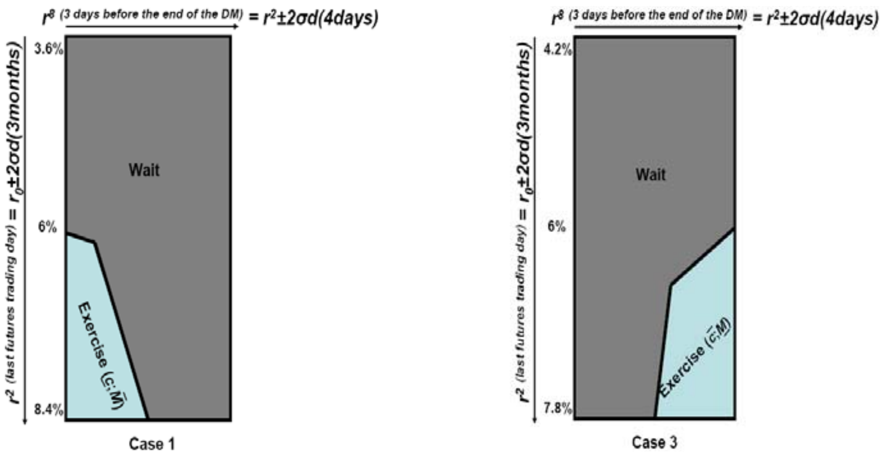

Figure 7: Optimal delivery strategy 3 days before the end of the DM for possible variations of interest rates (Vasicek)

Optimal delivery strategy on day 22 of the DM (Case 1)

Optimal delivery strategy on day 22 of the DM (Case 3)

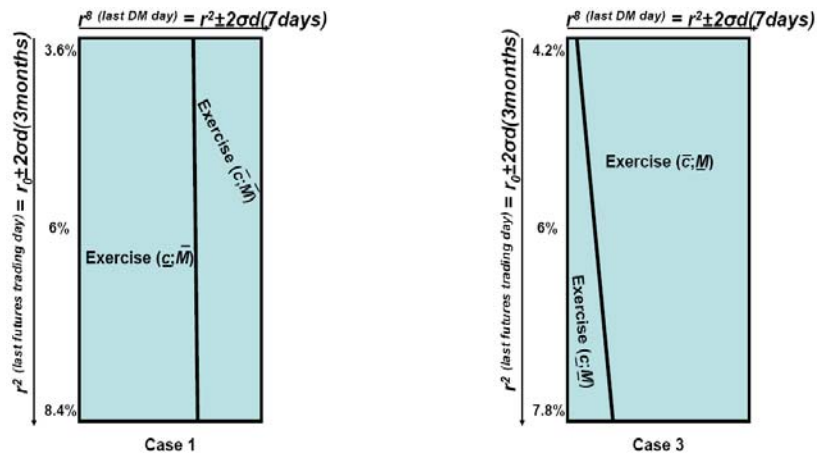

Figure 8: Optimal delivery strategy on the last day of the DM for possible variations of interest rates (Vasicek)

4.4 Sensitivity of the options values to the input parameters

In this last section, we study the impact of a change in the interest-rate model input parameters on the quality and timing option values. The base case parameter values are $\bar{r} = 0.06$, $\kappa = 0.5$ and

$\sigma = 0.02$.

Notice that, for both interest-rate dynamics considered in this work, the timing option is always downward sloping with respect to the short-term interest rate. However, the quality option does not exhibit such a specific relation (see for instance Figure 2).

We find that the quality option in the presence of timing is increasing with the distance between the long-run mean reversion level $\bar{r}$ and some rate around the reference coupon rate. Therefore, we study the sensitivity of the quality option value to the parameters of the interest-rate process when the long-term mean $\bar{r}$ is in the neighborhood of the coupon rate of the reference bond (the analysis is carried out for a reference rate of 6% but we verified that the same qualitative results are obtained for other reference rates).

In Figures 9, 10 and 11, corresponding to a mean reversion speed of 0.2, 0.5 and 0.8 respectively and a volatility of 0.01 for the Vasicek model and 0.02 for the CIR model, we report on the quality option value when both the long-run mean reversion level and the short-term interest rate vary in the neighborhood of 6%. These figures show that for both interest-rate dynamics considered here, the quality option value has a minimum with respect to the long-term mean when the value of this parameter is around 6%. We can see from these figures that increasing the speed of adjustment makes this minimum move towards the level of 6%. The same effect is obtained as the volatility level is decreased and is illustrated in Figure 12 which represents the sensitivity of the quality option with respect to the long-term mean for different levels of the volatility and for a high level of the mean reversion speed (kappa=0.8) for the Vasicek and CIR dynamics. So, for simultaneous high levels of the mean reversion speed and low levels of the interest-rate volatility, the quality option shows a minimum around the level of 6% for the long-run mean. This behavior of the quality option could be explained by the fact that low levels of volatility as well as high mean reversion speed contribute to obtaining a nearly flat term structure at $\bar{r}$. In addition, for the specific case of a flat term structure at the reference coupon level, all eligible bonds are equal for delivery and therefore the quality option is worthless. The minimum observed in the stochastic case may similarly be explained by the fact that the deliverable basket is the most homogenous for that given combination of the parameters, without however being all equal for delivery.

Figure 12 also shows that as we move away from the level of the long-term mean for which the minimum is achieved, the value of the quality option increases since the deliverable basket becomes more and more heterogenous. Furthermore, we see from this figure that the relation between the quality option and the volatility could be either negative or positive depending on the level of the long-term mean. These positive/negative relations of the quality option with respect to the volatility can be better seen in Figure 13. Such relations can be explained by the impact of volatility changes on futures and bonds prices. If the CTD is the bond with the highest maturity (as it is generally the case when $\overline{r}$ is larger than the reference rate), an increase in the level of the volatility will increase the price of the futures embedding the quality option more than the price of the straight contract, which consequently lowers the value of the quality option. The opposite effect of an increase in the volatility level on futures prices is observed for levels of $\overline{r}$ less than the reference coupon rate since, in this case, shortest maturity bonds are cheapest-to-deliver, resulting in a positive relation between the quality option and the volatility. This is illustrated in Figure 14 where some examples of the impact of changes in volatility on futures prices are presented.

We also study the simultaneous effect of a variation in the mean reversion speed and the long-term mean on the value of the quality option for a given level of volatility. Results are presented in Figure 15 and we notice that the relation between the quality option and the mean reversion speed is negative for levels of the long-term mean less than the reference coupon rate while this relation becomes positive in the opposite case. This result is the opposite of the relation we find between the quality option and the volatility, and is consistent since an increase in the volatility could be balanced by a reduction in the mean reversion speed.

Finally, Figure 16 illustrates that the impact of the volatility of interest rates on the timing option differs according to parameters and interest-rate models.

Quality option (Vasicek) Quality option (CIR)

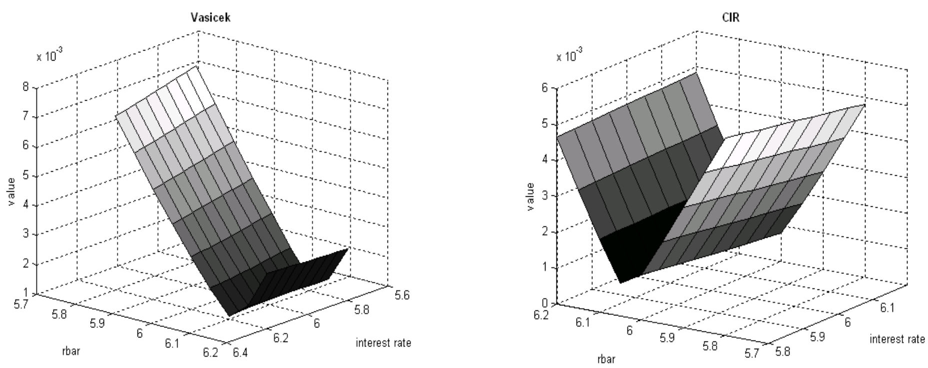

Figure 9: Quality option sensitivity to rbar and interest rates (kappa=0.2)

Quality option (Vasicek) Quality option (CIR)

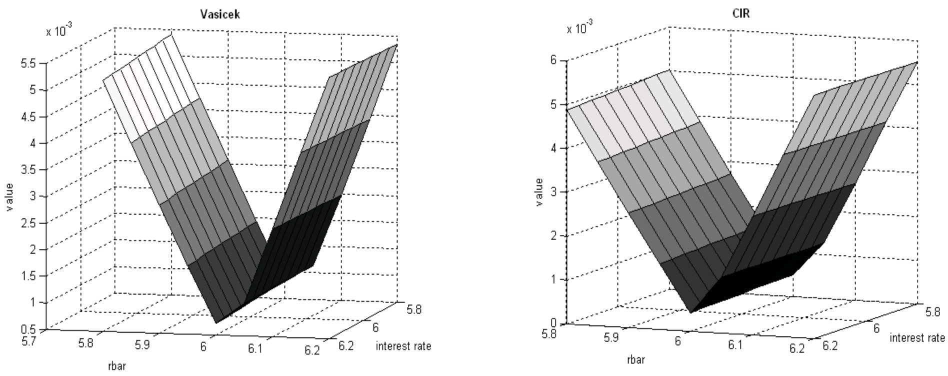

Figure 10: Quality option sensitivity to rbar and interest rates (kappa=0.5)

