Modeling Defaultable Securities with Recovery Risk

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Abstract

Although the Basel Committee has identified recovery risk as an important source of risk in relation to default, the impact of recovery rates on defaultable securities is not yet well understood. This paper proposes a discrete-time reduced form approach for pricing defaultable securities that incorporates stochastic recovery rates. We provide pricing formulas for risky bonds and credit default swap contracts in the case of an economy with an affine state vector. Faced with rich and realistic econometric representations of the state variables, the model stays tractable and can be estimated using standard techniques. We estimate a five-factor Gaussian model on BBB and B Standard & Poor's yield indices. Our analysis indicates that for both indices, the stochastic recovery model significantly outperforms a nested model where the unconditional recovery rate is estimated as a constant. The model is able to capture two important stylized facts of defaultable securities: The positive correlation between the loss given default, and the intensity of default and the negative correlation between the intensity of default and the risk-free interest rate.

JEL Classification: G12, G13

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1 Introduction

Two strands exist in the credit risk literature for modeling defaultable securities: Structural models and reduced-form models. In the former, credit events are triggered by changes in the firm's value relative to some barrier. This approach was pioneered by Merton (1974), who assumes that default occurs once the firm exhausts all its assets.[1] However, the structural approach is not straightforward to implement because, generally, the firm's assets are neither observable nor tradeable. Under the reduced-form approach, the value of the firm and its capital structure are not explicitly modeled, and the credit events are specified in terms of a jump process. This approach considers the default event as an unpredictable stopping time. Existing reduced form models (see Jarrow, Lando and Turnbull (1997), Lando (1998), Duffie and Singleton (1999)) use a continuous time framework and show that the price of any defaultable contingent claim can be computed as a conditional expectation with a modified discount factor. This facilitates the use of standard risk-free term structure tools for both theoretical and empirical purposes. It also renders reduced-form models particularly suitable for empirical implementation using risky bonds and credit default swaps (CDS).

The aim of this paper is to develop a discrete-time reduced form framework for the pricing of defaultable securities which incorporates a stochastic risk free interest rate, default intensity and recovery rate.[2] The tractable nature of our framework allows us to capture two important stylized facts characterizing the term structure of defaultable bonds. The first stylized fact is the time varying nature of the recovery rate. Empirical evidence, including Altman (2002) and Altman, Brady, Resti and Sironi (2005), suggests that recovery rates exhibit substantial variability and are negatively correlated with default rates. Acharya, Bharath and Srinivasan (2005) also show that the recovery rate is lower in a distressed economy than in a healthy economy. The second stylized fact is the correlation between recovery rates, the risk free term structure and default risk. Until very recently, most of the literature focused on the link between default risk and the risk free term structure (see Duffee (1998)) and ignored the impact of recovery rates on the term structure of credit spreads. A potential reason behind this lack of attention toward recovery rates is probably the ensuing computational burden. As a consequence, simplifying assumptions abound in the literature. For example, Jarrow, Lando and Turnbull (1997), Duffee (1999) and Driessen (2005) assume that the recovery rate is constant.[3] Das and Tufano (1995) model the time varying nature of the recovery rate but assume that the default risk is independent of the risk free term

[1] In Merton's model, default can only occur at maturity. Black and Cox (1976) relax this assumption by allowing default to occur when the firm's value reaches a lower barrier. Many papers extended the work of Merton (1974) and Black and Cox (1976), among them: Leland (1994), Leland and Toft (1996) and Collin-Dufresne and Goldstein (2001).

[2] See Rubinstein (1976) and Brennan (1976) for early studies on contingent claim valuation in discrete-time, Turnbull and Milne (1991) for the case of an economy where interest rates are stochastic, and Das and Sundaram (2000) for a discrete-time reduced form model.

[3] It is usually assumed that the constant recovery rate is arround 40% for corporate bonds and credit default swaps, and arround 25% for sovereigns.

structure to keep the model tractable.

In light of the recommendations of Basel II and the strong empirical evidence regarding the time varying nature of recovery rates, a recent literature on the joint identification of default and recovery risk has started to develop. A first strand focuses on the extraction of recovery rates implied by defaultable instruments. Zhang (2003) first showed that it is possible to extract the implied unconditional level of the recovery rate within a reduced-form framework. Using Argentinian CDS data, he demonstrates that the unconditional level of the recovery rate is around 25%, which roughly corresponds to the market practice. Pan and Singleton (2005) estimate a reduced-form model on sovereign CDS data and find that the estimated unconditional recovery is rather different from the usual market practice. Bakshi, Madan and Zhang (2006) explicitly model a stochastic recovery rate and derive pricing solutions for risky bonds. In a multifactor setting, these solutions prove rather complex and computationally intensive. This effectively limits the empirical implementation of the model to the single-factor case thereby constraining the correlation between the default intensity, the recovery rate and the level of the risk-free interest rate. Das and Hanouna (2006) develop a cross-sectional approach for extracting recovery rates from the CDS spreads that does not rely on time series information. Another strand in the literature focuses on realized recovery rates. Renault and Scaillet (2004) use data on defaulted debt to non-parametrically estimate the density of recovery rates. Guo, Jarrow and Zeng (2005) develop a reduced-form approach for modeling the realized recovery rate on defaulted debt and provide analytic expressions for both the pre and post defaulted bond. Chava, Stefanescu and Turnbull (2005) also perform a joint estimation of losses and default intensity using data on defaulted debt.

