Rewarding Trading Skills Without Inducing Gambling*

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Abstract

This paper develops a model of active portfolio management in which fund managers may secretly gamble in order to manipulate their reputation and attract more funds. We show that such trading strategies may expose investors to severe losses and are more likely to occur when fund managers are impatient, their trading skills are scalable and generate a high profit per unit of risk. We study long-term contracts that deter this behavior. Contracts with a guaranteed minimal compensation that increases each time the manager's reputation reaches a new high-water mark eliminate risk-shifting incentives for sufficiently well-established managers. Contracts that simultaneously increase and defer the manager's expected fee after abnormally high returns curb gambling incentives as well.

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"Hedge funds are investment pools that are relatively unconstrained in what they do. They are relatively unregulated, charge very high fees, will not necessarily give you your money back when you want it, and will generally not tell you what they do. They are supposed to make money all the time, and when they fail at this, their investors redeem and go to someone else who has recently been making money. Every three or four years they deliver a one-in-a-hundred year flood."

-Cliff Asness, Journal of Portfolio Management, 2004.

1 Introduction

The last thirty years have witnessed two important evolutions in financial markets. First, the management of large amounts of capital has been delegated to agents, such as hedge funds, who are subject to very few trading restrictions and disclose very little about their trading strategies. As a result, investors seeking to allocate their funds across such vehicles have to rely predominantly on the history of realized returns to assess future performance. Second, a rapid pace of financial innovation has made it possible to slice and combine a large variety of risks by trading a rich set of financial instruments.

These evolutions create room for a particular type of agency problem. Managers running out of genuine arbitrage opportunities may find it tempting to secretly take exposure to exotic risk factors or even gamble in order to temporarily improve their reputation. Strategies that generate frequent small positive excess returns, and rarely experience very large losses are especially appealing as they help disguise luck as skill. The spectacular demises of LTCM, Amaranth, and several other large hedge funds have left many investors with nothing, and suggested that this type of risk shifting was at play. Moreover, following the recent severe financial crisis, many observers argue that perverse incentives led to similar excessive and inefficient risk taking by financial institutions.

The goal of this paper is to develop a new framework for studying this risk-shifting problem. We investigate whether it is possible to eliminate risk-shifting incentives given the vast array of trading instruments available to modern fund managers who enjoy limited liability. Implicit in the view of some policymakers calling for a ban on so-called toxic instruments is that risk-shifting incentives are impossible to curb via contracting when speculators have access to exotic derivative instruments.

Our model builds upon the frictionless benchmark of Berk and Green (2004), who study career concerns in delegated fund management. As in their model, a fund manager and investors discover the manager's skills by observing her realized returns. Expected excess rates of returns increase with the manager's skill, but decrease with fund size. The fund size that optimally trades off scale and unit return increases with respect to the manager's skill. Competitive investors supply funds to the manager until they earn a zero net (after fees) expected return. In the beginning of each period, the manager sets fees that enable her to reach the optimal fund size and extract the entire surplus that she generates. Thus, social learning and competition among investors imply that both fund flows and managerial compensation strongly depend on the trading record.

We add a risk-shifting friction to this model. In contrast to many earlier papers on risk shifting, we consider a general setting in which the fund manager can secretly choose to trade fairly priced contracts with any arbitrary payoff distribution. This captures the large set of trading opportunities available to modern managers, and therefore, is an important case to tackle. By tampering with the distribution of realized returns, the manager may manipulate her reputation in order to increase future funds under management. We show that when risk shifting occurs, it takes a simple and natural form, a binary payoff analogous to the sale of deep out-of-the-money options. This greatly simplifies our analysis of risk-taking incentives.

Three factors that are conducive to risk shifting emerge from our analysis. The first one is the difference between the Sharpe ratios of skilled and unskilled managers. If this difference is large, the history of returns has a large impact on investors' beliefs about the manager's ability to generate future excess returns. The second factor is the scalability of trading skills - that is, the sensitivity of expected excess returns to fund size. If trading skills are scalable, a good reputation translates into large future fund size and thus into large future profits. Finally, because the manager can manipulate her reputation only temporarily, she finds it more valuable to do so when she is more impatient. These three factors determine the convexity of future expected gains as a function of realized returns, and thus affect inefficient risk-bearing incentives. In particular, the model predicts that "fallen star" managers (managers with a high initial potential who realize disappointing returns) are particularly prone to gambling. For a calibration consistent with that by Berk and Green (2004), we find that their frictionless equilibrium with short-term contracts breaks down in the presence of the risk-shifting problem.

We then study long-term contracts that address this friction. We analyze the case in which investors can fully commit to a contract, while the fund manager is unable to commit and can approach new competitive investors at any time. This is a common assumption in labor economics. It seems particularly relevant for hedge-fund managers who can easily operate in different jurisdictions and raise funds worldwide. We show that two types of contracts are useful to cope with the risk-shifting problem in this case.

First, investors can commit to provide some insurance to the manager if she does poorly. This reduces her incentive to gamble for resurrection. However, the scope of this insurance is severely limited by the manager's option to renegotiate better terms if her track record beats investors' expectations. Thus only well-established fund managers with a sufficiently high reputation can be offered such a contract. The contract consists in a guaranteed minimal compensation that increases each time the manager's reputation reaches a new high-water mark. Notice that guaranteed bonuses have been under severe criticism following the 2008 financial crisis. Our analysis implies that they may actually be stabilizing. Without them, a star manager would be tempted to gamble for resurrection after a series of bad shocks damages her reputation.

Second, we solve for an optimal contract that deters gambling even when the manager has a low initial reputation. The contract is designed to discriminate skills from luck. It exploits the fact that the impact of gambling on investors' beliefs vanishes in the long-run, when true skills are eventually revealed. Under this contract, the manager is entitled to receive at some future terminal date everything that she is expected to earn from the start of the contract. The key feature of the contract is that this future date is contingent on the trading record. The promised payment is further postponed and capitalized at the risk-free rate every time the manager's return looks suspiciously high given history. In light of the recent call for postponing a larger fraction of traders' compensation, our analysis suggests that deferring compensation only in case of abnormally high performance suffices to deter excessive risk taking.

Our paper bridges two strands of literature: the literature on risk-shifting and the literature on career concerns. The risk-shifting friction was first introduced by Jensen and Meckling (1976) as a source of value destruction within overly leveraged firms. Arguably, this friction is particularly relevant in the context of sophisticated financial institutions that can swiftly alter their risk profiles. Accordingly, a large asset-pricing literature studies the impact of nonconcave objective functions on the risk-shifting incentives of fund managers who have access to dynamically complete markets. Contributions include Basak, Pavlova, and Shapiro (2007), Carpenter (2000), and Ross (2004). Like this literature, we seek to identify the risk-taking strategies that optimally respond to nonconcave objectives. Within a simpler framework of risk-neutral agents, we extend this line of research in two directions. First, nonconcavities in the manager's objective arise endogenously in our model from managerial career concerns. Second, we study optimal contracting in the presence of this friction.

Biais and Casamatta (2000), Cadenillas, Cvitanić, and Zapatero (2007), Diamond (2001), Hellwig (2009), Ou-Yang (2003), and Palomino and Prat (2003) also study how compensation contracts should be structured when a fund manager can secretly increase risk. Unlike our paper, in which risk-shifting is problematic because career concerns introduce convexity in continuation utility, they study the interplay of risk-shifting with the need to elicit effort from the manager. Thus, these papers deliver very different predictions. While optimal contracts in their settings always feature some degree of pay-for-performance, we show that it can have devastating consequences for investors when career concerns enter the picture.

Our paper is also related to Goetzmann, Ingersoll, Spiegel and Welch (2007) who study manipulation-proof measures of managerial performance. They show that to be manipulation-proof a measure should take the form of a concave utility function averaged over return history. We also show that if the fund manager has nonconcave continuation utility she can engage in inefficient risk shifting, and that optimal contracts aim at concavifying the manager's objective.

Finally, our study relates to two recent extensions of the Berk and Green (2004) model. First, Berk and Stanton (2007) apply the Berk and Green setup to closed-end funds. In this case, learning affects the net asset value of the fund and not its size, which is fixed by construction. Berk and Stanton (2007) show that the impact of learning explains several features of the closed-end fund discount, and that the behavior of this discount crucially depends on the nature of the compensation contract. In particular, they find that the contract that we prove to be optimal for well established managers has plausible implications for the dynamics of the closed-end fund discount. Second, Dangl, Wu, and Zechner (2010) study an extension of Berk and Green in which a management company can fire a manager if her performance is not good enough. They restrict the analysis to short-term compensation contracts and solve for the optimal firing rule.

The paper is organized as follows. Section 2 defines and characterizes inefficient risk taking in a static framework. Section 3 introduces risk shifting in the dynamic model of delegated asset management set forth by Berk and Green (2004). Section 4 analyzes optimal contracting. Section 5 discusses extensions. Most proofs are relegated to the appendix.

2 Risk Shifting: Definition and Characterization

One of the central questions in delegated portfolio management is how much risk and what type of risk the manager is willing to take. If investors and the manager have different risk preferences, then contracting imperfections may result in less than optimal risk taking. Situations that are blatantly inefficient are those in which the manager has incentives to choose second-order dominated risk profiles. As is well-known, no risk-averse investor would agree with such a risk profile because it loads unnecessarily on risk without being rewarded for it (see Rothschild and Stiglitz (1970)). This section provides a general characterization of the objective functions that lead a manager to engage in such inefficient risk taking, and solves for the resulting risk profile in a simple static environment.