Quality option (Vasicek) Quality option (CIR)

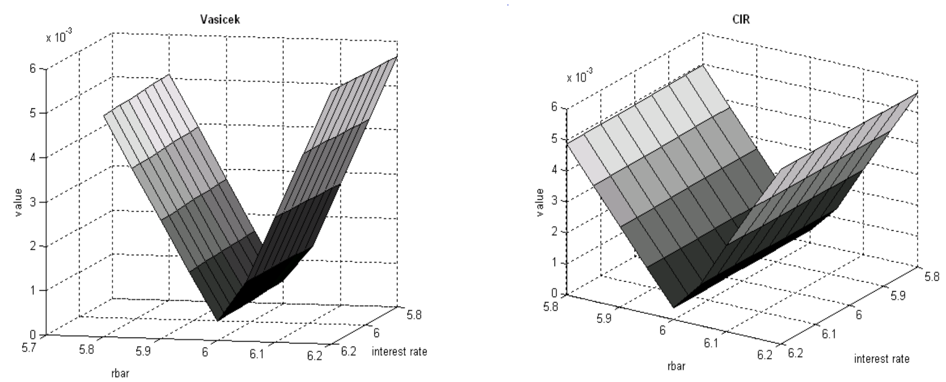

Figure 11: Quality option sensitivity to rbar and interest rates (kappa=0.8)

Quality option (Vasicek) Quality option (CIR)

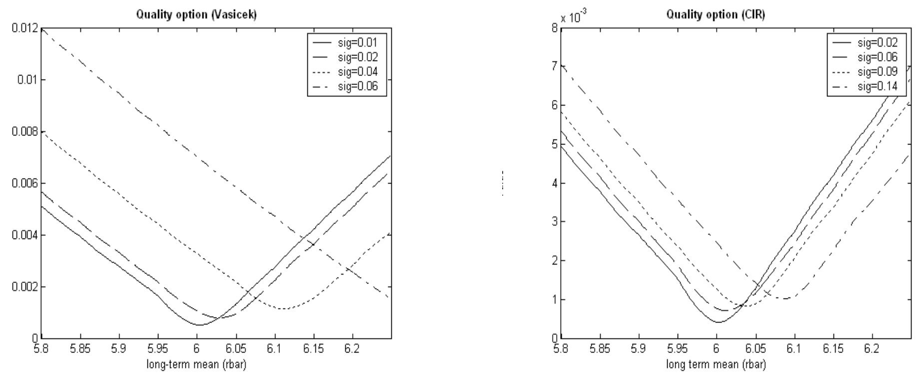

Figure 12: Quality option sensitivity to rbar and sigma (kappa=0.8)

Quality option (Vasicek) Quality option (CIR)

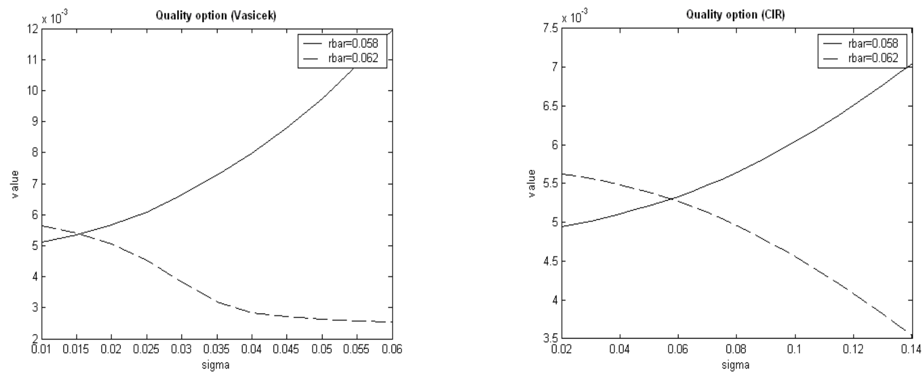

Figure 13: Quality option sensitivity to rbar and sigma (kappa=0.8)

Futures prices (rbar=0.05, kappa=0.8, Vasicek)

Futures prices (rbar=0.07, kappa=0.8, Vasicek)

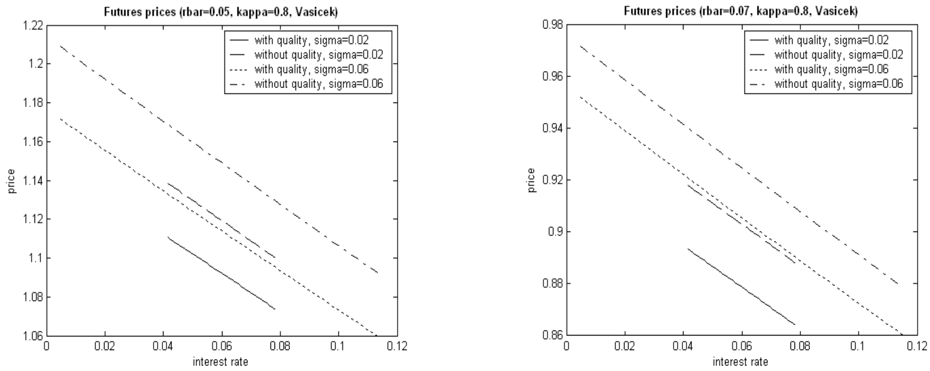

Figure 14: Impact of the volatility on futures prices

Quality option (Vasicek)

Quality option (CIR)

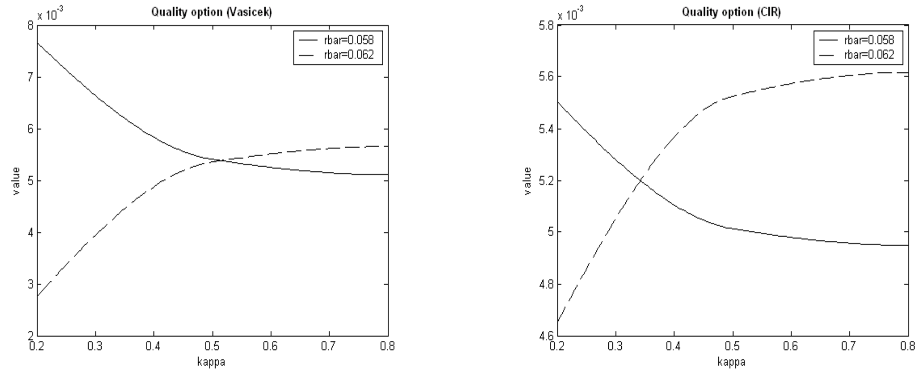

Figure 15: Quality option sensitivity to kappa and rbar

Timing option (Vasicek) Timing option (CIR)

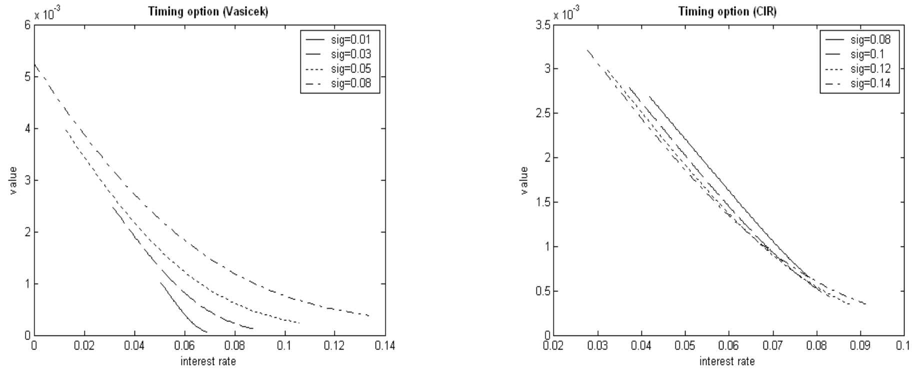

Figure 16: Timing option value sensitivity to sigma (kappa=0.5, rbar=0.06)

5 Prices of the CBOT T-Bonds Futures: An Empirical Investigation

This section is devoted to an empirical investigation of the CBOT T-bonds futures pricing model proposed in Section 2 and given by the dynamic program (5)-(16) when the instantaneous spot interest rate moves according to the Hull-White (1990) model. This model has been used in the literature dealing with the pricing of futures on governmental bonds but in its trinomial discrete version by Chen, Chou and Lin (1999) and applied to value the Japanese long-term Government Bond futures. In this section, this model is used in its continuous-time version for the valuation of the CBOT T-bonds futures. Under this model, the transition parameters in (21) and (22) are time-dependant but can still be expressed in closed-form.