In this paper, we extend the literature on the joint time series estimation of default and recovery implied by defaultable instruments by modeling a stochastic recovery rate in addition to the pricing kernel and the hazard rate, as well as their respective correlations within a fairly rich econometric set-up. In the existing reduced-form models, allowing for stochastic recovery rates in a multifactor set-up results in rather complex pricing solutions as explicitly recognized by Bakshi, Madan and Zhang (2006). As an alternative, we propose a discrete-time framework for modeling defaultable instruments in which we adopt a finite horizon economy with discrete-time trading.[4] We model the pricing kernel, the hazard rate and the recovery rate as discrete-time adapted processes and show that a discrete-time methodology not only retains much of the intuition underlying the continuous time valuation framework but also proves more tractable than existing models. Pricing formulas are derived for defaultable bonds and CDS contracts when the recovery rate is stochastic. We also examine different recovery assumptions for risky bonds: recovery of Treasury (RT), recovery of face value (RFV) and recovery of market value (RMV). Provided that the conditional Laplace transform of the state vector is analytically known, our approach admits closed form solutions for prices of CDS contracts and risky bonds under RT and RFV. Under the assumption of RMV, Monte Carlo simulation must be used to price the defaultable bond, but our model allows us to

[4] See Gouriéroux, Monfort and Polimenis (2006) for an analysis of credit risk modeling in discrete-time.

empirically disentangle variations of recovery and hazard rates, unlike the continuous-time RMV case. In addition to the substantial gain in tractability, these pricing solutions can be utilized within a rich class of discrete-time processes. The discrete-time affine family contains a large variety of dynamics including the essentially affine Gaussian process proposed by Ang and Piazzesi (2003) and the Markov Gamma process introduced by Gouriéroux, Monfort and Polimenis (2002). It also easily accommodates the inclusion of lagged macro variables as in Ang and Piazzesi (2003) or a regime switching economy as in Dai, Singleton and Yang (2003).[5] Recently, Dai, Le and Singleton (2006) extend the analysis of Gouriéroux, Monfort and Polimenis (2002) to hybrid processes, non-linear dynamics and generalized market prices of risk specifications.Our methodology can be used in all these contexts, thereby offering a great flexibility for capturing both the unconditional and conditional correlation between the risk free term structure, the default intensity and the recovery rate.

In order to illustrate the implications of a stochastic recovery rate on the term structure of spreads. We first provide a very simple numerical example that demonstrates the impact of the recovery rate on the level of spreads. It is shown in several places in the literature (see Houweling and Vorst (2003) for instance) that it is hard to identify both recovery and intensity using bonds' prices when the recovery rate is constant.[6] Our numerical illustration shows that this is not the case when the recovery rate is allowed to vary stochastically. We then estimate a five-factor Gaussian model on BBB and B Standard & Poor's yield indices under the RT assumption. BBB and B rated yield indices are representative of the investment grade and the high yield markets and are also expected to display different sensitivities toward recovery rates. Our analysis reveals that for both ratings, the stochastic recovery model significantly outperforms an alternative model where the unconditional recovery rate is estimated as a constant parameter. The size of the improvement in terms of the root mean squared error is 20% for BBB yields and 14% for B yields. We also find a positive correlation between the default intensity and the loss given default process, which is consistent with the findings of Altman, Brady, Resti and Sironi (2005). The implied loss given default process varies around 27% for the BBB index and 74% for the B spreads, which proves that the assumption of the a constant recovery rate for all rating classes is quite restrictive.

The rest of the paper is structured as follows. In section 2, we discuss the tractability of reduced-form models in the presence of recovery risk. Section 3 presents a general framework for pricing bonds and CDS contracts under recovery risk. In section 4, we explore pricing solutions for the family of discrete-time affine processes and prove that our methodology is tractable. Section 5 introduces the data and discusses the econometric model and the estimation technique. Section 6 presents the empirical results, and section 7 provides some concluding remarks. Proofs are collected in the appendix.

[5] Macro variables are also very often used to forecast recovery rates (see Gupton, Hamilton and Berthault (2001) and Gupton and Stein (2002)).

[6] Pan and Singleton (2005) show that it is possible to achieve identification using CDS spreads.

2 Modeling Recovery Rates in Reduced-Form Models

The choice between continuous time and discrete-time models is often a question of tractability. Continuous time models tend to prove very attractive for the purpose of valuation because closed form solutions are available for many applications. However, when used in models with stochastic recovery rates, continuous time models yield pricing solutions that are difficult to solve. To motivate the discrete-time methodology of this paper, this section summarizes the continuous time framework of Lando (1998) and Duffie and Singleton (1999). In particular, our results show that in this context, the pricing of bonds and CDS contracts under recovery risk in a fairly rich econometric specification becomes computationally intensive and thus hardly tractable for empirical implementation.

2.1 General Pricing Solutions for Defaultable Securities

2.1.1 Bond Pricing

Let the process $\{r_t, t \geq 0\}$ represent the risk free short rate and $\{\lambda_t, t \geq 0\}$ be the hazard rate of the default time. Assuming that the risky bond has a maturity denoted by T, a face value equal to \$1, and a random recovery payment at the time of default of the form Z_τ, Lando (1998) shows that the price of a zero-coupon bond can be computed as follows

$$\overline{B}(t,T) = E_t^Q \left[\exp\left(-\int_t^T (r_s + \lambda_s)\, ds \right) \right] + E_t^Q \left[\int_t^T Z_s \lambda_s \exp\left(-\int_t^s (r_u + \lambda_u)\, du \right) ds \right], \qquad (1)$$

where Q denotes the Equivalent Martingale Measure with respect to the money market account.

The first term accounts for the promised face value and the second conditional expectation reflects the recovery value that the bondholder receives upon default. The literature contains three different assumptions to model the recovery payment: recovery of face value (RFV) (see Brennan and Schwartz (1980) and Lando (1998)), recovery of Treasury (RT) (see Longstaff and Schwartz (1995), Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997) and Duffee (1999)) and recovery of market value (RMV) (Duffie and Singleton (1999)).

Under the RFV assumption, the recovery payment is a fraction of the face value

$$Z_s = (1 - L_s)$$

where $\{L_s, s > t\}$ is an adapted process bounded by 1. This process corresponds to the loss given default, whereas $\{(1 - L_s), s > t\}$ represents the recovery process.