Suppose that a manager has one unit of capital to invest at date 0 and that her utility is some function U over the date 1 gross return R. At date 0, she can choose any distribution μ over the date 1 gross return subject to preserving its mean, which we normalize to one w.l.o.g. Formally, let M denote the set of Borelian probability measures over $[0, +\infty)$. Then the manager solves the following problem:

$$\begin{aligned} &\sup_{\mu \in M} \int_0^\infty U(R) d\mu(R) \\ &s.t. \quad \int_0^\infty R d\mu(R) = 1. \end{aligned} \tag{2.1}$$

Let $P(U)$ denote the solution of this problem. Clearly, if U is concave then $P(U) = U(1)$ and the solution is attained with the risk-free return, i.e., $R \equiv 1$. On the other hand, whenever $P(U) > U(1)$, the solution involves exposure to a risk which carries no risk premium. Accordingly, throughout the paper, we deem an objective function U to be conducive to risk shifting if and only if $P(U) > U(1)$.

Aumann and Perles (1965) have studied a very similar class of problems, and have established conditions under which the problem associated with a given objective function U has the same solution as the one associated with the concavification of U.[1] This section shows that the dual approach generates a simple and practical determination of $P(U)$.

Let $P^*(U)$ denote the solution of the dual problem, which given the primal problem (2.1), takes the following form:

$$\begin{aligned} P^*(U) &\equiv \inf_{(z_1, z_2) \in \mathbb{R}^2} z_1 + z_2 \\ s.t. \quad \forall y &\geq 0, z_1 + y z_2 \geq U(y). \end{aligned} \tag{2.2}$$

The dual problem minimizes the value at 1 of a straight line that is above the graph of U. Proposition 1 shows that under the mild restriction (2.3) on the utility function

[1] The main difference is that they optimize over functions instead of measures.

U, solutions to the primal and dual problems coincide. The role of condition (2.3) is to ensure that the infimum is reached in the dual problem (2.2). Graphically, it means that the function U does not have an asymptote with a strictly positive slope.

Proposition 1 *Let* $U : [0, +\infty) \to \mathbb{R}$ *be a continuous function such that*

$$\lim_{y \to +\infty} \frac{U(y)}{y} = 0, \tag{2.3}$$

then

$$P(U) = P^*(U).$$

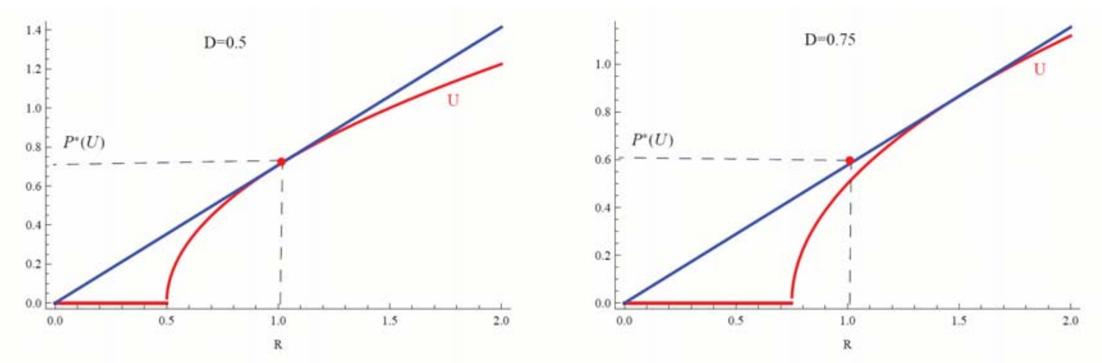
Proof. See the Appendix.■

Consider the following elementary example in which

$$U(R) = u(\max(R - D, 0)),$$

where u is increasing, concave, $u(0) = 0$, and $D > 0$. This corresponds to the situation in which the manager is protected by limited liability and faces a noncontingent liability equal to D at its consumption date. As is well-know, excessive leverage may create risk-shifting incentives. Figure 1 shows that if leverage is sufficiently low (low D), then there is no risk shifting. The manager still prefers to invest in the risk-free asset. However, as leverage increases, risk shifting occurs. Notice, that in this example, the

Figure 1: *Risk Shifting and Leverage*



optimal risk profile has a binary payoff: either 0 or a value which is strictly larger than 1, with appropriately assigned probabilities. This is so because the solution to (2.2) and hence to (2.1) is given by the value at 1 of the tangent to the objective that goes through the origin. The following proposition establishes that the solution to the general problem (2.1) also has this simple structure - either no gambling or a binary gamble. This significantly reduces the dimensionality of the problem, and simplifies the analysis of risk-shifting incentives.

Proposition 2 *Assume U satisfies (2.3), and is continuous, nondecreasing. If there is risk shifting, $P(U)$ can be attained with a binary payoff. Moreover, if U is twice differentiable and has at most one inflexion point, then a necessary and sufficient condition for risk shifting to occur is that $U'(1) > U(1) - U(0)$, and the binary payoff in case of risk shifting is of the form $\{0, R\}$, where $R > 1$.*

Proof. See the Appendix.■

Intuitively, as in the elementary example above, if the straight line corresponding to the solution of $P^*(U)$ is tangent to U in 1, then there is no risk shifting. If this is not the case, then this straight line must have at least one intersection with the graph of U on the left of 1, and at least one on the right. In this case, a binary gamble that pays off the abscissae of two such intersections solves $P(U)$. If the intersection on the left of 1 is at 0 (as is the case in the elementary example above), then the optimal risk profile consists in earning a superior return most of the time or in losing everything with a small probability. This payoff is similar to that obtained from the sale of deep-out-of-the-money options or the underwriting of any type of catastrophe insurance.

This result shows that if a manager's objective is conducive to risk shifting then she may adopt a risk profile that exposes investors to severe potential losses. This suggests that endowing sophisticated managers with inappropriate incentives may have dramatic implications for investors' welfare and financial stability. There are several reasons why a fund manager might have a nonconcave objective. For instance, in the presence of a standard moral-hazard problem - either an ex ante moral hazard problem as in Holmstrom (1979), or an ex post moral hazard as in Townsend (1979)) - providing the manager with incentives typically involves nonconcave payoffs. In the remainder of the paper, we focus instead on career concerns as the source of nonconcavities in the manager's preferences.

3 Risk Shifting and Career Concerns

As mentioned in the introduction, there is ample empirical evidence that investors chase past performance when allocating their funds across managers. This means that a manager's reputation has an important impact on her future assets under management and, in turn, her future fees. Thus the manager has strong incentives to control her reputation even if it comes at some cost. The goal of this section is to study the interplay of managers' career concerns with the risk-shifting problem introduced in Section 2. In Subsection 3.1, we present a frictionless model of career concerns in delegated asset management that closely follows Berk and Green (2004). In Subsection 3.2, we introduce the risk-shifting friction and study its impact on the equilibrium.

3.1 The Berk and Green (2004) model

Time is discrete and is indexed by $\{n\Delta t\}$, where $n \in \mathbb{N}$ and $\Delta t > 0$. There is a single consumption good which serves as the numéraire. Agents are of two types: a manager and many investors. Agents live forever, are risk-neutral, and discount future consumption at the instantaneous rate $r > 0$. The manager is protected by limited liability: She cannot have negative consumption. Investors receive a large endowment of the consumption good at each date $n\Delta t$, the manager does not. The manager has exclusive access to an investment technology. If the manager invests q_t consumption units at date t using her technology, she generates $q_{t+\Delta t}$ units at date $t + \Delta t$ such that

$$q_{t+\Delta t} = q_t e^{\left(r + a\theta - c(q_t) - \frac{\sigma^2}{2}\right)\Delta t + \sigma(B_{t+\Delta t} - B_t)}, \tag{3.1}$$

where $(B_t)_{t\geq 0}$ is a standard Wiener process, c is a continuous, nonnegative, and nondecreasing function with $c(0) = 0$, $\lim_{q\to+\infty} c(q) = +\infty$, and a and σ are strictly positive numbers. The parameter $\theta \in \{0; 1\}$ measures the manager's skills. This parameter is unobservable by both the manager and investors. All agents share the common date-0 prior that the manager is endowed with high skills - that is, $\theta = 1$ - with probability $\pi_0 \in (0, 1)$. The parameter a is the spread that a skilled manager can generate over the risk-free rate with her first dollar. As in Berk and Green (2004), the function c captures that many arbitrage opportunities or informational rents in financial markets are not perfectly scalable.

Except for the manager's skills, which nobody observes, each action and the manager's realized returns are publicly observable at each date $n\Delta t$. Let $(\pi_{n\Delta t})_{n\geq 0}$ denote the evolution of agents' beliefs about the probability that the manager is highly skilled - her perceived skills.

Contractual Environment

Investors are competitive and can fully commit to a contract. On the other hand, at the end of any period, the fund manager is free to walk away from a contract and sign a new one with competing investors. More precisely, at the end of each period $[t, t + \Delta t]$, after returns are realized and all contractual transfers for the period are made, the fund manager is free to terminate a contract without financial obligation to its investors, and to enter a new one with new investors starting at period $[t + \Delta t, t + 2\Delta t]$. In other words, commitment is one-sided. That a manager may quit to accept a better job elsewhere we take as an exogenous feature of the environment, reflecting prohibitions against involuntary servitude. This is a usual assumption in labor economics. We find it to be all the more plausible in the financial services industry. Hedge fund managers can swiftly move across jobs and financial centres because their activity requires few specific investments. Limited cross-border enforcement precludes covenants that would make such moves costly.