5.1 The Hull-White model (extended Vasicek)

The Hull-White model, also called the extended Vasicek model, was introduced by Hull and White (1990). This model assumes that the instantaneous short-term interest-rate process evolves under the risk-neutral probability measure according to

$$dr_t = \kappa(\bar{r}\,(t) - r_t)dt + \sigma dB_t, \quad \text{for } t \geq 0, \tag{32}$$

where $\{B_t,\ t \geq 0\}$ is a standard Brownian motion, κ is the mean reversion speed, $\overline{r}(t)$ is the long-term mean and σ is the volatility. As in the original Vasicek model, the process has a constant positive volatility and exhibits mean reversion with a constant positive speed of adjustment, but in the extended version, the long-term level is a deterministic function of time.

Given the current (time 0) term structure and a differentiable function $t \mapsto f(t)$ representing the associated instantaneous fitted forward-rate curve, the term structure of interest rates in the Hull-White model with

$$\overline{r}(t) = f(t) + \frac{1}{\kappa}\frac{\partial f(t)}{\partial t} + \frac{\sigma^2}{2\kappa^2}\left(1 - e^{(-2\kappa t)}\right) \tag{33}$$

will be identical to the current term structure of interest rates.

This model requires the use of market data to obtain a fitted zero-coupon yield curve. Several non-parametric fitting techniques can be used to model the yield curve. These are general curve-fitting families including, for example, B-splines and Nelson and Siegel (1985) (henceforth NS) curves that do not derive from an interest-rate model. In this paper, we choose to use the Augmented Nelson and Siegel yield-curve fitting model proposed by Björk and Christensen (1999), which extends the NS family curves by the addition of an exponential decay term. These authors study the question as to when a given parametrized family of forward-rate curves is consistent with the dynamics of a given arbitrage-free interest-rate model, in the sense that the model actually will produce forward-rate curves belonging to the considered family. Björk and Christensen (1999) show that one needs to add an exponential decay term in order to make the original NS family consistent with the extended Vasicek model. The main reason for this consistency requirement, as mentioned by the authors, is that if a given interest-rate model is subject to daily recalibration, it is important that, on each day, the parameterized family of forward-rate curves, which is fitted to bond market data, be general enough to be invariant under the dynamics of the term-structure model; otherwise, the marking-to-market of an interest-rate contingent claim would result in value changes attributable not to interest-rate movements, but rather to model inconsistencies.

With five parameters $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$, the Augmented NS forward-rate curves are

$$f(t) = \gamma_0 + \left(\gamma_1 + \gamma_2 \frac{t}{\gamma_4}\right) e^{(-t/\gamma_4)} + \gamma_3 e^{(-2t/\gamma_4)}. \tag{34}$$

The resulting equation for the zero-coupon yield curve is then

$$z(t) = \gamma_0 + \gamma_1 \left(\frac{1 - e^{(-t/\gamma_4)}}{(t/\gamma_4)}\right) + \gamma_2 \left(\frac{1 - e^{(-t/\gamma_4)}}{(t/\gamma_4)} - e^{(-t/\gamma_4)}\right) + \gamma_3 \left(\frac{1 - e^{(-2t/\gamma_4)}}{(2t/\gamma_4)}\right) \tag{35}$$

where $z(t)$ is the yield of a zero-coupon bond of maturity t.

Closed-form formulas for the transition parameters and discount factor in the extended Vasicek model are given in the Appendix.

5.2 The Data

Our interest rates data consist of 3-month maturity Treasury-Bill rates covering the period from January 1, 1982 to September 30, 2001 and 1-month maturity Treasury-Bill rates covering the period from October 1, 2001 to March 31, 2008 (315 observations) obtained from the Federal Reserve Statistical Release. The frequency is monthly. The interest rates are given in percentage and annualized form. We interpret these rates as proxies for the instantaneous riskless interest rate. Using 1-month or 3-month maturity T-Bill yields as proxies for instantaneous short rates is unlikely to create a significant proxy bias as shown in Chapman, Long and Pearson (1999).

In Table 4, we presents summary descriptive statistics for the short rate r_t, the short-rate changes $\Delta r_t = r_t - r_{t-1}$ and squared changes $(\Delta r_t)^2$. In this table, ACF(s) denotes the value of the autocorrelation function of order s.

Table 4: Summary statistics on short rate

Variable	r_t	Δr_t	$(\Delta r_t)^2$
Mean	5.3599	−0.0376	0.106390
Standard deviation	2.5984	0.3245	0.495339
Skewness	0.4989	−2.4606	14.188107
Kurtosis	0.4669	19.3280	226.953564
Minimum	0.85	−2.86	0
1st Quartile	3.6575	−0.1400	0.002500
Median	5.1950	0.0000	0.019600
3rd Quartile	6.9500	0.1400	0.078400
Maximum	14.28	1.36	8.1796
ACF (1)	0.9744	0.3602	0.2492
ACF (2)	0.9386	0.1326	0.0267
ACF (3)	0.9059	0.1055	0.0613
ACF (10)	0.7143	−0.0907	0.0330
ACF (30)	0.3439	−0.0398	0.0321
ACF (50)	0.1671	−0.0542	0.0000

We consider 73 futures contracts traded in the period between January 1, 1990 and March 31, 2008 representing the quarterly delivery cycles of the nearby futures contract. Futures prices are obtained from the Chicago Board of Trade. It is worthwhile mentioning that, prior to March 2000, the coupon of the notional bond was equal to 8%. The basket of the deliverable bonds is determined based on the information about the issue dates of the 30-year US Treasury bonds available on the CBOT web site.

The properties of the deliverable basket are provided in the Appendix in Tables 5-7 for each futures contract over the period of study.

In order to estimate the parameters of the Augmented Nelson and Siegel model, we use yield curves (Treasury yields for maturities ranging between 3 months and 30 years) for the period between January 1, 1990 and March 31, 2008. Figure 17 shows the estimated spot-yield surface computed 2 months prior to the first day of the delivery month of the nearest expiring futures contract during the period of study. Throughout almost the entire sample period, the spot-yield curve presented a positive slope, although it was approximately flat for some periods. Nevertheless, it is clear that the period under analysis includes a wide variety of term-structure shapes.

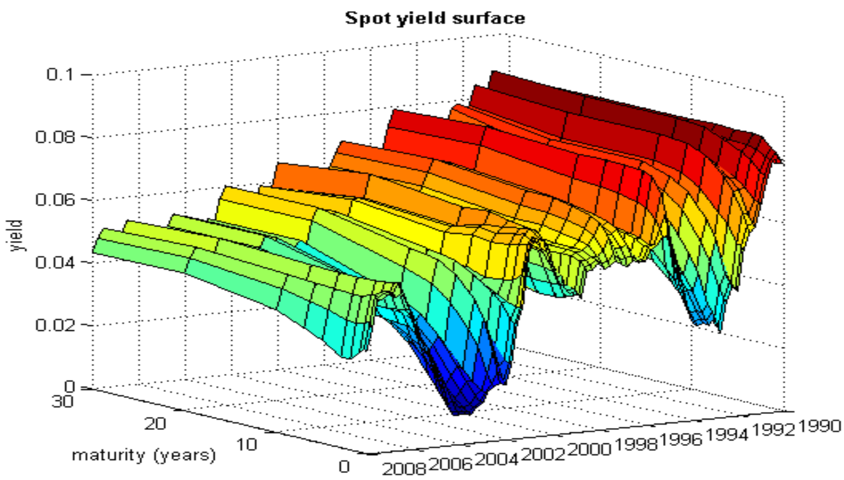

Figure 17: Spot-yield surface.

5.3 Empirical results

In the Hull-White model, the mean reversion speed and the volatility are obtained from the calibration of the Vasicek model to the short-term interest-rate data, using the maximum likelihood estimation technique, over the period from January 1, 1982 to inception (two months prior to the nearby futures contract) for each contract. These estimated values are provided in the Appendix in Tables 5-7 for each futures contract over the period of study.

Figures 18 and 19 plot, at inception and on day 1 of the delivery month respectively, a comparison of the observed and theoretical futures prices obtained using our DP pricing algorithm. These figures show a very good correlation between theoretical and observed prices; they also show that observed futures prices are generally lower than theoretical prices. According to our empirical findings, market futures prices are on average 2% lower than theoretical futures prices over the 1990-2008 time period, priced two month prior to the first day of the nearby delivery month. If the Hull-White model accurately describes the movements of the short rate, the market overvalues the embedded strategic delivery options. This is consistent with some empirical studies arguing that market futures prices are lower than what they should be when the term structure of interest rates is upward sloping. Arak and Goodman (1987) conclude that T-bond futures prices are too low and therefore believe that the market overvalues the embedded delivery options. Barnhill (1990) provides empirical evidence that futures prices are more often too low than too high. Most of the instances in which futures were too high occurred during a period in which the term structure of interest rates was downward sloping. He argues that during this period, the expected risk and cost of financing daily resettlement cash flows may have affected futures prices. For example, Gay and Manaster (1986) find that futures prices are too high over the period 1977-1983 and conclude that prices do not adequately value the seller strategic delivery options. They also find that short traders do not behave optimally in exercising their options, suggesting that the high futures prices reflect shorts' actual behavior, not the way they should optimally behave. Recall that over our period of study, the spot yield curve presented generally a positive slope as shown in Figure 17.