Under the RT assumption, the recovery payment is a fraction of a Treasury bond with maturity T and face value equal to \$1

$$Z_s = (1 - L_s) \times B(s,T)$$

Under the RMV assumption, Z_s is given by

$$Z_s = (1 - L_s) \times \overline{B}(s^-,T)$$

where $\overline{B}(s^-, T)$ denotes the price of the risky bond just before default.

The choice between these recovery assumptions depends on the legal structure of the instrument to be priced. If, for instance, one assumes liquidation at default and the absolute priority rule applies, then recovery of face value or recovery of Treasury value is more realistic. However, if liquidation is avoided, then one can use the recovery of market value assumption. More generally, different credit events such as bankruptcy, repudiation and restructuring affect the value of defaultable securities with different probabilities and thus result in different recovery rates. As pointed out by Duffie and Singleton (1999), in a continuous time framework, the choice between different recovery assumptions may also be potentially motivated by computational constraints. In general, r_s, λ_s and L_s are all functions of some state variables in the economy. A key ingredient in computing $\overline{B}(t, T)$ is the conditional joint distribution of the random vector (r_s, λ_s, L_s) . Analytical expressions are generally unavailable unless one makes specific assumptions about the dynamics of (r_s, λ_s, L_s) . Accordingly, simplifying assumptions, such as independence between the short rate and the hazard rate processes as in Jarrow and Turnbull (1995), or a constant recovery process as in Jarrow, Lando and Turnbull (1997) and Duffee (1998), have emerged in the literature.

In addition to the computational tractability, an important feature differentiating these recovery assumptions is their ability to identify the impact of the hazard and the recovery rate. While the RFV and RT allow for identification of the impact of the recovery rate, this is not the case under the RMV assumption. Recall that under the RMV assumption, equation (1) is equivalent to a recursive stochastic integral equation whose solution is, for a zero-coupon risky bond

$$\overline{B}^{RMV}(t, T) = E_t^Q \left[\exp \left(- \int_t^T (r_s + L_s \lambda_s) \, ds \right) \right].$$

The RMV assumption is extremely convenient for pricing credit derivatives, since it only relies on the explicit modeling of the short rate and the so-called "mean loss rate" ($L_s \lambda_s$). However, $\{L_s, s > t\}$ cannot be allowed to vary stochastically, otherwise it becomes impossible to distinguish between variations in the hazard rate and variations in the recovery rates, as explicitly recognized by Duffie and Singleton (1999). Consequently, although the RMV assumption has become the standard assumption in credit risk modeling, the few papers that explicitly models the recovery rate utilize the RT and RFV assumptions.

2.1.2 CDS Pricing

A popular instrument which offers a protection against credit events is the credit default swap (CDS) contract. The mechanism of a CDS contract works as follows: Consider two companies 'A' (the buyer) and 'B' (the seller) who enter into a contract which terminates at the time of a credit event or at a specified maturity, whichever occurs first. A credit event could be, for instance, the default of a third company 'C', called the reference company. It also includes other events such as bankruptcy, downgrade, failure to pay, repudiation or restructuring of the reference company. If a credit event occurs before the specified maturity, then company 'B' pays company 'A' a certain compensation in the form of a cash amount. In exchange of this insurance offered by company 'B', company 'A' periodically pays to company 'B' a fixed amount S, called the CDS spread or premium. If the termination of the contract is triggered by a credit event, then a default also elicits an accrued credit-swap premium from the buyer.

The price of a CDS contract with semi-annual premium payments and an accrued credit-swap premium is (see Duffie and Singleton (2003))

$$\begin{aligned} CDS(t,n) &= -\sum_{k=1}^{2n} E_t^Q \left[\exp\left(-\int_t^{t+\frac{k}{2}} (r_u+\lambda_u)\, du \right) \right] \times \frac{S}{2} \\ &\quad + \int_t^{t+n} E_t^Q \left[\exp\left(-\int_t^{s} (r_u+\lambda_u)\, du \right) \lambda_s \left(L_s - (\tau - t^\star) \frac{S}{2} \right) ds \right], \end{aligned}$$

where n is the maturity of the contract and $t^\star$ represents the first premium payment date that follows the credit event.

Because trading is in continuous-time, it is difficult to deal with the first premium date following the credit event. For this reason, modified payoffs are often considered with continuous time models. One possibility is to consider that rather than receiving the compensation L upon default, the buyer receives payments at the next premium date that follows default. Then, the accrued credit-swap premium is no longer necessary. Another solution is to simply eliminate the accrued credit swap premium from the payoff. Although the impact of the approximations is negligible for small default probabilities, this does not hold if one consider a higher likelihood of default as is the case for low rated firms or for some sovereign CDS.

Let us consider the case of a payoff in the absence of accrued interests. The price of the CDS is then

$$\begin{aligned} CDS(t,n) &= -\sum_{k=1}^{2n} E_t^Q \left[\exp\left(-\int_t^{t+\frac{k}{2}} (r_u+\lambda_u)\, du \right) \right] \times \frac{S}{2} \\ &\quad + \int_t^{t+n} \lambda_s L_s E_t^Q \left[\exp\left(-\int_t^{s} (r_u+\lambda_u)\, du \right) ds \right]. \end{aligned} \tag{2}$$

The pricing formula shares many of its features with the bond pricing solution. Computing the price of a CDS relies therefore on the same tools as for risky bonds. In the next section, we evaluate the tractability of the continuous time framework for the family of affine diffusion processes.