This baseline model is essentially identical to the model of Berk and Green (2004). The main modelling difference is that the distribution of skills is binomial in our setup while it is Gaussian in theirs. With our specification, the model is stationary in perceived skills π, and is therefore more tractable.

The optimal fund size is achieved when a manager with perceived skills π at date t receives a quantity of funds $q(\pi)$ that maximizes the net expected surplus that she creates over $[t, t + \Delta t]$:

$$q(\pi) = \arg\max_q q \left(\pi e^{(a-c(q))\Delta t} + (1-\pi)e^{-c(q)\Delta t} - 1 \right),$$

and thus

$$\lim_{\Delta t \to 0} q(\pi) = \arg\max_q q(\pi a - c(q)). \tag{3.2}$$

In this Section 3, we assume, as Berk and Green do, that the manager uses one-period linear contracts. We will study more general forms of interactions between investors and the manager in the following.

Assumption 1 *At each date t, the manager quotes a fee. The fee is the fraction of the date-$t + \Delta t$ assets under management (before new inflows/outflows of funds) that accrues to the manager.*

The following proposition solves for the optimal fund size, fees, and surplus generated as a function of the manager's perceived ability. It shows that, under Assumption 1, the manager is able to extract the entire first-best surplus since she acts as a monopolist facing a perfectly elastic demand at the rate of return r.

Proposition 3 *Under Assumption 1, let $q(\pi)$ denote the fund size, and $w(\pi)\Delta t$ and $v(\pi)\Delta t$ denote respectively the fee and expected profit over $[t, t+\Delta t]$ when the manager's* *perceived skills at date t are π. We have*

$$\begin{aligned}\lim_{\Delta t \to 0} q(\pi) &= \arg\max_q q(\pi a - c(q)), \\ \lim_{\Delta t \to 0} w(\pi) &= \pi a - c(q(\pi)), \\ \lim_{\Delta t \to 0} v(\pi) &= \max_q q(\pi a - c(q)). \end{aligned} \tag{3.3}$$

At each period, the manager generates and extracts the entire first-best surplus.

Proof. If a manager with perceived skills π quotes a fee $w\Delta t$, competitive investors will supply q as long as their net expected rate of return is equal to r. Thus, the fund supply $q(w)$ solves

$$(1 - w\Delta t)\left(\pi e^{(r+a-c(q))\Delta t} + (1-\pi)e^{(r-c(q))\Delta t}\right) = e^{r\Delta t},$$

therefore for a given w, as $\Delta t \to 0$, $q(w)$ is the solution (if any) to

$$\pi a = c(q(w)) + w. \tag{3.4}$$

The manager chooses w such that:

$$w = \arg\max_w w\Delta t \times q(w) = \arg\max_w (\pi a - c(q(w))) \times q(w).$$

Further,

$$\max_w (\pi a - c(q(w))) \times q(w) = \max_q q(\pi a - c(q)),$$

meaning that the manager extracts the entire first-best surplus. ■

Corollary 1 *If there exists $\alpha \geq 1$, $\beta > 0$ such that $c(q) = \beta q^{\frac{1}{\alpha-1}}$, then*

$$\lim_{\Delta t \to 0} v(\pi) = \beta^{1-\alpha}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}(a\pi)^{\alpha}. \tag{3.5}$$

This power specification for c and thus v will play an important role in what follows because it allows to model the scalability of trading skills with just one parameter α. As α increases, the manager's skills become more scalable and therefore, her expected profit becomes more sensitive to her reputation. In the hedge fund universe, global macro strategies are typically quite scalable. On the other hand, strategies based on shareholder activism may be more difficult to spread over increasing amounts of capital.

Berk and Green specify a linear cost function c corresponding to $\alpha = 2$. They show that their model matches quantitatively well the empirically observed relationship between mutual funds realized returns and inflows/outflows. In the limiting case in which α tends to 1, $v(\pi)$ tends to a linear surplus π.

Berk and Green's model assumes frictionless delegation of asset management. In reality, agency problems give rise to numerous conflict of interest between investors and managers.[2] Thus, while the Berk and Green model represents a very useful baseline case, we expand on it to study one of the most important agency problems in delegated asset management: risk shifting. The ability to change risk exposure can be especially problematic in hedge funds, which face far fewer portfolio restrictions and disclosure rules than mutual funds. Managers employed by hedge funds or banks' proprietary desks typically have a free hand at taking a very large variety of bets that outsiders do not continuously monitor.

3.2 Risk Shifting

Assumption 2 *(Risk shifting). At each date, the manager can secretly invest all or part of her funds in an alternative technology whose returns are perfectly scalable and independent from the returns on the technology described in (3.1). This technology enables her to generate a one-period gross return with any arbitrary distribution over* $[0, \infty)$ *with mean* $e^{r'\Delta t}$*, where* $r' \leq r$*. Investors observe returns realized at the reporting and contracting dates* $n\Delta t$*, at which the manager's position is marked-to-market.*

Assumption 2 adds informational asymmetry to the baseline model. This alternative technology allows the manager to take secret bets in order to manipulate her reputation. One can view the manager's technology described in (3.1) as a risky arbitrage opportunity in a new segment of the market which is still inefficient, and whose risk is not yet spanned by existing, fairly priced instruments. At the same time, it is conceivable that the manager can also secretly invest in a rich set of fairly priced liquid instruments. We allow for the possibility that concealing these trades from investors comes at a cost $r - r'$. Since such secret trades yield less in expectation than the manager's efficient technology (3.1), they would be undesirable absent career concerns. The assumption that these trades are independent from the efficient technology ensures that they cannot be used for arbitrage purposes. Finally, we assume perfect scalability of these bets for expositional simplicity, and this plays no role in our results.

The goal of this section is to determine the parameter values under which the introduction of this risk-shifting problem affects the outcome in the Berk and Green benchmark. We first outline two results that yield a tractable analysis of the manager's incentives to gamble. First, consider a situation in which the manager does not gamble

[2] See, e.g., Stracca (2006) for a recent survey.

and investors believe so. Assume that if investors believe that the manager is skilled with probability π_t at date t, then the manager collects an expected surplus $v(\pi_t)\Delta t$ for the $[t, t+\Delta t]$-trading round, where v is a continuous function. Proposition 4 gives the expected surplus $V(\pi)$ of the manager over her life-time when $\Delta t \to 0$ if she starts out with skills π.

Proposition 4 *Let*

$$V(\pi, \Delta t) = E_0 \left(\sum_{n=0}^{\infty} e^{-rn\Delta t} v(\pi_{n\Delta t}) \Delta t \right), \quad \pi_0 = \pi, \tag{3.6}$$

and

$$V(\pi) = \lim_{\Delta t \to 0} V(\pi, \Delta t).$$

We have

$$V(\pi) = \int_0^1 G(\pi, x)\, v(x) dx, \tag{3.7}$$

where

$$G(\pi, x) = \frac{2\sigma^2}{\psi a^2 x^2 (1-x)^2} \begin{cases} g(1-\pi) g(x) & \text{if} \quad 0 \le x \le \pi \\ g(\pi) g(1-x) & \text{if} \quad \pi \le x \le 1 \end{cases}, \tag{3.8}$$

and

$$\begin{aligned} \psi &= \sqrt{1 + \frac{8r\sigma^2}{a^2}}, \\ g(u) &= u^{\frac{1}{2}+\frac{1}{2}\psi} (1-u)^{\frac{1}{2}-\frac{1}{2}\psi}. \end{aligned} \tag{3.9}$$

Convergence of $V(\pi, \Delta t)$ *to* $V(\pi)$ *when* $\Delta t \to 0$ *is uniform over* $\pi \in (0, 1)$.

Proof. See the Appendix.■

The function $G(\pi, x)$ has an intuitive interpretation: It is a weighted probability that the manager will have a perceived ability x over her career if she starts out with perceived skills π. Note that if $r \to 0$, then $\psi \to 1$ and all the contribution to the continuation utility comes from what the manager gets when her skill level is either 0 or 1. This is again intuitive because in the long-run, there is a complete revelation of the manager's skills.

Suppose now that the manager gambles during her first trading round, realizes a return R, and from then on no longer gambles. Suppose that investors believe instead that she has never gambled. Let $\pi_{n\Delta t}$ be the manager's (correct) belief about her skills at date $n\Delta t$ and let $\pi^R_{n\Delta t}$ be investors' (incorrect) perception of her skills at date $n\Delta t$. We have

Proposition 5

$$\pi_t^R \equiv \lim_{\Delta t \to 0, n\Delta t \to t} \pi_{n\Delta t}^R = \frac{\pi_t R^{\frac{a}{\sigma^2}}}{1 - \pi_t + \pi_t R^{\frac{a}{\sigma^2}}}. \tag{3.10}$$

Proof. See the Appendix.■

Remark. In principle, the realization of $R = 0$ should perfectly reveal to investors that the manager gambled because the efficient technology delivers strictly positive returns with probability one. We assume instead that π_t^R is continuous at zero, so that investors infer $\pi = 0$ from observing $R = 0$. Equivalently, we could assume that traders who get caught gambling are excluded from the market.

In the remainder of the paper, all results will be established for Δt sufficiently small. To ease the exposition and to focus on economic intuition, we prove the results using properties of $V(\pi)$. That the function $V(\pi, \Delta t)$ also satisfies these properties if Δt is sufficiently small follows from the uniform convergence established in Proposition 4. The following heuristic derivations illustrate the impact of each parameter on the manager's incentives to gamble.