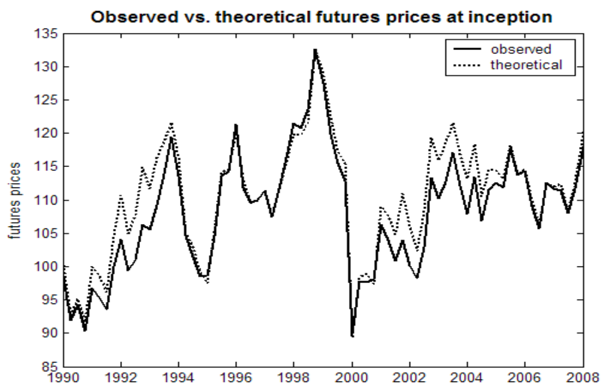

Figure 18: Observed vs. theoretical futures prices at inception.

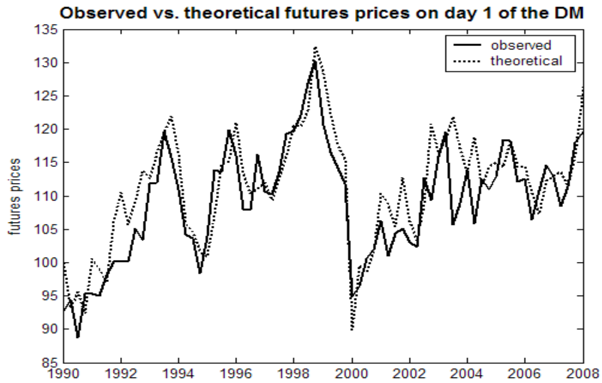

Figure 19: Observed vs. theoretical futures prices on day 1 of the DM.

We present in Figures 20-22 the differences between market and theoretical futures prices at inception and on the first day of the delivery month, separately and on the same figure. These figures show that pricing differences can be positive or negative and are generally small (for some contracts less than 0.05), especially at inception. Therefore, the model can be used to forecast the futures prices with a good level of accuracy. Notice that theoretical futures prices on day 1 of the delivery month are obtained using inception date yield curve, which explains the higher differences on the first day of the delivery month. All futures prices are reported in Tables 8 and 9 given in the Appendix. We also present in Figure 23 the yield curves at inception corresponding to the highest undervaluation of futures prices (maximum negative difference), obtained for December 1992 contract, and the highest overvaluation of futures prices (maximum positive difference), obtained for September 1998 contract. We verified that the highest undervaluation corresponds to the yield curve with the highest positive slope among all the yield curves considered in this study. However, the highest overvaluation is associated with a nearly flat yield curve as shown in Figure 23. These results are consistent with the findings of Barnhill (1990).

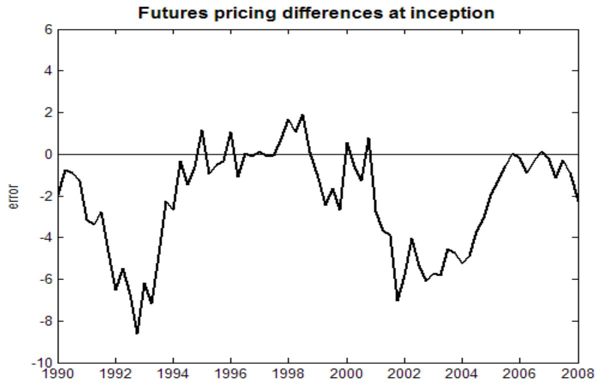

Figure 20: Futures pricing errors at inception.

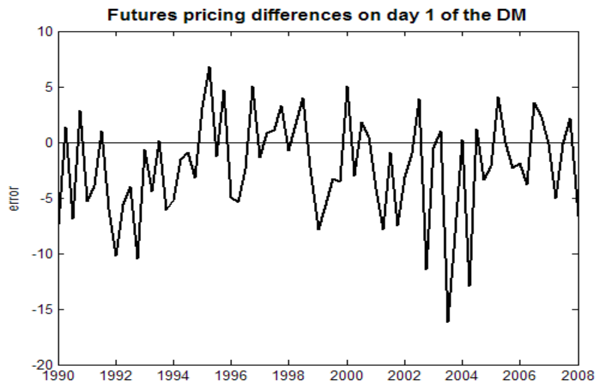

Figure 21: Futures pricing errors on day 1 of the DM.

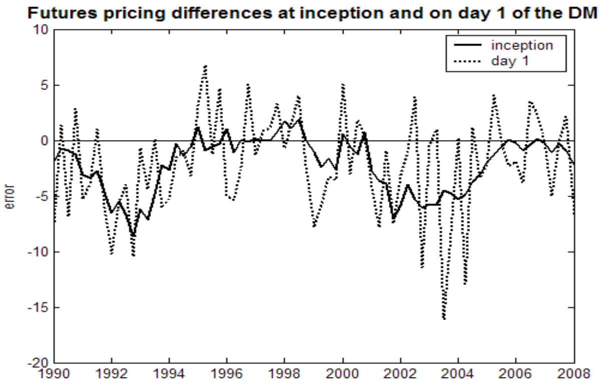

Figure 22: Futures pricing errors at inception and on day 1 of the DM.

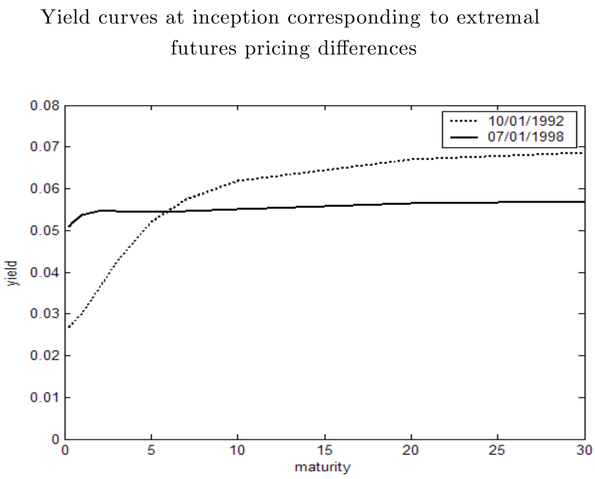

Figure 23: Yield curves at inception corresponding to extremal futures pricing differences.

6 Conclusion

This paper presents an efficient numerical method for the pricing of the CBOT T-bond futures contract, and for the identification of optimal exercise strategies, under stochastic interest rate dynamics, and accounting for the interaction of all the inter-dependent delivery options. This numerical algorithm, which combines dynamic programming, finite elements approximation, analytical integration and fixed point evaluation, is to our knowledge the first to tackle all the complexities of the CBOT futures contract in a stochastic interest-rate framework. The numerical and empirical illustrations are provided here under the Vasicek, CIR and Hull-White models, but the model we propose is flexible and can be used with any specification for multi-factor interest-rate dynamics, provided the transition parameters and discount factor can be obtained in closed-form or approximated efficiently. Our numerical investigations show that the interaction between the quality and timing options in a stochastic environment makes the delivery strategies complex, and not easy to characterize. Empirical results show that futures prices are generally undervalued, which means that the market overvalues the embedded delivery options. According to our findings, observed futures prices are on average 2% lower than theoretical futures prices over the 1990-2008 time period, priced two months prior to the first day of delivery months.

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7 Appendix

7.1 Transition parameters

We give, for the Vasicek (1977), CIR (1985) and Hull-White (1990) models, the closed-form formulas for the transition parameters $A_{k,i}^{t,t+\delta} \equiv A_{k,i}^{\delta}$ and $B_{k,i}^{t,t+\delta} \equiv B_{k,i}^{\delta}$ defined respectively in (21) and (22) as well as for the discount factor $\rho(r, t, t + \delta)$ defined in (1). For all models, the derivation of these closed-forms starts from the distribution of the random vector

$$\left(r_{t+\delta}, \int_t^{t+\delta} r_u du \right) \tag{36}$$

conditional on the value of r_t, for $0 \leq t \leq t + \delta$. For proofs and more details about the derivation of these closed-forms we refer to Ben-Ameur *et al.* (2007).