2.2 Pricing Solutions in Continuous-Time Affine Models

To illustrate the trade-off between the econometric richness and the analytical tractability of continuous time models in the general case of an affine state vector, let us assume that the economy is driven by a $N-$dimensional affine state vector Y_t (see Duffie, Pan and Singleton (2000)). The conditional probability distribution of the state vector is characterized through the Laplace and the extended Laplace transforms of the pair $\left(\int_t^T Y_s ds, Y_T\right)$. Under some technical conditions, the Laplace transform of this pair is defined as follows

$$\begin{aligned} G(t,T,u,v) &= E_t^Q\left[\exp\left(\int_t^T u'Y_s ds + v'Y_T\right)\right] \\ &= \exp\left(\alpha\left(t,T,u,v\right) + \beta\left(t,T,u,v\right)Y_t\right), \end{aligned} \tag{3}$$

where $(u,v) \in \left(\mathbb{R}^N, \mathbb{R}^N\right)$ and α and β satisfy a set of $(N+1)$ ordinary differential equations (ODEs) that can be solved numerically.

The extended Laplace transform is

$$\begin{aligned} J(t,T,u,v,w) &= E_t^Q\left[\exp\left(\int_t^T u'Y_s ds + v'Y_T\right)(wY_T)\right] \\ &= G(t,T,u,v)\left(A\left(t,T,u,v,w\right) + B\left(t,T,u,v,w\right)Y_t\right), \end{aligned}$$

where A and B satisfy another set of $(N+1)$ ODEs.

Furthermore, we specify the short rate, the hazard rate and the recovery rate as follows

$$r_t = \delta_0 + \delta_1 Y_t,$$
$$\lambda_t = \gamma_0 + \gamma_1 Y_t \text{ and } (1 - L_s) = \exp\left(-\phi Y_t\right)$$

In order to guarantee that the recovery rate is bounded by one,[7] the coefficients of ϕ_1 are constrained to be positive. Consistent with the recent empirical literature (Dai and Singleton (2000), Duffee (2002)), a three-factor model is usually needed for a good cross-sectional and times-series fit of the risk-free term structure. The hazard rate and the recovery rate can contain one or several additional factors which can be viewed as idiosyncratic. To illustrate the complexity of the pricing solutions in affine models, we compute the price of a zero-coupon risky bond under the assumption of RFV. Under some technical conditions, the latter is

$$\overline{B}(t,T) = G(t,T,-\left(\delta_1+\gamma_1\right),0)$$

$$+\int_t^T \exp(-\left(\delta_0+\gamma_0\right)(s-t))\left(\gamma_0\left(G(t,s,-\left(\delta_1+\gamma_1\right),0) - G(t,s,-\left(\delta_1+\gamma_1\right),-\phi)\right)ds \tag{4}$$

$$+\int_t^T \exp(-\left(\delta_0+\gamma_0\right)(s-t))\left((J(t,s,-\left(\delta_1+\gamma_1\right),0,\delta_1) - J(t,s,-\left(\delta_1+\gamma_1\right),-\phi,\delta_1)\right)ds \tag{5}$$

For a given maturity, the first term is available in closed form up to the solution of a set of N ODE's. However, the second and third terms involve the computation of $2(N+2)$ integrals and each of these integrals relies on the numerical solutions of a set of $2N$ ODEs. While fast numerical solutions of the set of Ricatti equations are available, the frequent use of numerical integrations techniques substantially increases the computational complexity. This complexity also holds for

[7] The positivity of the factors affecting the recovery rate is also a necessary condition to insure that the latter is bounded by one. When some factors are Gaussian, positivity is no longer guaranteed but the probability of obtaining negative values can be assumed to be negligible once the model is estimated.

CDS pricing solutions. The second term in the pricing solution of a CDS (2) does resemble the second term in equation (1). Consequently, when the loss given default is stochastic, computing the price of the CDS in a continuous time setting results in the same computational complexity as for risky bonds.

At this point, it would be interesting to see if the availability of a closed form solution of the Laplace transform would reduce the computational complexity and improves the tractability of the model. Indeed this is the case for some affine diffusions, including the correlated square-root process and the multifactor Gaussian process. Intuitively, if the Laplace transform is available in closed form and thus does not require the numerical solution of ODEs, one would expect the model to gain in tractability. As will be clear in this example, this is not the case and the computational burden still prevails even when the Laplace transform is available in closed form.

Suppose that $\alpha(t,T,u,v)$ and $\beta(t,T,u,v)$ are available in closed form. The extended Laplace transform can then be expressed as

$$\begin{aligned} J(t,T,u,v,w) &= E_t^Q\left[\exp\left(u\int_t^T R_s ds + vY_T\right)(wY_T)\right] \\ &= \sum_{i=1}^N w_i \times \frac{\partial G(t,T,u,v)}{\partial v_i} \\ &= \sum_{i=1}^N w_i \times \left(\frac{\partial \alpha(t,T,u,v)}{\partial v_i} + \sum_{i=1}^N \frac{\partial \beta_i(t,T,u,v)}{\partial v_i} Y_{it}\right). \\ &\quad \times \exp\left(\alpha(t,T,u,v) + \beta(t,T,u,v)\, Y_t\right). \end{aligned}$$

This pricing solution involves the numerical evaluation of $2\left(N(N+1)+2\right)$ integrals which demonstrates that the computational cost is still significant even though the Laplace transform is known analytically. The result underlines the importance of building a tractable model for pricing defaultable securities under recovery risk. In the next section, we introduce our model and derive general pricing solutions for risky bonds and CDS contracts with recovery risk.