From (3.10) we can see that

$$\frac{d^2\pi_0^R}{dR^2}\Big|_{R=1} = -\frac{a}{\sigma^2}\pi_0(1-\pi_0)\left(1 - \frac{a}{\sigma^2} + 2\frac{a}{\sigma^2}\pi_0\right). \tag{3.11}$$

This suggests that all other things being equal, the incentive to manipulate beliefs is strongest when π_0 is low, and that it decreases with π_0, because reputation becomes more concave in R as π_0 increases. Suppose that the manager tries to "pick up nickels in front of a steamroller", that is, she gambles and realizes an instantaneous return of $1+\varepsilon$ with probability $1/(1+\varepsilon)$, where ε is small, or loses everything. Then from (3.10) for π_0 small, in case of success, her new reputation is approximately

$$\pi^R \simeq \pi_0 \left(1 + \frac{a}{\sigma^2}\varepsilon\right).$$

If $v(\pi) = cst \times \pi^\alpha$ and the discount rate is not too small then the continuation utility of the manager also behaves as π^α.[3] Therefore, the manager's net expected gain from the gamble is approximately

$$\frac{1}{1+\varepsilon}\left(\pi^R\right)^\alpha - \pi_0^\alpha \simeq \pi_0^\alpha\left(\frac{\alpha a}{\sigma^2} - 1\right)\varepsilon. \tag{3.12}$$

This suggests that whether there is risk shifting or not depends on whether the ratio $\frac{\alpha a}{\sigma^2}$ is greater or less than 1. The next proposition formalizes this broad intuition.

[3]See Lemma 4 in the Appendix for a formal proof.

Proposition 6 *Suppose that Assumption 1 holds and that* $c(q) = \beta \times q^{\frac{1}{\alpha-1}}$, $\alpha \geq 1$, $\beta > 0$.

i) If

$$\frac{a}{\sigma^2} \leq \frac{1}{\alpha} \quad \textit{and} \quad r' \leq r - \sigma^2/2 \tag{3.13}$$

then there exists an equilibrium in which the manager extracts the entire first-best surplus and does not engage in risk shifting.

ii) If

$$\frac{a}{\sigma^2} > \frac{1}{\alpha} \quad \textit{and} \quad r > \frac{\alpha(\alpha-1)a^2}{2\sigma^2} \tag{3.14}$$

then such an equilibrium does not exist.

Proof. The proof is relegated to the Appendix.

The next corollary shows that this result does not depend on the assumption of linear one-period contracts. It holds as long as contracts last only for one period.

Corollary 2 *Proposition 6 holds wihout Assumption 1 as soon as investors and the manager write one-period contracts.*

Proof. The only part of Assumption 1 that is used in the proof of Proposition 6 is that the manager extracts the entire expected surplus at each period in an equilibrium without risk shifting. Given that investors are competitive and the manager lacks commitment, this is more generally the case as long as contracts last for one period only.■

The two conditions in 3.13 play very distinct roles. Condition $a\sigma^2 \leq 1/\alpha$ implies that the continuation utility of the manager is concave in realized returns, which discourages risky bets. This is exactly the intuition outlined in (3.12). Condition $r' \leq r - \sigma^2/2$ ensures that the manager does not find it worthwhile to secretly invest in the risk-free asset. At the same time, if condition (3.14) is satisfied then the manager is tempted to gamble regardless of the value of r' because gambles become instantaneous fair lotteries as $\Delta t \to 0$.

Proposition 6 shows that risk shifting is particularly appealing when three conditions are met. First, a skilled manager generates high excess returns relative to an unskilled manager per unit of risk (a/σ^2 is large). Second, the manager's skills are sufficiently scalable (α is large). Finally, the manager should be sufficiently impatient (r is large). The intuition is the following. If a skilled manager generates high unit excess returns with low volatility and her strategy is scalable, then small good news about her skill translate into large expected future fund size, and thus into large future expected profits. This creates strong incentives to boost returns with a gamble. If the manager is patient, however, she cares only for the long run in which she ends up with the reputation that she deserves regardless of earlier gambling attempts. In this regard, it is worth noticing that condition $r > \frac{\alpha(\alpha-1)a^2}{2\sigma^2}$ in (3.14) is only a sufficient condition for risk shifting. Numerical analysis suggests that the manager is tempted to gamble under much milder conditions on r when condition $a/\sigma^2 > 1/\alpha$ is satisfied.

To assess whether the risk-shifting friction is likely to be important in practice, we consider a calibration consistent with that of Berk and Green. We set $\alpha = 2$, $a = 5\%$, $\sigma = 25\%$, and $r = 5\%$. Simple calculations then show that condition (3.14) is satisfied. More generally, condition $a/\sigma^2 > 1/\alpha$ is very likely to be satisfied in practice. It holds whenever the Sharpe ratio of a portfolio strategy is larger than its volatility ($a/\sigma > \sigma$), which is true for almost all investment strategies. In sum, this suggests that risk shifting matters in this model for parameter values that are empirically plausible.

4 Contracting Risk Shifting Away

The previous section shows that if the manager can write only one-period contracts and cannot commit not to extract the whole surplus then there is no equilibrium in which the fund manager does not gamble if conditions (3.14) hold. Situations that involve some degree of risk shifting are undesirable for several reasons. First, by assumption, gambles generate a lower expected return than the manager's efficient technology, which is clearly unappealing even for risk-neutral investors. Second, it leads to second-order dominated risk-profiles. This may not be costly for the risk-neutral agents. However, any risk-averse investor would despise it. Lastly, risk shifting in equilibrium would add extra noise in realized returns and thus reduce the speed at which investors learn about managerial skills. This in turn should lead to a suboptimal allocation of capital.[4] Therefore, we study in this section whether optimal contracting can help avoid risk shifting and restore the first-best under conditions (3.14).

Our previous restriction to one-period contracts corresponds to the situation of investors with high liquidity needs, who require the option to redeem their claims and receive their fair value in cash at the end of each period. In practice, however, many hedge funds successfully impose lock-ups to their investors, and/or require a significant advance notice before withdrawal. Also, the manager's compensation is often history-dependent, as evidenced by high-water marks. Accordingly, we study in this section whether multi-period arrangements help alleviating the risk-shifting problem. We study the particular case in which $\alpha = 1$. This case is particularly tractable because

[4]See section 5 for a discussion of equilibria with risk shifting.

the first-best continuation utility of the manager $V(\pi_t)$ is proportional to π_t, and is therefore a martingale as π_t is.

We exhibit two different contracts that, although drastically different in nature, both implement the first best. The first one seeks to provides as much insurance as possible to the manager so as to concavify its continuation utility. The second one exploits the fact that the impact of gambles on reputation vanishes in the long run. Throughout this section, we assume that $r' \leq r - \sigma^2/2$. As established in Proposition 6, this ensures that the manager will never be tempted to secretly invest in the risk-free asset.

4.1 Optimal Contracts for Well Established Managers

Clearly, if the manager could commit to a contract, then fully insuring him with a fixed wage equal to $rV(\pi_0)$ would eliminate any incentives to gamble. However, with limited commitment on the manager's side, such full insurance is not feasible. The manager would walk away from such a contract as soon as her reputation improves. Thus, a contract that provides some insurance to the manager must be structured such that the continuation utility of the manager is at any date at least as large as its outside options given its current reputation. This implies that the optimal contract for a given initial π_0 cannot be determined in isolation. Instead, all contracts for all initial skill levels depend on each other through the channel of outside options. If, however, we exhibit a contract such that the manager generates the first best - that is, he never gambles, and such that he always expects to earn at least $V(\pi_t)$ by staying in the contract, then we know that this contract should prevail in equilibrium. We now show that such a contract exists for managers that start out with a sufficiently high initial reputation π_0.

Harris and Holmstrom (1982) identified the contract that firms use to provide maximal consumption smoothing to a worker subject to a similar commitment problem as ours. Under this contract, the worker receives a wage that is downward rigid and increases each time her reputation reaches a new historical high. The next proposition shows that this contract deters risk shifting for sufficiently well-established managers.

Proposition 7 *i) Suppose that $\alpha = 1$, $\sigma^2 < a$, and $\frac{2r}{a} < \frac{\sigma^2}{a} + 1$. Then there exists $\overline{\pi} < 1$ such that a manager with initial skills above $\overline{\pi}$ can be hired to implement the first-best action. The manager extracts the entire associated surplus. The contract* *consists in paying a wage per unit of time equal to* $w(M_t)$*, where*

$$\begin{aligned} M_t &= \sup_{0 \le s \le t} \pi_s, \\ w(x) &= \frac{x}{x + (1-x)\frac{\psi+1}{\psi-1}}, \end{aligned} \tag{4.1}$$

where ψ *is defined in 3.9.*

ii) There also exists $\underline{\pi} \in (0,1)$ *such that a manager with initial skills below* $\underline{\pi}$ *would find gambling desirable when being offered this contract.*

Proof. See the Appendix.■

This simple closed-form characterization of the Harris and Holmstrom (1982) contract in continuous time is novel, to our knowledge. It can be useful in other applications. This contract is reminiscent of financial contracts with high-water marks that grant performance fees only when cumulative returns reach a new maximum. It provides more insurance than typical 2/20 hedge-fund contracts with high-water marks because the payment per unit of time is downward rigid, but it has a similar history dependence. The downward rigidity of payments is akin to guaranteed bonuses.