7.1.1 The Vasicek model

Under the risk-neutral probability measure, the interest-rate process is the solution to the following stochastic differential equation

$$dr_t = \kappa(\overline{r} - r_t)dt + \sigma dB_t, \quad \text{for } t \geq 0, \tag{37}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion, κ is the mean reversion speed, $\overline{r}$ is the long-term mean and σ is the volatility. For the Vasicek model, the distribution of the random vector (36) conditional on $r_t = r$ is bivariate normal with mean

$$\mu(r,\delta) = (\mu_1(r,\delta), \mu_2(r,\delta)) = \left(\overline{r} + e^{-\kappa\delta}(r-\overline{r}), \overline{r}\delta + \frac{1-e^{-\kappa\delta}}{\kappa}(r-\overline{r})\right) \quad (38)$$

and covariance matrix

$$\sum(\delta) = \begin{bmatrix} \sigma_1^2(\delta) & \sigma_{12}(\delta) \\ \sigma_{21}(\delta) & \sigma_2^2(\delta) \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa\delta}\right) & \frac{\sigma^2}{2\kappa^2}\left(1-2e^{-\kappa\delta}+e^{-2\kappa\delta}\right) \\ \sigma_{21} & \frac{\sigma^2}{2\kappa^3}\left(-3+2\kappa\delta+4e^{-\kappa\delta}-e^{-2\kappa\delta}\right) \end{bmatrix}. \quad (39)$$

The discount factor and the transition parameters are then given by

$$\rho(r,t,t+\delta) = \exp(-\mu_2(r,\delta) + \sigma_2^2(\delta)/2), \quad (40)$$

$$A_{k,i}^{t,t+\delta} = e^{-(\mu_2(a_k,\delta)+\sigma_2^2(\delta)/2)}\left[\Phi(x_{k,i}) - \Phi(x_{k,i-1})\right], \quad (41)$$

and

$$\begin{aligned} B_{k,i}^{t,t+\delta} &= e^{-(\mu_2(a_k,\delta)+\sigma_2^2(\delta)/2)}\left[(\mu_1(a_k,\delta) - \sigma_{12}(\delta))(\Phi(x_{k,i}) - \Phi(x_{k,i-1}))\right. \\ &\qquad \left. -\sigma_1(\delta)(e^{-x_{k,i}^2} - e^{-x_{k,i-1}^2})/\sqrt{2\pi}\right], \end{aligned} \quad (42)$$

where

$$\begin{aligned} x_{k,i} &= (a_i - \mu_1(a_k,\delta) + \sigma_{12}(\delta))/\sigma_1 \text{ for } i = 0, ..., q, \\ x_{k,-1} &= -\infty \end{aligned} \quad (43)$$

and Φ is the standard normal distribution function.

7.1.2 The CIR model

Under the risk-neutral probability measure, the interest-rate process is the solution to the following stochastic differential equation

$$dr_t = \kappa(\overline{r} - r_t)dt + \sigma\sqrt{r_t}dB_t, \quad \text{for } t \geq 0. \quad (44)$$

For the CIR model, the distribution of the random vector (36) conditional on $r_t = r$ is characterized by its Laplace transform:

$$\begin{aligned} & E\left[\exp(-\omega \int_t^{t+\delta} r_u du - \upsilon r_{t+\delta}) \mid r_t = r\right] \\ = \; & \exp(X(\delta,\omega,\upsilon) - rY(\delta,\omega,\upsilon)), \end{aligned} \tag{45}$$

where

$$X(\delta,\omega,\upsilon) = \frac{2\kappa\bar{r}}{\sigma^2}\log\left[\frac{2\gamma(\omega)e^{(\gamma(\omega)+\kappa)\delta/2}}{(\upsilon\sigma^2+\gamma(\omega)+\kappa)(e^{\gamma(\omega)\delta}-1)+2\gamma(\omega)}\right], \tag{46}$$

$$Y(\delta,\omega,\upsilon) = \frac{\upsilon(\gamma(\omega)+\kappa+e^{\gamma(\omega)\delta}(\gamma(\omega)-\kappa))+2\omega(e^{\gamma(\omega)\delta}-1)}{(\upsilon\sigma^2+\gamma(\omega)+\kappa)(e^{\gamma(\omega)\delta}-1)+2\gamma(\omega)} \text{ and} \tag{47}$$

$$\gamma(\omega) = \sqrt{\kappa^2+2\omega\sigma^2}. \tag{48}$$

For the CIR model, the discount factor and the transition parameters are given by

$$\rho(r,t,t+\delta) = \exp(X(\delta,1,0) - rY(\delta,1,0)), \tag{49}$$

$$A_{k,i}^{t,t+\delta} = \rho(a_k,t,t+\delta)\sum_{u=0}^{\infty} e^{-\lambda_\kappa/2}\frac{(\lambda_\kappa/2)^u}{u!}\left[F_{d+2u}(\frac{a_{i+1}}{\eta}) - F_{d+2u}(\frac{a_i}{\eta})\right], \tag{50}$$

and

$$\begin{aligned} B_{k,i}^{t,t+\delta} = \; & \rho(a_k,t,t+\delta)\eta\sum_{u=0}^{\infty} e^{-\lambda_\kappa/2}\frac{(\lambda_\kappa/2)^u}{u!}\left[-2(a_{i+1}f_{d+2u}(\frac{a_{i+1}}{\eta}) - a_i f_{d+2u}(\frac{a_i}{\eta})\right. \\ & \left. +(d+2u)(F_{d+2u}(\frac{a_{i+1}}{\eta}) - F_{d+2u}(\frac{a_i}{\eta}))\right], \end{aligned} \tag{51}$$

where F_{d+2u} and f_{d+2u} are the distribution and the density functions of a chi-square random variable with $d+2u$ degrees of freedom,

$$\gamma = \sqrt{\kappa^2+2\sigma^2}, \tag{52}$$

$$\eta = \frac{\sigma^2\left(e^{\gamma\delta}-1\right)}{2\left((\gamma+\kappa)(e^{\gamma\delta}-1)+2\gamma\right)}, \tag{53}$$

$$d = \frac{4\kappa\bar{r}}{\sigma^2} \text{ and} \tag{54}$$

$$\lambda_k = \frac{8\gamma^2 e^{\gamma\delta} a_k}{\sigma^2\left[(\gamma+\kappa)(e^{\gamma\delta}-1)+2\gamma\right](e^{\gamma\delta}-1)}. \tag{55}$$

7.1.3 The Hull-White model

For the extended Vasicek model, the distribution of the random vector (36) conditional on $r_t = r$ is bivariate normal with mean

$$\mu^*(r,t,\delta) = (\mu_1^*(r,t,\delta), \mu_2^*(r,t,\delta)), \tag{56}$$

where

$$\mu_1^*(r,t,\delta) = e^{(-\kappa\delta)}\left(rf\left(t+\delta\right) - f\left(t\right)\right) + \frac{\sigma^2}{2\kappa^2}\left(\left(1-e^{(-\kappa\delta)}\right) + e^{(-\kappa t)}\left(e^{(-\kappa(t+\delta))} - e^{(-\kappa t)}\right)\right) \text{ and } \tag{57}$$

$$\begin{aligned}\mu_2^*(r,t,\delta) &= (r - f\left(t\right))\frac{\left(1-e^{(-\kappa\delta)}\right)}{\kappa} + (t+\delta)\, y\left(t+\delta\right) - ty\left(t\right) \\ &\quad + \frac{\sigma^2}{2\kappa^2}\delta - \frac{\sigma^2}{2\kappa^3}\left(1-e^{(-\kappa\delta)}\right) + \frac{\sigma^2}{4\kappa^3}\left(e^{(-\kappa(t+\delta))} - e^{(-\kappa t)}\right)^2.\end{aligned} \tag{58}$$

The covariance matrix, the discount factor and the transition parameters are given by the same expressions as in the Vasicek model, but with μ replaced with μ^*. It is worthwhile mentioning that, unlike the Vasicek and CIR models, the transition parameters under the Hull-White model are not only function of the time interval δ, but also function of time t and therefore need to be updated at each time t.

7.2 Futures pricing algorithm

1. **Initialization:**

   Define $\mathcal{G}$. Define ε. Set $\widetilde{v}_{\overline{n}}^h(a_{k'}, a_k) = 0$ for all $a_{k'}, a_k \in \mathcal{G}$.

2. **Step 1:** (end-of-the-month, $m = n, ..., \overline{n}$)

   2.1 Set $m = \overline{n}$. Set $k' = 1$.

   2.2 Set $g_n\left(a_{k'}\right)$ using (29) with $T = t_{\overline{n}+1}^5$ and $m' = t_n^2$.

   2.3 Compute $\widetilde{g}_n^*\left(a_{k'}\right)$:

   2.3.1 Apply the optimization procedure at $(m, a_{k'}, a_k)$ for all $a_k \in \mathcal{G}$ yielding

$$\widetilde{v}_m^a(a_{k'}, a_k) = v_m^a\left(a_{k'}, a_k\right).$$

2.3.2 Apply the interpolation procedure, setting $h(a_k) = \widetilde{v}^a_m(a_{k'}, a_k)$, $a_k \in \mathcal{G}$, yielding

$$\widehat{v}^a_m\left(a_{k'}, r\right) = \widehat{h}(r),$$

and apply the expectation procedure to $\widehat{h}(r) = \widehat{v}^a_m\left(a_{k'}, r\right)$ at $t = t^8_m$ and $\tau = t^5_{m+1}$ for all $a_k \in \mathcal{G}$, yielding

$$\widetilde{v}^e_m\left(a_{k'}, a_k\right) = \widetilde{h}\left(t^8_m, t^5_{m+1}, a_k\right) \quad \text{for all} \quad a_k \in \mathcal{G}.$$

2.3.3 Compute

$$\widetilde{v}_m\left(a_{k'}, a_k\right) = \max\left[\widetilde{v}^e_m\left(a_{k'}, a_k\right), \widetilde{v}^h_m\left(a_{k'}, a_k\right)\right] \quad \text{for all } a_k \in \mathcal{G}$$

and apply the interpolation procedure, setting $h(a_k) = \widetilde{v}_m(a_{k'}, a_k)$, $a_k \in \mathcal{G}$, yielding

$$\widehat{v}_m\left(a_{k'}, r\right) = \widehat{h}(r).$$

2.3.4 While $m \geq n$, apply the expectation procedure to $\widehat{h}(r) = \widehat{v}_m\left(a_{k'}, r\right)$ at $t = t^8_{m-1}$ and $\tau = t^8_m$ for all $a_k \in \mathcal{G}$, yielding

$$\widetilde{v}^h_{m-1}\left(a_{k'}, a_k\right) = \widetilde{h}\left(t^8_{m-1}, t^8_m, a_k\right) \quad \text{for all } a_k \in \mathcal{G},$$

set $m = m - 1$ and go to step 2.3.1,

Else, apply the expectation procedure to $\widehat{h}(r) = \widehat{v}_n\left(a_{k'}, r\right)$ at $t = t^2_n$ and $\tau = t^8_n$, yielding $\widetilde{v}^2_n\left(a_{k'}\right) = \widetilde{h}\left(t^2_n, t^8_n, a_{k'}\right)$.