3 A Tractable Framework for Pricing Defaultable Securities with Recovery Risk

3.1 Market Structure

We assume an economy with a frictionless financial market where trading follows a discrete-time sequence $t, t+1, ..., t+n$. In this economy, two classes of bonds are traded: Risk-free bonds and defaultable bonds. The uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The probability of default is modeled as in reduced-form models but with one slight modification. We assume that default can occur any time between two consecutive trading dates and that its time, τ, corresponds to the first time jump of a Cox process with hazard rate $\{\lambda(s), 0 \le s \le T\}$. In this probabilistic framework, the filtration $\{\mathcal{F}_{t+i}, i = 0, ..., n\}$ contains the information on whether default occurred before $t+i$ in addition to information on the risk free term structure and any other relevant state variables. If we denote by $\{\mathcal{G}_{t+i}, i = 0, ..., n\}$ the basic filtration without the information on the occurrence of default, then $\mathcal{F}_{t+i}$ can be written as

$$\mathcal{F}_{t+i} = \mathcal{G}_{t+i} \vee \sigma\left(\tau < s, s \leq t+i\right),$$

where the sigma field $\sigma(\tau < s, s \leq t+i)$ holds the information on whether default occurred before $t+i$.

Under technical conditions (see Lando (1998)), the probability of no jump is then defined as

$$\begin{aligned}
\Pr\left(\tau > t+k \mid \mathcal{F}_t\right) &= E\left[\exp(-\int_t^{t+k} \lambda_s ds) \mid \mathcal{G}_t\right] \\
&= E_t\left[\exp(-\sum_{i=1}^{k} \Lambda_{t+i})\right],
\end{aligned}$$

where the subscript t denotes the information contained in the basic filtration [8] $\{\mathcal{G}_{t+i}, i = 0, ..., n\}$ and

$$\Lambda_{t+i} = \int_{t+i-1}^{t+i} \lambda_s ds < \infty.$$

In this paper, rather than modeling $\lambda(u)$, we model $\{\Lambda_{t+i}, i = 1, ..., n\}$ as a discrete-time process. This assumption can also be viewed as piecewise constant modeling of the hazard rate.

For pricing defaultable securities, we need information on whether default occurred prior to any trading date $t+i$ and thus conditioning on $\mathcal{F}_{t+i}$ is necessary. For any asset with price X_t and cumulative dividend payment D_t, the following Euler equation has to be satisfied at time t

$$X_t = E\left[M_{t,t+1}\left(X_{t+1} + D_{t+1}\right) \mid \mathcal{F}_t\right], \tag{6}$$

where $\left\{M_{t+i-1,t+i} = \frac{U'(c_t)}{U'(c_{t+1})}, i = 1, ..., n\right\}$ is a discrete-time $\mathcal{G}_{t+i}$ measurable process that corresponds to the pricing kernel between $t+i-1$ and $t+i$.

Modeling the pricing kernel is equivalent to specifying the risk free term structure. Using equation (6), the price at time t of a risk free zero-coupon bond with maturity $t+n$ is

$$\begin{aligned}
B(t, t+n) &= E\left[\prod_{i=1}^{n} M_{t+i-1,t+i} \mid \mathcal{G}_t\right] \qquad (7) \\
&= E_t\left[M_{t,t+n}\right].
\end{aligned}$$

We now turn to the third component of the term structure of defaultable bonds: The recovery rate. We assume that default can occur at any time between two consecutive trading dates. In contrast to continuous time models, our model assumes that the recovery payment is received at the next trading date following the default time. This hypothesis is fundamental and makes the

[8] Throughout this paper, the subscript t indicates a conditioning on the filtration $\mathcal{G}_t$.

framework very attractive for pricing CDS contracts and other credit derivatives. Furthermore, it provides the flexibility necessary to obtain tractable closed form solutions for prices of risky bonds. For simplicity, we consider a defaultable security that yields \$1 at $t+n$ if no default has occurred. Upon default, the security's payment (at the next trading date) is described by an adapted stochastic process $\mathcal{Z} = \{Z_{t+i}, i = 1, ...n\}$.

In the course of the next two sections, we assemble all these building blocks and provide general pricing solutions for bonds and CDS contracts under recovery risk.

3.2 Risky Bond Pricing with Recovery Risk

Let us consider the case of a constant coupon-bearing risky bond, with coupon dates following the sequence $\mathcal{T}_c = \{t, t+p, t+2p, ..., t+n\}$. If we denote the constant coupon by C, the price of a risky bond with a recovery payment at the next trading date following the default time is obtained using the Euler equation (6). The value of the risky bond, $\overline{B}(t,n)$, verifies the following recursive equation

$$\overline{B}(t,t+n) = E\left[M_{t,t+1}\left[\left(\overline{B}(t+1,t+n) + C \times \mathbf{1}_{(t+1\in\mathcal{T}_c)}\right)\mathbf{1}_{(\tau>t+1)} + Z_{t+1}\mathbf{1}_{(\tau<t+1)}\right] \mid \mathcal{F}_t\right], \qquad (8)$$

where the boundary condition is

$$\overline{B}(t+n-1,t+n) = E\left[M_{t+n-1,t+n}\left[\left(\mathbf{1}_{(\tau>t+n)} + C\right) + Z_{t+n}\mathbf{1}_{(\tau<t+n)}\right] \mid \mathcal{F}_{t+n-1}\right]. \qquad (9)$$

Under the RFV assumption, the recovery payment is

$$Z_{t+i} = (1 - L_{t+i}),$$

where $\{L_{t+i}, i = 0, 1, ..., n\}$ is a discrete-time $\mathcal{G}_{t+i}$ adapted process which is bounded by 1.

Under the RT assumption, the recovery payment is

$$Z_{t+i} = (1 - L_{t+i})\, B(t+i, t+n),$$

where $B(t+i,n)$ is the price at $t+1$ of a Treasury bond with maturity $t+n$.

Under the RMV assumption, the recovery payment is

$$Z_{t+i} = (1 - L_{t+i})\, \overline{B}^{RMV}(t+i, t+n),$$

where, $\overline{B}^{RMV}(t+i,n)$ denotes the price of the defaultable bond under the assumption of recovery of market value.

Solving equations (8) and (9) recursively yields the price of the risky bond under each of these assumptions. The following proposition summarizes the pricing solutions for risky bonds under the three recovery assumptions described above.