The Harris and Holmstrom contract works because it reduces the sensitivity of the manager's continuation utility to her performance. This reduces her incentive to gamble for resurrection after she does poorly. However, the scope of this insurance is severely limited by the manager's option to quit if she exhibits a good track-record. The lower the reputation, the higher is the value of this option to quit after beating expectations. One way to see this is to notice that the ratio of the current perceived ability x to the wage under the contract in (4.1) when the new performance maximum is reached, $w(x)$, monotonically decreases from $(\psi+1)/(\psi-1)$ to one, as a function of perceived ability. Thus, the Harris and Holmstrom contract supplies enough insurance to eliminate risk shifting only if initial reputation is sufficiently high.

4.2 Optimal Contracts With Stochastic Payment Date

Of course, a high reputation does not come out of thin air, but rather has to be earned. It is therefore important to study whether there are other contracts that allow for the employment of managers with lower initial reputations. We now show that there exist gambling-proof contracts that provide employment for arbitrary small initial reputation and grant a compensation arbitrary close to the first-best surplus. These contracts exploit the fact that the impact of gambling on investors' beliefs vanishes in the long run, when true skills are eventually revealed. We still consider the case $\alpha = 1$, which is analytically simpler since the continuation utility, equal to π_t/r, is a martingale. The following lemma introduces the key ingredient of such contracts.

Lemma 1 *Suppose that at date 0 the manager and investors share a common belief that the manager is skilled with prob $\pi_0 \in (0,1)$. Suppose that the manager gambles and realizes gross return $R > 1$. Then there exists $T(\pi_0, R)$ such that for any $t > T(\pi_0, R)$ the manager's expectation of investors' expectation of his perceived skills at time t is less than $R\pi_0$:*

$$\frac{1}{R} E_0 \left(\pi_t^R\right) < \pi_0, \tag{4.2}$$

where π_t^R is defined in (3.10).

Proof. As $t \to \infty$, $E_0 \left(\pi_t^R\right) \to \pi_0$.■

Let us fix an arbitrarily small $\varepsilon > 0$. Next we describe the gambling-proof contract for the manager with initial skill level π_0. If $\pi_0 < \varepsilon$ then the manager is not hired. Otherwise, she is hired and the contract is such that at each date t she is promised a future payment of

$$\frac{e^{rT_t} \pi_{t+T_t} (1-\varepsilon)}{r} \tag{4.3}$$

at date $t + T_t$, where T_t is defined recursively. T_0 is chosen sufficiently large that two conditions are met. First, it is large enough that the manager with initial skill level ε will not find it optimal to gamble and obtain reputation π_0. This T exists by Lemma 1. Second, it is large enough that investors are expected to earn at least $\pi_0 (1-\varepsilon)/r$ until time T.

Then $T_{t+\Delta t}$ is defined as follows. Let $R_{t+\Delta t}$ denote the gross return that the manager realizes over $[t, t+\Delta t]$. Then

$$\begin{cases} T_{t+\Delta t} = T_t - \Delta t, & \text{if} \quad R_{t+\Delta t} \leq 1 \\ T_{t+\Delta t} = \max\{T_t, T(\pi_t, R)\}, & \text{if} \quad R_{t+\Delta t} > 1, \end{cases} \tag{4.4}$$

In words, each time the manager realizes a positive excess return, the payment is sufficiently postponed that the manager never finds it ex ante appealing to gamble. Once the payment is made the contract is over.

By construction, the contract is gambling-proof against binary gambles that deliver return R with probability $1/R$ and 0 otherwise. If the manager does not engage in risk shifting, her continuation utility does not change since the expected payment is simply postponed and capitalized at the risk-free rate. Notice that paying the manager at a very remote date that is not contingent on her performance does not solve the risk shifting problem since the manager with a low reputation will be tempted to gamble just before this date. On the other hand, postponing the payment while conditioning on her future reputation deters gambling since investors' beliefs can be manipulated only temporarily. Finally, we restrict the attention to binary gambles with weight on 0 because direct computations show that the manager's utility satisfies the condition of Proposition 2.

These contracts are such that no investor can deviate and offer a new contract that increases the surplus by more than $\varepsilon > 0$ for a fund manager with any skill level. Since such contracts exist for any arbitrarily small $\varepsilon > 0$, in the limit they lead to situations that are arbitrarily close to the first-best. In other words, arbitrarily small costs of offering a new contract guarantee that these contracts define an equilibrium.

That such contracts are optimal in our setup clearly owes to risk neutrality and a universal discount rate. Still, we believe that they feature a novel mechanism that could help deter inefficient risk taking in practice. Consider a trader who is employed to beat a given benchmark. The idea behind our contracts is simply that whenever this trader largely beats the benchmark by a sufficiently large margin, her promised bonus should be postponed and based on a longer sample of her excess returns. This is a simple and intuitive way to prevent inefficient gambling without imposing excessively remote payments to traders with smooth and consistent track records. In practice, implementation takes the form of a simultaneous increase both in the fraction of assets under management that accrue to the manager and in the future date at which the value of these assets will be computed when she realizes an abnormally high performance.

5 Discussion

Equilibria with risk-shifting

We established that appropriately designed long-term contracts eliminate the risk-shifting problem caused by career concerns and manager's inability to commit not to renegotiate the contract. Our very broad formulation of risk-shifting has the advantage that the gambling-proof contracts that we design are robust to any form of gambling. This is important given the strong ability of the financial services industry to innovate and create new markets. When only one-period contracts are available and condition (3.14) holds then all equilibria feature some degree of risk shifting. The reader may wonder what type of risk shifting takes place in this case. We have shown that a manager with low reputation will be tempted to take binary gambles if investors believe that the manager does not gamble. This is why equilibria without gambling cannot be sustained in this case.

Equilibrium gambles, however, cannot be such binary bets in equilibria in which risk-shifting occurs (if such equilibria exist at all). Conditionally on gambling, equilibrium gambles must be random variables whose distribution has no atoms other than at zero. Otherwise, an optimal contract could punish the associated realizations with a zero payoff at no cost. Also, an equilibrium with gambling behavior can only happen in mixed strategies since if the manager gambled with probability one then investors would not update their beliefs at all. Thus in such an equilibrium, the manager must be indifferent between gambling and not gambling. This suggests that equilibrium gambles may still take a form of out-of-the-money options where small positive excess returns come at the expense of possibly loosing everything. The exact characterization of the equilibrium gambling behaviour in our fairly general dynamic setup, however, is a difficult problem.

The fund manager can commit

Although we find it probably less relevant in practice, it is worthwhile discussing the situation in which commitment power is on the other side of the relationship. Suppose that condition (3.14) holds and investors can enter only into one-period contracts. Suppose also now that the manager can fully commit to any future contingent action plan.

It is clear that in this case, if the manager wants to generate the first-best surplus, then she can do so: She can, for instance, commit to work for free, and thus credibly never gamble in the future. It is also clear that if the manager wants to implement the first-best, then she will be unable to extract the entire associated surplus. If the manager extracts the entire surplus then she must extract the entire surplus at each period because contracts last only one-period. In this case, however, she will gamble as established in Corollary 2. This implies that a manager willing to implement the first-best must leave some persistent excess returns to the competitive investors.

Notice that any contract that leaves some rent to competitive investors cannot be implemented only by quoting fees because competitive investors always supply funds until they earn zero expected excess returns. The manager, however, can implement the contract by quoting both a fee and a fund size at each period. To extract less than the first-best surplus, the manager must commit to a fee per period that is smaller than the first-best fee described in Proposition 3 so as to leave some of the total surplus to investors. Since investors would supply more funds than the optimum fund size in Proposition 3 with such smaller fees, the manager must also impose a restriction on the fund's size and turn down inflows.

The above observation may be empirically relevant. First, it provides a new rationale for the existence of caps on funds sizes. Second it is consistent with Jagannathan et al. (2006), who find that the best hedge funds are able to generate persistent after-fees excess returns for their investors. It is important to note, however, that these conclusions depend on the assumption that the manager will not find it to her advantage to gamble. This seems to be likely, given that risk shifting destroys the joint surplus and the manager can commit to any incentive compatible sharing rules. However, a formal proof of this fact seems to be difficult in our very general formulation of risk shifting. Conditions under which this holds might be easier to find if one assumes a smaller space of possible gambles.[5]

6 Conclusion

This paper studies risk-shifting incentives in delegated asset management. Using Berk and Green's (2004) model of the fund management industry as a benchmark, we show that career concerns create gambling incentives that can lead to devastating losses for investors if not curbed properly. Our model predicts that in order to improve their reputations, fund managers may opt for trading strategies that generate small excess returns at the cost of rare but very large losses, which is consistent with anecdotal evidence. The paper also investigates how compensation contracts might be structured to prevent such behavior. We find that the first-best can always be achieved with appropriate long-term contracts.

In general, there are several ways to deal with risk shifting in delegated asset management. Possible solutions include increased transparency, restrictions on the set of instruments that managers can trade, and compensation contract design. In this paper we focus on solving the risk-shifting problem using compensation design only. This makes the problem more challenging. Future research could combine optimal contracting with these additional means of addressing the risk-shifting problem. On the other hand, we assume that positions are always valued at a fair market price. Instruments that are more difficult to value such as illiquid securities or exotic derivative contracts would likely provide fund managers with additional risk-shifting incentives if trading losses can be concealed for some time. Future research could further explore these channels to engage in risk-transformation.

[5] See e.g., Acharya et al. (2010) and Malliaris and Yan (2010) who consider a case in which only one gamble with exogenous payoffs is available.