2.3.5 While $\left|\widetilde{v}_n\left(a_{k'}\right)\right| > \varepsilon$, apply the root finding procedure to update $g_n\left(a_{k'}\right)$ and go to step 2.3.1.

Else, set $\widetilde{g}^*_n\left(a_{k'}\right) = g_n\left(a_{k'}\right)$.

2.4 While $k' < q$, set $k' = k' + 1$ and go to step 2.2.

2.5 Apply the interpolation procedure, setting $h(a_k) = \widetilde{g}^*_n\left(a_k\right)$, $a_k \in \mathcal{G}$, yielding

$$\widehat{g}^*_n\left(r\right) = \widehat{h}(r),$$

and apply the expectation procedure to $\widehat{h}(r)$ at $t = t^8_{n-1}$ and $\tau = t^2_n$, for all $a_k \in \mathcal{G}$, yielding

$$\widetilde{g}^8_{n-1}(a_k) = \widetilde{h}\left(t^8_{n-1}, t^2_n, a_k\right) \quad \text{for all} \quad a_k \in \mathcal{G}.$$

3. **Step 2** (delivery month, $m = \underline{n}, ..., n-1$)

 3.1 Set $m = n - 1$.

 3.2 Set $k' = 1$.

 3.3 Set $g_m(a_{k'})$ using (29) with $T = t^5_{m+1}$and $m' = t^2_m$.

 3.4 Compute $\widetilde{g}^*_m(a_{k'})$:

  3.4.1 Apply the optimization procedure at $(m, a_{k'}, a_k)$ for all $a_k \in \mathcal{G}$ as in step 2.3.1, yielding $\widetilde{v}^a_m(a_{k'}, a_k) = v^a_m(a_{k'}, a_k)$.

  3.4.2 Apply the interpolation and expectation procedures at $t = t^8_m$ and $\tau = t^5_{m+1}$ as in step 2.3.2, setting $h(a_k) = \widetilde{v}^a_m(a_{k'}, a_k)$, $a_k \in \mathcal{G}$, yielding $\widetilde{v}^e_m(a_{k'}, a_k) = \widetilde{h}\left(t^8_m, t^5_{m+1}, a_k\right)$ for all $a_k \in \mathcal{G}$.

  3.4.3 Using (14), compute

$$\widetilde{v}^h_m(a_{k'}, a_k) = g_m(a_{k'})\,\rho\left(a_k, t^8_m, t^2_{m+1}\right) - \widetilde{g}^8_m(a_k) \quad \text{for all } a_k \in \mathcal{G}.$$

  3.4.4 Compute

$$\widetilde{v}_m(a_{k'}, a_k) = \max\left[\widetilde{v}^e_m(a_{k'}, a_k), \widetilde{v}^h_m(a_{k'}, a_k)\right] \quad \text{for all } a_k \in \mathcal{G}$$

  and apply the interpolation procedure as in step 2.3.3, setting $h(a_k) = \widetilde{v}_m(a_{k'}, a_k)$, $a_k \in \mathcal{G}$, yielding $\widehat{v}_m(a_{k'}, r) = \widehat{h}(r)$.

  3.4.5 Apply the expectation procedure to $\widehat{h}(r) = \widehat{v}_m(a_{k'}, r)$ at $t = t^2_m$ and $\tau = t^8_m$ as in step 2.3.4, yielding $\widetilde{v}^2_m(a_{k'}) = \widetilde{h}\left(t^2_m, t^8_m, a_{k'}\right)$.

  3.4.6 While $|\widetilde{v}_m(a_{k'})| > \varepsilon$, apply the root finding procedure to update $g_m(a_{k'})$ and go to step 3.4.1,

  Else set $\widetilde{g}^*_m(a_{k'}) = g_m(a_{k'})$.

 3.5 While $k' < q$, set $k' = k' + 1$ and go to step 3.3.

 3.6 Apply the interpolation procedure as in step 2.5, setting $h(a_k) = \widetilde{g}^*_m(a_k)$, $a_k \in \mathcal{G}$, yielding $\widehat{g}^*_m(r) = \widehat{h}(r)$.

3.7 While $m \geq \underline{n}$, apply the expectation procedure to $\widehat{h}(r)$ as in step 2.5 at $t = t^8_{m-1}$ and $\tau = t^2_m$, for all $a_k \in \mathcal{G}$, yielding $\widetilde{g}^8_{m-1}(a_k) = \widetilde{h}\left(t^8_{m-1}, t^2_m, a_k\right)$ for all $a_k \in \mathcal{G}$, set $m = m - 1$ and go to step 3-2.

Else, apply the expectation procedure to $\widehat{h}(r)$ as in step 2.5 at $t = t^2_{m-1}$ and $\tau = t^2_m$, for all $a_k \in \mathcal{G}$, yielding $\widetilde{g}^2_{m-1}(a_k) = \widetilde{h}\left(t^2_{m-1}, t^2_m, a_k\right)$ for all $a_k \in \mathcal{G}$.

4. **Step 3** (before the delivery month, $m = -1, \ldots, \underline{n} - 1$)

4.1 Set $m = \underline{n} - 1$.

4.2 Using (16), compute
$$\widetilde{g}^*_m(a_k) = \frac{\widetilde{g}^2_m(a_k)}{\rho(a_k, t^2_m, t^2_{m+1})}.$$

4.3 Apply the interpolation procedure as in step 3.6, setting $h(a_k) = \widetilde{g}^*_m(a_k)$, $a_k \in \mathcal{G}$, yielding $\widehat{g}^*_m(r) = \widehat{h}(r)$.

Apply the expectation procedure to $\widehat{h}(r)$ as in step 3.7 at $t = t^2_{m-1}$ and $\tau = t^2_m$, for all $a_k \in \mathcal{G}$, yielding $\widetilde{g}^2_{m-1}(a_k) = \widetilde{h}\left(t^2_{m-1}, t^2_m, a_k\right)$ for all $a_k \in \mathcal{G}$.

4.4 While $m \geq -1$, set $m = m - 1$ and go to step 4.2.

7.3 Hull-White input parameters

Table 5: Hull-White input parameters (1)