Proposition 1 *The price at t of a risky coupon-bearing bond with maturity $t+n$ has to satisfy the following general pricing solutions:*

1. *Under the assumption of RT*

$$\overline{B}^{RT}(t,t+n) = E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right] + \sum_{k=1}^{\frac{n}{p}} E_t\left[M_{t,t+kp}\exp(-\sum_{i=1}^{kp}\Lambda_{t+i})\times C\right]$$
$$+E_t\left[M_{t,t+n}\left(\sum_{k=0}^{n}\exp(-\sum_{i=0}^{k}\Lambda_{t+i}\mathbf{1}_{(k>0)})((L_{t+k}\mathbf{1}_{(k>0)}+\mathbf{1}_{(k=0)})-(L_{t+k+1}\mathbf{1}_{(k<n)}+\mathbf{1}_{(k=n)}))\right)\right]. \tag{10}$$

2. *Under the assumption of RFV*

$$\overline{B}^{RFV}(t,t+n) = E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right] + \sum_{k=1}^{\frac{n}{p}} E_t\left[M_{t,t+kp}\exp(-\sum_{i=1}^{kp}\Lambda_{t+i})\times C\right]$$
$$+\sum_{k=1}^{n} E_t\left[M_{t,t+k}\exp\left(-\sum_{i=1}^{k-1}\Lambda_{t+i}\mathbf{1}_{(k>1)}\right)\left(1-\exp(-\Lambda_{t+k})\right)\left(1-L_{t+k}\right)\right]. \tag{11}$$

3. *Under the assumption of RMV*

$$\overline{B}^{RMV}(t,t+n) = E_t\left[\prod_{i=1}^{n} M_{t+i-1,t+i}\left(1-L_{t+i}\left(1-\exp(-\Lambda_{t+i})\right)\right)\right]$$
$$+\sum_{k=1}^{\frac{n}{p}} E_t\left[\prod_{i=1}^{k} M_{t+i-1,t+i}\left(1-L_{t+i}\left(1-\exp(-\Lambda_{t+i})\right)\right)\times C\right]. \tag{12}$$

Proof. *See Appendix.* ■

Intuitively, this Proposition states that the value of the risky bond is a linear combination of expectations of discounted cash flows where the discount factor includes the intensity of default and the recovery rate in addition to the pricing kernel. Although these pricing solutions are established in their most general form, one can already draw some preliminary conclusions. As in case in the continuous time framework, under the RT and RFV assumptions, the price of the risky bond equates to that of a risky bond that has zero recovery in the case of default in addition to terms that compensates for the existence of a positive recovery on default. Under the assumption of RMV, the price of the risky bond is the discrete-time equivalent of the Duffie and Singleton (1999) formula. It is worth emphasizing, however, that in the discrete-time framework it is not necessary to fix the loss given default in order to achieve identification. Unlike the continuous time case, the loss given default and the hazard rate do not enter the pricing equation symmetrically which allows the loss given default to be potentially identified and the assumption of RMV to be empirically investigated.

Obviously, a key ingredient in using these general solutions is the specification of the conditional joint distribution of the state variables. Once the conditional distribution of the state variable is characterized, the price of the risky bond is computed using (10) , (11) and (12). That the term structure of credit spreads is modeled adequately requires the relationship between these state variables be defined in a way that allows the model to be consistent with some stylized facts. More precisely, the model has to be able to provide some new insights about the relationship between the three components of the term structure of defaultable bonds: The risk free term structure, the default risk and the recovery rate. Moreover, the model should provide a framework suitable for pricing credit derivatives.

3.3 CDS Pricing with Recovery Risk

In this section, we show how this model can be applied to the valuation of one of the most popular credit derivatives: a CDS contract. Once again, we provide pricing formula in the context of a stochastic recovery rate.As for risky bonds, we assume that the face value of the reference obligation is equal to \$1. We again assume that trading follows a discrete-time sequence of dates denoted by $\{t, t+1, ..., t+n\}$. The premium of the CDS, S, is paid every p periods, and the payment dates then follow the sequence $\mathcal{T}_c = \{t, t+p, t+2p, ..., t+n\}$. The contract starts at time t and has a maturity equal to n periods. If default occurs between two consecutive trading dates, then the recovery payment is received at the next trading date following the default time. As is customary, we use the assumption of RFV to describe the recovery payment upon default.

Let us consider two consecutive premium payment dates $t+(k-1)p$ and $t+kp$. Assuming no default prior to $t+(k-1)p$, the buyer will pay the premium at $t+kp$ if no credit event occurs in the interval $[t+(k-1)p, t+kp]$. If between any two consecutive trading dates $t+i-1$ and $t+i$, where $(k-1)p+1 \leq i \leq kp$, a credit event is documented, then the buyer receives an amount of cash L_{t+i} and pays the accrued credit-swap premium $\frac{i}{p}S$. The payoff of the buyer in the interval $[t+(k-1)p, t+kp]$ can be summarized in this diagram

No default between $t+(k-1)p$ and $t+kp$	$-S$
Default occurs between two consecutive trading dates in $[t+(k-1)p, t+kp]$	$L_{t+i} - \frac{i}{p}S$

The discounted payoff of the buyer between $t+(k-1)p$ and $t+kp$ is then

$$-M_{t,t+kp}\mathbf{1}_{(\tau>t+kp)} \times S + \sum_{i=(k-1)p+1}^{kp} M_{t,t+i}\mathbf{1}_{(t+i-1<\tau<t+i)}\left(L_{t+i} - \frac{i}{kp}S\right)$$

Applying the same reasoning between t and $t+n$, the price of the CDS contract is established in the following proposition.