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Appendix

Proof of Proposition 1

Lemma 2 *There exists* $(z_1, z_2) \in \mathbb{R} \times [0, +\infty)$ *such that* (z_1, z_2) *satisfies (2.2) and* $P^*(U) = z_1 + z_2$.

Proof. Condition (2.3) implies that the set $Z = \{(z_1, z_2) : \forall y \geq 0, z_1 + yz_2 \geq U(y)\}$ is nonempty. It is closed, and there exists K such that

$$(z_1, z_2) \in Z \to z_1 \geq K, z_2 \geq K.$$

The function $(z_1, z_2) \to z_1 + z_2$ is continuous. Thus, there exists $(z_1,\, z_2) \in Z$ such that $P^*(U) = z_1 + z_2$. Condition (2.3) readily implies that $z_2 \geq 0$.■

Let μ be a probability measure that satisfies (2.1), and $(z_1,\, z_2) \in \mathbb{R}^2$ defined as in Lemma 2. We have

$$z_1 + z_2 = \int_0^\infty (z_1 + Rz_2)\, d\mu(R) \geq \int_0^\infty U(R) d\mu(R).$$

This implies that

$$P^*(U) \geq P(U).$$

Let us show that the reverse inequality also holds. Establishing the reverse inequality for U with compact support is without loss of generality: for all U satisfying (2.3), there clearly exists $V \in C_c([0, \infty))$ such that $V \leq U$ and $P^*(V) = P^*(U)$. We omit the straightforward proof of the following lemma.

Lemma 3 **a.** $P^*(U_1) \leq P^*(U_2)$ *for* $U_1,\, U_2 \in C_c([0, +\infty))$ *such that* $U_1 \leq U_2$,

b. $P^*(\lambda U) = \lambda P^*(U)$ *for* $U \in C_c([0, +\infty))$ *and* $\lambda \in [0, +\infty)$,

c. $P^*(U_1 + U_2) \leq P^*(U_1) + P^*(U_2)$ *for* $U_1,\, U_2 \in C_c([0, +\infty))$.

From Lemma 3, $P^*(\cdot)$ is a positively homogeneous and subadditive functional on $C_c([0, +\infty))$. The Hahn-Banach Theorem therefore implies that for any $U \in C_c([0, \infty))$, there exists a positive linear functional L_U on $C_c([0, \infty))$ such that $L_U \leq P^*$ and $L_U(U) = P^*(U)$. By the Riesz representation Theorem, there exists a Borelian measure μ_U on $[0, \infty)$ such that for all $V \in C_c([0, \infty))$

$$L_U(V) = \int_0^\infty V(R) d\mu_U(R).$$

For $M > 1$, let $u_M, v_M \in C_c([0,\infty)) \times C_c([0,\infty))$ such that

$$\begin{aligned} u_M(x) &= 1 \text{ on } [0,M], \, x \geq M \rightarrow u_M \leq 1, \\ v_M(x) &= x \text{ on } [0,M], \, x \geq M \rightarrow v_M \leq M. \end{aligned}$$

Clearly,

$$P^*(u_M) = P^*(v_M) = 1$$

Then

$$\begin{aligned} L_U(u_M) &= \int_0^\infty u_M(R)\, d\mu_U(R) \leq P^*(u_M) = 1, \\ L_U(v_M) &= \int_0^\infty v_M(R)\, d\mu_U(R) \leq P^*(v_M) = 1. \end{aligned}$$

Letting $M \rightarrow +\infty$ implies

$$\int_0^\infty d\mu_U(R) \leq 1, \int_0^\infty R d\mu_U(R) \leq 1,$$

and thus

$$P^*(U) = L_U(U) = \int_0^\infty U(R) d\mu_U(R) \leq P(U).$$

■

Proof of Proposition 2

Let a continuous nondecreasing function U satisfy condition (2.3). Let $(z_1, z_2) \in \mathbb{R}^2$ defined as in Lemma 2. Clearly, if $z_2 = 0$ then there is no gambling. If $z_2 > 0$, let

$$S = \{y \geq 0 : z_1 + z_2 y = U(y)\}.$$

That U is continuous together with condition 2.3 implies that S is a nonempty compact set. Let

$$y_1 = \min S, \; y_2 = \max S.$$

We now proceed in two steps.

Step 1. We show that $y_2 \geq 1 \geq y_1$.

Proof. We show that $y_2 \geq 1$. The proof that $y_1 \leq 1$ is symmetric. Assume that $y_2 < 1$. For some $\varepsilon \in (0, 1-y_2)$, let

$$\eta(\varepsilon) = \min_{y \geq y_2+\varepsilon} \left\{ \frac{z_1 - U(y)}{y} + z_2 \right\}.$$

By condition 2.3,

$$\lim_{y\to+\infty}\frac{z_1-U(y)}{y}+z_2=z_2>0,$$

and by definition of y_2

$$y\geq y_2+\varepsilon\to\frac{z_1-U(y)}{y}+z_2>0,$$

continuity of U therefore implies that $\eta(\varepsilon)>0$. Let (z_1', z_2') defined as

$$z_1'=z_1+(y_2+\varepsilon)\,\eta(\varepsilon)\,,\ \ z_2'=z_2-\eta(\varepsilon)\,.$$

First, the pair (z_1', z_2') satisfies (2.2). To see this, notice that

$$z_1'+yz_2'=z_1+yz_2+\eta(\varepsilon)\,(y_2+\varepsilon-y)\,.$$

Thus $z_1'+yz_2'>z_1+yz_2\geq U(y)$ for $y<y_2+\varepsilon$. Further $z_1'+yz_2'\geq z_1+yz_2-\eta(\varepsilon)\,y\geq U(y)$ for $y\geq y_2+\varepsilon$ by definition of $\eta(\varepsilon)$. Second, we have

$$z_1'+z_2'=z_1+z_2+(y_2+\varepsilon-1)\,\eta(\varepsilon)<z_1+z_2,$$

which contradicts the definition of (z_1, z_2). Thus it must be that $y_2\geq 1$.■

Step 2. Now, if $y_1=y_2$, then step 1 implies that $S=\{1\}$, and there is no gambling. If $y_1<y_2$, from

$$\begin{aligned} z_1+y_1z_2 &= U(y_1),\\ z_1+y_2z_2 &= U(y_2),\end{aligned}$$

we have

$$z_1+z_2=\frac{1-y_1}{y_2-y_1}U(y_2)+\frac{y_2-1}{y_2-y_1}U(y_1). \tag{A1}$$

From (A1), $P(U)=P^*(U)$ is attained with a payoff equal to y_1 with probability $\frac{y_2-1}{y_2-y_1}$ and y_2 with probability $\frac{1-y_1}{y_2-y_1}$. These probabilities are well-defined from step 1. Notice that there is gambling if and only if $y_1<1$ and $y_2>1$.

We now prove the last part of the proposition. Together with (2.3), that U has at most one inflexion point implies that U is either concave - in which case there is no gambling - or convex then concave. Consider the latter case. If there is no risk shifting then $P^*(U)$ should be solved by the tangent to U at 1, which requires $U'(1)\leq U(1)-U(0)$. Suppose now that $U'(1)\leq U(1)-U(0)$. It implies that U is concave over $[1,+\infty)$ (because $\frac{U(y)-U(0)}{y}$ is decreasing in $y=1$). In this case the tangent to U at 1 solves $P^*(U)$. Finally, if there is risk shifting, the solution to $P^*(U)$ is attained with a tangent that starts at 0.■

Proof of Proposition 4

We first show point-wise convergence. That is, we establish (3.7) for a fixed $\pi_0 = \pi$. By Bayes' theorem, $\pi_{n\Delta t}$, the perceived skills at date $n\Delta t$, satisfy

$$\pi_{n\Delta t} = \frac{\pi_0 \varphi_{n\Delta t}}{1 - \pi_0 + \pi_0 \varphi_{n\Delta t}}, \tag{A2}$$

where

$$\varphi_{n\Delta t} = \exp\left\{ \frac{a}{\sigma^2} \left(a \left(\theta - \frac{1}{2} \right) n\Delta t + \sigma B_{n\Delta t} \right) \right\} \tag{A3}$$

is the likelihood ratio process. Let us introduce the continuous-time process $(\pi_t)_{t\geq 0}$ that obeys

$$d\pi_t = \frac{a}{\sigma} \pi_t (1 - \pi_t) d\overline{B}_t, \quad \pi_0 = \pi,$$

where $\overline{B}_t = \frac{1}{\sigma}\left(\theta a t + \sigma B_t - a \int_0^t \pi_s ds\right)$. Then $(\overline{B}_t)_{t\geq 0}$ is a standard Wiener process under the agents' filtration (see Liptser and Shiryaev (1978)). Further, as $\Delta t \to 0$ and $n\Delta t \to t$, $\pi_{n\Delta t} \to \pi_t$ a.s. (see Liptser and Shiryaev (1978)). Hence, $V(\pi)$ can be re-written as

$$\begin{aligned} V(\pi) &= E_0 \int_0^\infty e^{-rt} v(\pi_t) dt, \\ s.to \quad d\pi_t &= \frac{a}{\sigma} \pi_t (1 - \pi_t) d\overline{B}_t, \quad \pi_0 = \pi. \end{aligned} \tag{A4}$$

By the Feynman-Kac formula, the function V solves the following linear second-order differential equation:

$$\frac{a^2}{2\sigma^2} \pi^2 (1 - \pi)^2 V''(\pi) - rV(\pi) + v(\pi) = 0. \tag{A5}$$

From (A4) it follows that

$$V(0) = v(0)/r, \quad V(1) = v(1)/r. \tag{A6}$$

The corresponding homogeneous equation

$$\frac{a^2}{2\sigma^2} \pi^2 (1 - \pi)^2 V''(\pi) - rV(\pi) = 0 \tag{A7}$$

has two regular singular points at 0 and 1. The solutions of the homogeneous equation are linear combinations of the two independent solutions

$$\begin{aligned} g(\pi) &= (1-\pi)^{\frac{1}{2}+\frac{1}{2}\psi}\pi^{\frac{1}{2}-\frac{1}{2}\psi}, \\ h(\pi) &= g(1-\pi). \end{aligned}$$

From here, formulas (3.7) and (3.8) are standard results in the theory of inhomogeneous differential equations. The function G is the Dirichlet-Green function for the differential operator associated with the homogeneous differential equation (see, e.g., Driver (2003)).