contract	r (inception)	r (day1)	γ_0	γ_1	γ_2	γ_3	γ_4	kappa	sigma	rbar	cmin	cmax	basket size
Mar-90	0.0783	0.0808	0.0799	-0.0052	0.0017	0.0038	2.2575	0.7426	0.0166	0.0725	0.0725	0.14	27
Jun-90	0.0805	0.0794	0.0864	0.0298	-0.0111	-0.0372	2.3047	0.7451	0.0164	0.0732	0.0725	0.14	28
Sep-90	0.08	0.0764	0.0844	-0.0261	0.014	0.022	1.1552	0.7453	0.0161	0.0732	0.0725	0.14	29
Dec-90	0.0737	0.0728	0.089	0.0439	-0.0527	-0.0633	0.5834	0.7428	0.0159	0.0727	0.0725	0.14	29
Mar-91	0.0666	0.0627	0.0811	-0.0654	0.0345	0.051	2.1857	0.728	0.0157	0.0715	0.0725	0.14	30
Jun-91	0.0594	0.0579	0.0829	-0.09	0.0326	0.067	1.0718	0.6837	0.0156	0.0694	0.0725	0.14	31
Sep-91	0.0576	0.0551	0.0838	-0.0491	0.0363	0.0207	2.2985	0.6506	0.0155	0.068	0.0725	0.14	32
Dec-91	0.0525	0.0451	0.0776	-0.1008	0.0498	0.0757	2.1391	0.6117	0.0153	0.0663	0.0725	0.14	33
Mar-92	0.0396	0.0414	0.0741	-0.1244	0.0469	0.0902	1.9885	0.4947	0.0152	0.0605	0.0725	0.14	33
Jun-92	0.0411	0.0382	0.0792	-0.1347	0.0435	0.0975	1.2261	0.4684	0.0151	0.0589	0.0725	0.14	33
Sep-92	0.0363	0.0322	0.078	-0.008	-0.0707	-0.0347	0.8084	0.4146	0.0149	0.0552	0.0725	0.14	34
Dec-92	0.0267	0.034	0.0721	0.0292	-0.1054	-0.0764	0.9013	0.3281	0.0148	0.047	0.0725	0.14	35
Mar-93	0.0319	0.0303	0.0723	-0.1279	0.0432	0.0875	1.5546	0.3442	0.0147	0.0487	0.07125	0.14	36
Jun-93	0.0296	0.0314	0.0687	-0.1004	0.0289	0.0606	1.9863	0.3125	0.0145	0.045	0.07125	0.14	36
Sep-93	0.0306	0.0308	0.0661	-0.0574	-0.004	0.0209	1.6562	0.3135	0.0144	0.0451	0.0625	0.14	36
Dec-93	0.0298	0.0319	0.0646	-0.0189	0.0062	-0.0174	4.8875	0.2986	0.0142	0.0431	0.0625	0.14	35
Mar-94	0.0316	0.0358	0.0675	-0.0125	0.0087	-0.0253	4.2684	0.3005	0.0141	0.0434	0.0625	0.14	35
Jun-94	0.0382	0.0428	0.0773	-0.0185	-0.0262	-0.0257	0.9551	0.3201	0.014	0.046	0.0625	0.14	34
Sep-94	0.0432	0.0467	0.0787	0.0257	-0.0327	-0.0682	1.1897	0.3462	0.0139	0.0491	0.0625	0.14	34
Dec-94	0.0505	0.0571	0.0799	0.0654	-0.0703	-0.1078	0.5045	0.3612	0.0138	0.051	0.0625	0.14	34
Mar-95	0.0595	0.0594	0.08	-0.0045	0.0084	-0.0273	0.5675	0.3845	0.0138	0.0541	0.0625	0.14	34
Jun-95	0.0594	0.0567	0.0761	0.0456	-0.0357	-0.065	1.9079	0.3872	0.0137	0.0546	0.0625	0.14	33
Sep-95	0.0568	0.0545	0.0691	-0.0196	-0.0052	0.0072	3.1737	0.3843	0.0135	0.054	0.0625	0.14	34
Dec-95	0.0553	0.0545	0.068	0.0158	-0.0258	-0.029	2.6534	0.3813	0.0134	0.0536	0.0625	0.14	33
Mar-96	0.052	0.0498	0.0625	-0.0097	-0.0149	-0.0004	2.538	0.3793	0.0133	0.0532	0.06	0.14	34
Jun-96	0.052	0.0523	0.0702	0.0147	-0.019	-0.0343	3.4813	0.3756	0.0132	0.0527	0.06	0.14	33
Sep-96	0.0527	0.0532	0.0716	0.0303	-0.0231	-0.0516	2.4899	0.3779	0.0131	0.053	0.06	0.14	34
Dec-96	0.051	0.0508	0.0712	0.0316	-0.0284	-0.0547	1.9028	0.3781	0.013	0.053	0.06	0.1325	34

Table 6: Hull-White input parameters (2)

contract	r (inception)	r (day1)	γ_0	γ_1	γ_2	γ_3	γ_4	kappa	sigma	rbar	cmin	cmax	basket size
Mar-97	0.0519	0.0524	0.0698	0.0224	-0.0212	-0.0425	2.1868	0.3743	0.0129	0.0525	0.06	0.1325	35
Jun-97	0.0532	0.0507	0.0731	0.0487	-0.0312	-0.0721	2.1524	0.3787	0.0128	0.0531	0.06	0.1325	35
Sep-97	0.0518	0.0521	0.0705	0.0562	-0.0402	-0.0772	3.0884	0.375	0.0127	0.0526	0.06	0.1325	36
Dec-97	0.051	0.0527	0.0657	0.0406	-0.0327	-0.057	2.6386	0.3755	0.0126	0.0526	0.06	0.1325	36
Mar-98	0.0532	0.0526	0.0606	0.0281	-0.025	-0.0363	2.7663	0.3792	0.0125	0.0531	0.06	0.1325	36
Jun-98	0.0512	0.0508	0.0615	0.0463	-0.0378	-0.058	2.641	0.377	0.0124	0.0527	0.06	0.1325	36
Sep-98	0.0509	0.0492	0.0579	0.0501	-0.036	-0.0588	2.0162	0.3761	0.0123	0.0526	0.0525	0.1325	37
Dec-98	0.0423	0.0449	0.0534	0.0823	-0.0852	-0.0943	1.8211	0.3692	0.0123	0.0517	0.0525	0.1325	38
Mar-99	0.0449	0.0471	0.0559	0.046	-0.0539	-0.0576	2.226	0.3662	0.0123	0.0511	0.0525	0.1325	39
Jun-99	0.0444	0.0477	0.0612	0.0395	-0.0394	-0.058	2.8272	0.3666	0.0122	0.0512	0.0525	0.125	38
Sep-99	0.0468	0.0497	0.0628	0.03	-0.0225	-0.0482	1.9255	0.369	0.0122	0.0514	0.0525	0.1175	37
Dec-99	0.0488	0.0527	0.0648	0.0327	-0.0214	-0.051	2.5613	0.3704	0.0121	0.0516	0.0525	0.1125	36
Mar-00	0.0548	0.0576	0.0682	0.0391	-0.0249	-0.057	1.1305	0.3787	0.012	0.0526	0.0525	0.10625	35
Jun-00	0.0587	0.0574	0.0591	0.0165	0.0094	-0.0193	1.235	0.3838	0.012	0.0535	0.0525	0.10625	36
Sep-00	0.06	0.0627	0.0599	0.0068	0.0047	-0.0075	1.1883	0.3843	0.0119	0.0536	0.0525	0.09875	35
Dec-00	0.0627	0.0623	0.0594	0.1173	-0.0525	-0.1301	0.3104	0.3864	0.0119	0.0542	0.0525	0.0925	34
Mar-01	0.0587	0.0484	0.055	-0.0295	-0.0128	0.0369	1.7184	0.3867	0.0118	0.054	0.0525	0.09125	34
Jun-01	0.0422	0.0367	0.0579	-0.0368	-0.0079	0.0218	1.9842	0.373	0.0119	0.0517	0.0525	0.09125	33
Sep-01	0.0367	0.0343	0.0593	-0.0478	-0.0085	0.0259	1.1523	0.3483	0.0119	0.0492	0.0525	0.09125	33
Dec-01	0.0226	0.0185	0.0573	0.0139	-0.0739	-0.0488	1.1885	0.3178	0.012	0.0463	0.0525	0.09125	32
Mar-02	0.0173	0.0178	0.0595	-0.0783	-0.0124	0.0362	0.9623	0.2715	0.012	0.0415	0.0525	0.09125	32
Jun-02	0.0179	0.0172	0.0611	-0.0732	-0.0105	0.0294	0.8037	0.2674	0.0119	0.041	0.0525	0.09125	31
Sep-02	0.0171	0.0169	0.059	-0.0733	-0.0111	0.0313	1.3337	0.2582	0.0118	0.0398	0.0525	0.09125	30
Dec-02	0.0163	0.0126	0.0548	-0.0751	-0.0173	0.0369	2.2753	0.2505	0.0117	0.0388	0.0525	0.09125	30
Mar-03	0.0118	0.012	0.0566	-0.0324	-0.0537	-0.0124	1.6375	0.2269	0.0117	0.0353	0.0525	0.09125	30
Jun-03	0.0117	0.0117	0.0549	-0.0738	-0.0274	0.0309	1.8526	0.221	0.0116	0.0343	0.0525	0.09	29
Sep-03	0.0089	0.0098	0.0516	0.0102	-0.0892	-0.0532	1.462	0.2084	0.0116	0.0321	0.0525	0.09	29
Dec-03	0.0088	0.0096	0.0521	-0.1604	0.0851	0.1172	4.0397	0.202	0.0115	0.0309	0.0525	0.08875	28