Proposition 2 *The discounted value, at time t, of a CDS contract with a maturity equal to $t+n$ is*

$$CDS(t,t+n) = \sum_{k=1}^{\frac{n}{p}} -E_t\left[M_{t,t+kp}\exp\left(-\sum_{i=1}^{kp}\Lambda_{t+i}\right)\right]\times S + \sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[\xi_i\left(L_{t+i}-\frac{i}{kp}S\right)\right], \quad (13)$$

where

$$\xi_i = M_{t,t+i}(\exp(-\mathbf{1}_{(j>0)}\sum_{j=1}^{i-1}\Lambda_{t+j}) - \exp(-\sum_{j=1}^{i}\Lambda_{t+j})).$$

The CDS premium is computed such that the discounted value of the CDS is equal to zero

$$S = \frac{\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[\xi_i L_{t+i}\right]}{\sum_{k=1}^{\frac{n}{p}} E_t\left[M_{t,t+kp}\exp\left(-\sum_{i=1}^{kp}\Lambda_{t+i}\right)\right] + \sum_{i=(k-1)p+1}^{kp} E_t\left[\xi_i\frac{i}{kp}\right]},$$

Proof. See Appendix. ∎

The discrete-time specification and the assumption of a recovery payment at the next trading date following default easily accommodate the existence of an accrued credit-swap premium. As is the case for risky bonds in the previous section, the pricing formula under recovery risk is written in its most general form. It involves a finite summation of conditional expectations and allows the spread to be easily inverted once the dynamics of the state vector are specified.

In the next section, we characterize the conditional joint distribution of the state variables using the conditional Laplace transform. We then demonstrate how knowing the Laplace transform of the state vector suffices to price bonds and CDS contracts under recovery risk.

4 Closed Form Solutions for Risky Bonds and CDS Contracts

Building on Propositions 1 and 2, we derive closed-form expressions for prices of risky bonds and CDS contracts for the general class of affine processes. Under some technical conditions, a process is affine if its conditional Laplace transform is an exponential affine function of the current values of the state variables. In our setup, the family of affine dynamics proves particularly attractive and offers an analytical tractability which allows the derivation of closed form solutions. Gouriéroux, Monfort and Polimenis (2002) also employ discrete-time affine processes in order to characterize the risk free term structure as in Duffie and Kan (1996). Prominent among affine processes are the Gaussian process and the Markov Gamma process introduced by Gouriéroux, Monfort and Polimenis (2002). The family of affine processes has been extended by Dai, Singleton and Yang (2003) to incorporate a stochastic volatility factor in the case of a regime-switching economy, which results in closed form solutions for bonds prices provided that some restrictions, analogous to those existing in affine continuous time models are imposed on the parameters. Dai, Le and Singleton (2006) recently established that mixtures of the Gaussian and the Markov Gamma processes with fairly general specifications for the market price of risk can be constructed to allow for stochastic volatility models.

In this section, we show that a discrete-time approach not only retains much of the intuition underlying the continuous time valuation framework described above, but it also furnishes great analytical tractability. We first derive the pricing solutions for an affine state vector and subsequently we focus on specific econometric formulations.

4.1 Pricing Solutions for a Discrete-Time Affine State Vector

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and make the assumption that the discrete-time economy described in the previous section is driven by a state vector Y_t in some state space included in $\mathbb{R}^N$. To build a term structure model, we need to specify the dynamics of the pricing kernel $M_{t,t+1} = M(Y_t, Y_{t+1})$ underlying the time t valuation payoffs at time $t+1$, the hazard rate $\Lambda_{t+1} = \Lambda(Y_{t+1})$ and the loss given default $L_{t+1} = L(Y_{t+1})$. Toward this goal, we make the following assumptions:

Assumption 1: The time t price of any security with an exponential payoff is for any $N \times 1$ real vector α

$$E_t\left[M_{t,t+1} \exp\left(\alpha' Y_{t+1}\right)\right] = \exp(a(\alpha) Y_t + b(\alpha)), \tag{14}$$

where $a(\alpha)$ is a $1 \times N$ real vector and $b(\alpha)$ is a scalar.

Assumption 2: Λ_t and $\mathcal{L}_t$, where $\mathcal{L}_t = -\log(L_t)$, are expressed as follows

$$\left[\begin{array}{c} \Lambda_t \\ \mathcal{L}_t \end{array}\right] = \left[\begin{array}{c} \gamma_0 + \gamma Y_t \\ \phi_0 + \phi Y_t \end{array}\right]. \tag{15}$$

The first assumption states that the state vector is affine under the Equivalent Martingale Measure. It is worth observing that this assumption does not restrict the dynamics of the state variables under the actual measure. This allows non-linear dynamics (see Dai, Le and Singleton) to be used as long as the state vector is affine under the risk-neutral measure. The no arbitrage restriction on the pricing kernel implies that

$$r_t = -a(0_{\mathbb{R}^N}) Y_t - b(0_{\mathbb{R}^N}).$$

The second assumption implies that each component of the vectors Λ_t and $\mathcal{L}_t$ is a linear combination of the state variables. The elements of γ and ϕ control the correlation between the default intensity and the loss given default. The source of this correlation depends on the state variables that the hazard rate and the loss given default have in common. A key benefit of this assumption is that it allows for firm specific or idiosyncratic factors that drive the default risk and the recovery rate. This is in line with the observation of Acharya, Bharath and Srinivasan (2004) that modeling the stochastic nature of the recovery rate must take into account firm-specific factors as well as industry-specific factors. The affine specification also accommodates the inclusion of observable macroeconomic variables. Recent empirical works show that macro variables play an important role in both the risk free and the credit spread term structure and it turns out that including macro variables in standard no-arbitrage models enhances the understanding of the relationship between economic business cycles and the term structures of Treasury yields and credit spreads. Ang and Piazzesi (2003) include macro variables in a discrete-time affine Gaussian model and prove that better yields forecasts are obtained when macro factors are added. Following the same methodology, Amato and Luisi (2005) find that macro variables significantly impact on the term structure of credit spreads. As pointed out by Piazzesi (2005), discrete-time models are able to incorporate higher order lags of macro variables, a feature that can be exploited in our framework.