We now show that $V(\pi_0, \Delta t)$ converges to $V(\pi_0)$ uniformly in π_0 as $\Delta t \to 0$. We have

$$V(\pi_0) - V(\pi_0, \Delta t) = E_0 \sum_{n=0}^{\infty} \int_{n\Delta t}^{(n+1)\Delta t} \left(e^{-rt} v(\pi_t) - e^{-rn\Delta t} v(\pi_{n\Delta t})\right) dt.$$

Thus it enough to show that $\forall \varepsilon > 0$, $\exists \overline{\Delta t}$ such that $\forall \Delta t < \overline{\Delta t}$ and $\forall \pi \in [0,1]$

$$\sup_{s \le \Delta t} \sup_{\pi_0 \in [0,1]} \left| E\left(v(\pi_s) - v(\pi_0)\right) \right| < \varepsilon. \tag{A8}$$

By change of variables (A8) can be re-written as

$$\sup_{s \le \Delta t} \sup_{\pi_0 \in [0,1]} \left| E\left(\widehat{v}(\pi_0, B_s, s) - \widehat{v}(\pi_0, 0, 0)\right) \right| < \varepsilon,$$

where

$$\widehat{v}(\pi_0, x, t) = v\left(\frac{\pi_0 \exp\left\{\frac{a}{\sigma^2}\left(a\left(\theta - \frac{1}{2}\right)t + \sigma x\right)\right\}}{1 - \pi_0 + \pi_0 \exp\left\{\frac{a}{\sigma^2}\left(a\left(\theta - \frac{1}{2}\right)t + \sigma x\right)\right\}} \right). \tag{A9}$$

Since v is continuous over [0,1] and thus uniformly continuous, it is enough to show that $\forall \pi_0 \in [0,1]$ and $\forall \varepsilon > 0$, $\exists \overline{\Delta t}$ such that $\forall \Delta t < \overline{\Delta t}$

$$\sup_{s \le \Delta t} \left| E\left(\widehat{v}(\pi_0, B_s, s) - \widehat{v}(\pi_0, 0, 0)\right) \right| < \varepsilon, \tag{A10}$$

which follows from the weak converge of the measures induced by B_s to the measure concentrated at 0 as $s \to 0$.■

Lemma 4 *Suppose condition 3.14 holds. Then there exists a finite limit*

$$0 < \lim_{\pi \to 0} V(\pi)\pi^{-\alpha} < \infty. \tag{A11}$$

Proof. We have

$$V(\pi)=\int_0^1 G(\pi,x)\,x^\alpha dx=\frac{2\sigma^2}{a^2\psi}\left[\begin{array}{l}(1-\pi)^{\frac{1}{2}(1+\psi)}\pi^{\frac{1}{2}(1-\psi)}\int_0^\pi(1-x)^{-\frac{1}{2}(\psi+3)}x^{\alpha+\frac{1}{2}(\psi-3)}dx\\+(1-\pi)^{\frac{1}{2}(1-\psi)}\pi^{\frac{1}{2}(1+\psi)}\int_\pi^1(1-x)^{\frac{1}{2}(\psi-3)}x^{\alpha-\frac{1}{2}(3+\psi)}dx\end{array}\right],$$

where ψ is defined in (3.9). Further $r\sigma^2>\frac{a^2}{2}\alpha(\alpha-1)$ implies that $\psi>2\alpha-1$. Therefore,

$$\exists\lim_{\pi\to0}(1-\pi)^{\frac{1}{2}(1-\psi)}\pi^{\frac{1}{2}(1+\psi)}\int_\pi^1(1-x)^{\frac{1}{2}(\psi-3)}x^{\alpha-\frac{1}{2}(3+\psi)}dx\times\pi^{-\alpha}<\infty,$$

and

$$\exists\lim_{\pi\to0}(1-\pi)^{\frac{1}{2}(1+\psi)}\pi^{\frac{1}{2}(1-\psi)}\int_0^\pi(1-x)^{-\frac{1}{2}(\psi+3)}x^{\alpha+\frac{1}{2}(\psi-3)}dx\times\pi^{-\alpha}<\infty.\blacksquare$$

Proof of Proposition 5

Suppose that the manager gambles and realizes a return R over $[0,\Delta t]$, and from then on invests in her own storage technology. Let $\left(\pi^R_{n\Delta t}\right)_{n\in\mathbb{N}}$ denote the process - *under the manager's filtration* - of her skills as perceived by investors who believe instead that she has invested in her storage technology at date 0. These investors believe that

$$R=e^{\left(r+\theta a-c(q_0)-\frac{\sigma^2}{2}\right)\Delta t+\sigma B_{\Delta t}}.$$

From (A2) and (A3) it follows that

$$\begin{aligned}\pi^R_{\Delta t}&=\frac{\pi_0R^{\frac{a}{\sigma^2}}e^{\frac{a}{\sigma^2}\left(\frac{\sigma^2}{2}+c(q_0)-r-\frac{a}{2}\right)\Delta t}}{1-\pi_0+\pi_0R^{\frac{a}{\sigma^2}}e^{\frac{a}{\sigma^2}\left(\frac{\sigma^2}{2}+c(q_0)-r-\frac{a}{2}\right)\Delta t}},\\ \forall n\geq0,\quad \pi^R_{(n+1)\Delta t}&=\frac{\pi^R_{\Delta t}\frac{\varphi_{(n+1)\Delta t}}{\varphi_{\Delta t}}}{1-\pi^R_{\Delta t}+\pi^R_{\Delta t}\frac{\varphi_{(n+1)\Delta t}}{\varphi_{\Delta t}}}.\end{aligned}\tag{A12}$$

As $\Delta t\to0$

$$\lim_{\Delta t\to0}\pi^R_{\Delta t}=\frac{\pi_0R^{\frac{a}{\sigma^2}}}{1-\pi_0+\pi_0R^{\frac{a}{\sigma^2}}},$$

and

$$\lim_{\Delta t\to0}\frac{\varphi_{(n+1)\Delta t}}{\varphi_{\Delta t}}=\frac{\pi_t}{1-\pi_t}\frac{1-\pi_0}{\pi_0}.$$

Therefore,

$$\lim_{\Delta t\to0,\,n\Delta t\to t}\pi^R_{n\Delta t}=\frac{\pi_tR^{\frac{a}{\sigma^2}}}{1-\pi_t+\pi_tR^{\frac{a}{\sigma^2}}}\blacksquare.\tag{A13}$$

Proof of Proposition 6

Part 1. We first show that if $\sigma^2 > \alpha a$ then the manager does not engage in risk shifting in the Berk and Green equilibrium. There are two steps. In step 1, we show that any risky gamble is not desirable. Then in step 2, we demonstrate that the manager will not invest in the alternative technology at the risk-free rate.

Step 1. Suppose the manager believes that she is skilled with probability π_0. At the same time, investors believe that the manager is skilled with probability π_0' and that the manager never engages in risk shifting. We show that the manager has no incentives to deviate by taking a one-shot risky gamble in this case. Let $W(\pi_0, \pi_0', R, \Delta t)$ be the expected utility of the manager conditional on realizing a first-period return R.

Similar to the proof of Proposition 5 one can show that investors' perception of the manager's skills π_t^R is

$$\pi_t^R = \frac{\pi_t R^{\frac{a}{\sigma^2}}}{(1-\pi_t)\frac{(1-\pi_0')\pi_0}{(1-\pi_0)\pi_0'} + \pi_t R^{\frac{a}{\sigma^2}}}. \tag{A14}$$

Proposition 4 then implies that

$$\lim_{\Delta t \to 0} W(\pi_0, \pi_0', R, \Delta t) = \int_0^1 G(\pi_0, x)\, v\left(\frac{x R^{\frac{a}{\sigma^2}}}{(1-x)\frac{(1-\pi_0')\pi_0}{(1-\pi_0)\pi_0'} + x R^{\frac{a}{\sigma^2}}}\right) dx. \tag{A15}$$

Differentiating twice w.r.t. R shows that this function is concave in R when $\sigma^2 \geq \alpha a$. Hence the manager has no incentives to take a one-shot risky gamble in this case. Because this holds for arbitrary heterogeneous priors π_0, π_0', this implies that multi-period deviations cannot be desirable by backward induction.