Table 7: Hull-White input parameters (3)

contract	r (inception)	r (day1)	γ_0	γ_1	γ_2	γ_3	γ_4	kappa	sigma	rbar	cmin	cmax	basket size
Mar-04	0.0088	0.0097	0.0573	-0.0769	-0.0123	0.0281	1.7499	0.1975	0.0114	0.0299	0.0525	0.0875	27
Jun-04	0.0095	0.0097	0.0537	-0.0631	-0.022	0.0186	1.9382	0.1964	0.0114	0.0297	0.0525	0.0875	27
Sep-04	0.0101	0.0143	0.0578	0.0253	-0.0665	-0.0747	1.4895	0.1961	0.0113	0.0297	0.0525	0.0875	26
Dec-04	0.0152	0.0206	0.054	0.0418	-0.0777	-0.0818	1.482	0.2078	0.0113	0.032	0.0525	0.0875	26
Mar-05	0.0199	0.0255	0.0513	0.088	-0.0953	-0.1212	1.2188	0.2161	0.0112	0.0336	0.0525	0.0875	25
Jun-05	0.0266	0.0279	0.0511	0.0394	-0.041	-0.0651	1.9096	0.229	0.0112	0.036	0.0525	0.0875	24
Sep-05	0.0302	0.0335	0.0459	0.0574	-0.0493	-0.0739	1.9063	0.2312	0.0111	0.0364	0.0525	0.08125	23
Dec-05	0.0322	0.0399	0.0465	0.0867	-0.0566	-0.1038	0.8517	0.2371	0.0111	0.0376	0.0525	0.08125	23
Mar-06	0.0405	0.0445	0.047	0.0708	-0.0518	-0.0783	1.3003	0.2434	0.0111	0.0388	0.045	0.08125	23
Jun-06	0.0466	0.0475	0.0504	0.0324	-0.0231	-0.0368	1.422	0.2512	0.0111	0.0407	0.045	0.08125	22
Sep-06	0.0481	0.0507	0.0533	0.0652	-0.0375	-0.0723	0.8645	0.2522	0.011	0.0411	0.045	0.08	21
Dec-06	0.0467	0.0521	0.0479	0.092	-0.058	-0.0948	0.8574	0.2533	0.011	0.0413	0.045	0.07625	20
Mar-07	0.0479	0.0525	0.0466	0.0196	0.0046	-0.0219	0.2489	0.2545	0.011	0.0416	0.045	0.07625	21
Jun-07	0.0512	0.048	0.0499	0.0561	-0.0512	-0.055	1.3296	0.2556	0.011	0.0423	0.045	0.07625	22
Sep-07	0.0455	0.0455	0.0508	0.0868	-0.0522	-0.0956	0.5122	0.2539	0.011	0.0412	0.045	0.07625	21
Dec-07	0.0352	0.0355	0.049	0.1264	-0.0953	-0.1431	0.7892	0.2503	0.011	0.0399	0.045	0.07625	20
Mar-08	0.0309	0.0199	0.0467	0.1384	-0.1338	-0.1555	1.0448	0.2408	0.0111	0.0378	0.04375	0.07625	20

7.4 Observed *vs.* theoretical futures prices

Table 8: Observed vs. theoretical futures prices (1)

Contract	Inception			Day 1 of the DM		
	Observed	Theoretical	Difference	Observed	Theoretical	Difference
Mar-90	98.40625	100.329807	-1.923557	92.65625	100.05593	-7.39968
Jun-90	91.90625	92.67514	-0.76889	94.375	93.056412	1.318588
Sep-90	94.28125	95.199369	-0.918119	88.75	95.617884	-6.867884
Dec-90	90.40625	91.701602	-1.295352	95.28125	92.387567	2.893683
Mar-91	96.8125	99.967477	-3.154977	95.25	100.534853	-5.284853
Jun-91	95.3125	98.675985	-3.363485	95.0625	98.93197	-3.86947
Sep-91	93.5	96.253235	-2.753235	98.15625	97.104595	1.051655
Dec-91	100.03125	104.719791	-4.688541	100.09375	106.118183	-6.024433
Mar-92	104.09375	110.596185	-6.502435	100.3125	110.491767	-10.179267
Jun-92	99.40625	104.912949	-5.506699	100.09375	105.631667	-5.537917
Sep-92	100.96875	107.687302	-6.718552	105.15625	109.185747	-4.029497
Dec-92	106.25	114.847873	-8.597873	103.40625	113.85578	-10.44953
Mar-93	105.53125	111.698345	-6.167095	111.875	112.527052	-0.652052
Jun-93	109.21875	116.370489	-7.151739	112	116.455788	-4.455788
Sep-93	113.90625	118.749529	-4.843279	119.53125	119.402965	0.128285
Dec-93	119.40625	121.677572	-2.271322	115.84375	121.953267	-6.109517
Mar-94	113.625	116.305612	-2.680612	110.96875	116.169335	-5.200585
Jun-94	104.625	104.963668	-0.338668	104.28125	105.79824	-1.51699
Sep-94	101.375	102.851986	-1.476986	103.59375	104.508871	-0.915121
Dec-94	98.59375	99.202812	-0.609062	98.375	101.544409	-3.169409
Mar-95	98.71875	97.561444	1.157306	103.9375	100.798213	3.139287
Jun-95	104.5625	105.494974	-0.932474	113.875	107.078705	6.796295
Sep-95	113.5625	114.104653	-0.542153	113.5625	114.82787	-1.26537
Dec-95	114.3125	114.630663	-0.318163	119.9375	115.224198	4.713302
Mar-96	121.34375	120.297266	1.046484	116.03125	120.981639	-4.950389
Jun-96	111.78125	112.847489	-1.066239	107.96875	113.342746	-5.373996
Sep-96	109.5625	109.546977	0.015523	107.90625	110.330758	-2.424508
Dec-96	109.90625	110.024529	-0.118279	116.25	111.167576	5.082424
Mar-97	111.375	111.235002	0.139998	110.65625	111.984539	-1.328289
Jun-97	107.46875	107.556113	-0.087363	110.15625	109.342355	0.813895
Sep-97	111.6875	111.730518	-0.043018	113.59375	112.535728	1.058022
Dec-97	116.34375	115.613803	0.729947	119.28125	115.948731	3.332519
Mar-98	121.46875	119.798768	1.669982	119.65625	120.38474	-0.72849
Jun-98	120.75	119.667794	1.082206	122.03125	120.407564	1.623686
Sep-98	123.53125	121.606848	1.924402	126.84375	122.818804	4.024946
Dec-98	132.65625	132.572808	0.083442	130.1875	132.418695	-2.231195

Table 9: Observed vs. theoretical futures prices (2)

Contract	Inception			Day 1 of the DM		
	Observed	Theoretical	Difference	Observed	Theoretical	Difference
Mar-99	127.46875	128.587439	-1.118689	120.59375	128.408532	-7.814782
Jun-99	119.9375	122.3878	-2.4503	116.5625	122.220973	-5.658473
Sep-99	115.53125	117.189526	-1.658276	114.1875	117.495451	-3.307951
Dec-99	112.84375	115.517199	-2.673449	111.875	115.412156	-3.537156
Mar-00	89.875	89.313724	0.561276	94.875	89.854988	5.020012
Jun-00	97.8125	98.395572	-0.583072	96.5	99.558029	-3.058029
Sep-00	97.6875	98.953069	-1.265569	100.5625	98.724865	1.837635
Dec-00	98.09375	97.319612	0.774138	102.0625	101.558027	0.504473
Mar-01	106.28125	109.032667	-2.751417	106.25	110.40985	-4.15985
Jun-01	103.875	107.554017	-3.679017	100.96875	108.792633	-7.823883
Sep-01	100.90625	104.820956	-3.914706	104.4375	105.342928	-0.905428
Dec-01	103.875	110.900584	-7.025584	105.15625	112.666146	-7.509896
Mar-02	100.125	105.928179	-5.803179	103	106.125005	-3.125005
Jun-02	98.25	102.272102	-4.022102	102.34375	103.376459	-1.032709
Sep-02	102.9375	108.283813	-5.346313	112.71875	108.80916	3.90959
Dec-02	113.3125	119.411898	-6.099398	109.34375	120.747683	-11.403933
Mar-03	110.15625	115.8982	-5.74195	115.875	116.32504	-0.45004
Jun-03	112.75	118.582212	-5.832212	119.5625	118.521818	1.040682
Sep-03	117.09375	121.634924	-4.541174	105.6875	121.851594	-16.164094
Dec-03	112.0625	116.822155	-4.759655	108.84375	117.052693	-8.208943
Mar-04	107.90625	113.170733	-5.264483	113.8125	113.559115	0.253385
Jun-04	113.46875	118.349956	-4.881206	105.84375	118.812429	-12.968679
Sep-04	106.8125	110.50898	-3.69648	112.625	111.444562	1.180438
Dec-04	111.46875	114.543795	-3.075045	110.96875	114.368071	-3.399321
Mar-05	112.53125	114.471798	-1.940548	112.90625	115.062629	-2.156379
Jun-05	111.78125	113.070535	-1.289285	118.40625	114.290928	4.115322
Sep-05	117.625	118.211076	-0.586076	118.25	118.326621	-0.076621
Dec-05	113.75	113.693861	0.056139	112.0625	114.364261	-2.301761
Mar-06	114.375	114.589959	-0.214959	112.5625	114.463788	-1.901288
Jun-06	109.03125	109.944224	-0.912974	106.46875	110.308925	-3.840175
Sep-06	105.75	106.11142	-0.36142	110.75	107.22811	3.52189
Dec-06	112.53125	112.425885	0.105365	114.625	112.32355	2.30145
Mar-07	111.71875	111.905101	-0.186351	112.875	113.04965	-0.17465
Jun-07	111.34375	112.482169	-1.138419	108.53125	113.612338	-5.081088
Sep-07	108.03125	108.31816	-0.28691	111.40625	111.507787	-0.101537
Dec-07	111.71875	112.631735	-0.912985	118	115.856367	2.143633
Mar-08	117.90625	120.165775	-2.259525	119.671875	126.34062	-6.668745