Step 2. We now show that investing in the alternative technology at the risk-free rate r' is also not desirable. If the manager invests in her efficient storage technology then π_t evolves according to (A4). If, on the other hand, she invests in the risk-free asset π_t evolves as

$$d\pi = \frac{a}{\sigma}\pi(1-\pi)\left(r' - r + \frac{\sigma^2}{2} - (\pi a - c(q_t))\right) dt.$$

Suppose that at time t the manager allocates x_t percentage of her funds to the efficient storage technology and invests the rest in the risk-free asset. Then her continuation utility is

$$V(\pi, x) = E_0 \int_0^\infty e^{-rt} v(\pi_t) dt, \tag{A16}$$

$$s.to \quad d\pi_t = \frac{a}{\sigma}\pi_t(1-\pi_t)\left((1-x_t)\left(r' - r + \frac{\sigma^2}{2} - (\pi a - c(q_t))\right) + x_t d\overline{B}_t\right), \quad \pi_0 = \pi.$$

The optimal investment policy x_t that maximizes ($A16$) satisfies the HJB equation:

$$\sup_{x\in[0,1]} x^2\psi_2^2 V'' + (1-x)\psi_1 V' - rV + v = 0, \tag{A17}$$

where

$$\begin{aligned}\psi_1 &= \frac{a}{\sigma}\pi_t(1-\pi_t)\left(r' - r + \frac{\sigma^2}{2} - (\pi a - c(q_t))\right) < 0,\\ \psi_2 &= \frac{a}{\sigma}\pi_t(1-\pi_t) > 0.\end{aligned}$$

If $x_t \equiv 1$ then π_t is a martingale and by Jensen's inequality ($v(\pi) = \pi^\alpha$)

$$Ev(\pi_t) \geq v(\pi_0).$$

Therefore, at the optimal investment policy x_t, $rV(\pi, x) \geq v(\pi)$. Thus (A17) implies that the optimal policy is indeed $x_t \equiv 1$.

Part 2. Here we show that if $\sigma^2 < \alpha a$, $r\sigma^2 > \frac{a^2}{2}\alpha(\alpha-1)$ then for π_0 small enough, there exists a one-period gamble which makes the manager better off. Let $R = (1-\rho)^{-\sigma^2/a}$, $\rho \in [0,1)$. Consider the following gamble:

$$\begin{cases} R & \text{Prob.} \quad 1/R \\ 0 & \text{Prob.} \quad 1-1/R, \end{cases}$$

From (A15) the expected net gain from the above one-period gamble over perpetual investment in the efficient storage technology is

$$\int_0^1 G(\pi, x)\, x^\alpha u(x,\rho)dx, \tag{A18}$$

where

$$u(x,\rho) = \frac{(1-\rho)^{\sigma^2/a}}{(1-\rho(1-x))^\alpha} - 1. \tag{A19}$$

Since $\sigma^2 < \alpha a$, there exists $\bar{x}$ and some $\widehat{\rho} \in (0,1)$ such that for all $x \in [0,\bar{x}]$, $\frac{(1-\widehat{\rho})^{\sigma^2/a}}{(1-\widehat{\rho}(1-x))^\alpha} > 1+\varepsilon$ for some $\varepsilon > 0$ and therefore, $u(x,\widehat{\rho}) > \varepsilon > 0$. Thus for π small enough

$$\int_0^1 G(\pi, x)\, x^\alpha u(x,\widehat{\rho})dx > \int_\pi^1 G(\pi, x)\, x^\alpha u(x,\widehat{\rho})dx.$$

Using (3.8) we have

$$\int_\pi^1 G(\pi, x)\, x^\alpha u(x,\widehat{\rho})dx = \frac{2\sigma^2}{\psi a^2} g(\pi)\int_\pi^1 x^{\alpha-\frac{3}{2}-\frac{1}{2}\psi}(1-x)^{-\frac{3}{2}+\frac{1}{2}\psi}u(x,\widehat{\rho})dx. \tag{A20}$$

$r\sigma^2 > \frac{a^2}{2}\alpha(\alpha-1)$ implies that $\psi > 2\alpha - 1$. Therefore, the integral

$$\int_{\pi}^{1} x^{\alpha-\frac{3}{2}-\frac{1}{2}\psi}(1-x)^{-\frac{3}{2}+\frac{1}{2}\psi}u(x,\widehat{\rho})dx$$

diverges as $\pi \to 0$. In this case, its sign is determined by the sign of $u(\cdot,\widehat{\rho})$ in the neighborhood of 0, which is positive. Thus, the net gain from the gamble is positive.■

Proof of Proposition 7

Step 1. First we solve for the Harris and Holmstrom contract. Let us remind that

$$\pi_t = \frac{\pi_0\phi_t}{1-\pi_0+\pi_0\phi_t}, \quad \phi_t = e^{\frac{a}{\sigma}\overline{B_t^\theta}}, \quad \overline{B_t^\theta} = \frac{a}{\sigma}\left(\theta - \frac{1}{2}\right)t + B_t.$$

Let

$$M_t^\theta = \sup_{0\le s\le t} \overline{B_t^\theta}. \tag{A21}$$

Let $p_{\theta,t}$ denote the density of M_t^θ and let

$$\widehat{p}_{\theta,t} = \int_0^\infty e^{-rt}p_{\theta,t}. \tag{A22}$$

From formula 1.1.2 of Borodin and Salminen (page 250) we can deduce that

$$\widehat{p}_{1,t}(y) = \frac{a}{2\sigma r}(\psi-1)\,e^{-\frac{a}{2\sigma}(\psi-1)y}, \tag{A23}$$

$$\widehat{p}_{0,t}(y) = \frac{a}{2\sigma r}(\psi+1)\,e^{-\frac{a}{2\sigma}(\psi+1)y}, \tag{A24}$$

where ψ is defined in (3.9).

Assume the wage w is a function of $\sup_{0\le s\le t}\pi_t$. Since the manager gets all the surplus we must have

$$\frac{\pi_0}{r} \equiv \int_0^\infty w\left(\frac{\pi_0 e^{\frac{a}{\sigma}y}}{1-\pi_0+\pi_0 e^{\frac{a}{\sigma}y}}\right)(\pi_0\widehat{p}_{1,t}(y) + (1-\pi_0)\widehat{p}_{0,t}(y))\,dy. \tag{A25}$$

Let $z = e^{\frac{a}{\sigma}y} \Rightarrow y = \frac{\sigma\ln(z)}{a}$ and $dy = \frac{\sigma}{az}$. Hence, (A25) becomes

$$2\pi_0 \equiv \int_1^\infty w\left(\frac{\pi_0 z}{1-\pi_0+\pi_0 z}\right)(\pi_0(\psi-1)z + (1-\pi_0)(\psi+1))\,z^{-\frac{3+\psi}{2}}dz. \tag{A26}$$

If

$$w(x) = \frac{x}{x+(1-x)\frac{\psi+1}{\psi-1}}. \tag{A27}$$

Then

$$w\left(\frac{\pi_0 z}{1-\pi_0+\pi_0 z}\right)\left(\pi_0\left(\psi-1\right)z+\left(1-\pi_0\right)\left(\psi+1\right)\right)=\pi_0 z(\psi-1). \tag{A28}$$

Therefore, the RHS of (A29) is always equal to the LHS of (A29).

Step 2. Next, we prove that for sufficiently high π_0 the Harris and Holmstrom contract does not induce risk shifting. It is clear that the manager has the strongest incentives to engage in risk shifting when her perceived ability achieves an all time maximum. Let it be π_0. Suppose that the manager deviates and gambles during her first trading round but investors believe instead that she has invested in her storage technology at date 0. Let $R \geq 0$ denote the return that she realizes. The present value of her future earnings after the return R is realized and assuming she will no longer gamble from then on is

$$W(\pi_0,R)=\frac{1}{2}\int_1^\infty \max\left\{\frac{\pi_0 R^{\frac{a}{\sigma^2}}z}{\pi_0 R^{\frac{a}{\sigma^2}}z+(1-\pi_0)\frac{\psi+1}{\psi-1}},\frac{\pi_0}{\pi_0+(1-\pi_0)\frac{\psi+1}{\psi-1}}\right\}\times$$

$$\times\left(\pi_0(\psi-1)z+(1-\pi_0)(\psi+1)\right)z^{-\frac{3+\psi}{2}}dz. \tag{A29}$$

Direct computations show that $W(\pi_0,R)$ as a function of R is first convex and then concave. Therefore, applying Proposition 2 we can see that the necessary and sufficient conditions for the absence of risk shifting in this case is that

$$W'_R(\pi_0,1)\leq W(\pi_0,1)-W(\pi_0,0). \tag{A30}$$

Direct computations show that (A30) takes the following form:

$$\frac{a}{2\sigma^2}(1-\pi_0)\pi_0\int_1^\infty \frac{(\psi^2-1)\,z^{-\frac{1+\psi}{2}}dz}{\pi_0(\psi-1)z+(1-\pi_0)(\psi+1)}\leq \pi_0-\frac{\pi_0}{\pi_0+(1-\pi_0)\frac{\psi+1}{\psi-1}}, \tag{A31}$$

or

$$\frac{1}{2}\int_1^\infty \frac{(\psi^2-1)\left(\pi_0(\psi-1)+(1-\pi_0)(\psi+1)\right)z^{-\frac{1+\psi}{2}}dz}{\pi_0(\psi-1)z+(1-\pi_0)(\psi+1)}\leq\frac{2\sigma^2}{a}. \tag{A32}$$

We are interested in π_0 sufficiently close to 1. Therefore, for no risk shifting we need

$$(\psi-1)<\frac{2\sigma^2}{a}\Leftrightarrow\frac{2r}{a}<\frac{\sigma^2}{a}+1. \tag{A33}$$

Step 3. In order to show ii), simply notice that (A32) becomes

$$1 + \psi < \frac{2\sigma^2}{a}$$

as $\pi_0 \to 0$, which cannot hold when $\sigma^2 < a$. ■