We now turn to the pricing of defaultable securities in an affine setting. Using assumptions (14) and (15), we characterize the conditional probability distribution of $(\sum_{i=1}^{p} Y_{t+i}, Y_{t+p})$ and $(\sum_{i=1}^{p} Y_{t+i}, Y_{t+1})$ via their Laplace transform in the following proposition. We then show that once these Laplace transforms are known analytically, the pricing problem described in equations (10), (11) and (13) yields an explicit solution.

Proposition 3 *Assuming that* (14) *and* (15) *hold, then for any* $p \geq 2$:

1. *The conditional Laplace transform of* $\left(M_{t,t+p}, \sum_{i=1}^{p} Y_{t+i}, Y_{t+p} \right)$ *is given by*

$$\begin{aligned} G^{1}_{t,p}(\alpha,\beta) &\equiv E_t\left[M_{t,t+p} \exp\left(\alpha' \sum_{i=1}^{p} Y_{t+i} + \beta' Y_{t+p} \right) \right] \\ &= \exp\left(A_{1,p}(\alpha,\beta) Y_t + B_{1,p}(\alpha,\beta) \right), \end{aligned} \tag{16}$$

where A_1 *and* B_1 *are computed recursively as follows*

$$A_{1,i}(\alpha,\beta) = a\left(A_{1,i-1} + \alpha\right) \text{ and } A_{1,1} = a(\alpha+\beta). \tag{17}$$

$$B_{1,i}(\alpha,\beta) = B_{1,i-1} + b(A_{1,i-1} + \alpha) \text{ and } B_{1,1} = b(\alpha+\beta). \tag{18}$$

2. *The conditional Laplace transform of* $\left(M_{t,t+p}, \sum_{i=1}^{p} Y_{t+i}, Y_{t+1} \right)$ *is given by*

$$\begin{aligned} G^{2}_{t,p}(\alpha,\beta) &\equiv E_t\left[M_{t,t+p} \exp\left(\alpha \sum_{i=1}^{p} Y_{t+i} + \beta Y_{t+1} \right) \right] \\ &= \exp\left(A_{2,p}(\alpha,\beta) Y_t + B_{2,p}(\alpha,\beta) \right), \end{aligned} \tag{19}$$

where

$$A_{2,p}(\alpha,\beta) = a\left(\alpha+\beta+A_{2,p-1}(\alpha)\right) \text{ and } B_{2,p}(\alpha,\beta) = B_{2,p-1} + b\left(\alpha+\beta+A_{2,p-1}(\alpha)\right), \tag{20}$$

and for any $i = 2, ..., p-1$, A_2 and B_2 are computed recursively as follows:

$$A_{2,i}(\alpha) = a(A_{2,i-1} + \alpha) \text{ and } A_{1,1} = a(\alpha). \tag{21}$$

$$B_{2,i}(\alpha) = B_{2,i-1}(\alpha) + b(\alpha + A_{2,i-1}(\alpha)) \text{ and } B_{1,1} = b(\alpha). \tag{22}$$

Proof. See Appendix ■

With $\alpha = \beta = 0_{\mathbb{R}^N}$, the Laplace transforms in (16) and (19) give the price at t of a risk free bond with maturity $t+n$:

$$\begin{aligned} B(t,n) &= G^1_{t,p}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) = G^2_{t,p}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) \\ &= \exp\left(A'_{1,n}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) Y_t + B_{1,n}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N})\right) \end{aligned} \tag{23}$$

The term structure of the risk free rate is affine, provided that the conditional Laplace transform (16) is known analytically. This result gives the discrete-time counterpart of the family of affine diffusions processes characterized by Duffie and Kan (1996) and later generalized to the case of affine jump diffusions processes by Duffie, Pan and Singleton (2000). The Laplace transform provides the appropriate tools to compute closed form solutions for the prices of risky bonds under the assumptions of RT and RFV and CDS contracts. These formulas are established in the following proposition.

If the state vector Y_t follows an affine process whose Laplace transform is known analytically, we obtain the following pricing solutions by combining (16) and (19) with the general solutions derived in (10), (11) and (13)

Proposition 4 *The price at time t of a risky coupon-bearing bond with maturity $t+n$ is:*

1. Under the assumption of RT

$$\begin{aligned} \overline{B}^{RT}(t,t+n) &= G^1_{t,n}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) - \exp(-\phi_0) G^2_{t,n}(0_{\mathbb{R}^N}, -\phi) + \exp(-\phi_0 - n\gamma_0) G^1_{t,n}(-\gamma, -\phi) \\ &\quad + \sum_{k=1}^{n/p} \exp(-kp\gamma_0) G^1_{t,kp}(-\gamma, 0_{\mathbb{R}^N}) \times C \\ &\quad + \sum_{k=1}^{n-1} \left[\exp\left(B_{1,n-k}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) - \phi_0 - k\gamma_0\right) G^1_{t,k}\left(-\gamma, \left(A_{1,n-k}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) - \phi\right)\right) \right. \\ &\quad \left. - \exp\left(B_{2,n-k}(0_{\mathbb{R}^N}, -\phi) - \phi_0 - k\gamma_0\right) G^1_{t,k}(-\gamma, A_{2,n-k}(0_{\mathbb{R}^N}, -\phi))\right]. \end{aligned} \tag{24}$$

2. Under the assumption of RFV

$$\overline{B}^{RFV}(t,t+n) = G^1_{t,n}\left(-\gamma, 0_{\mathbb{R}^N}\right) + \sum_{k=1}^{n/p} G^1_{t,kp}(-\gamma, 0_{\mathbb{R}^N}) \times C$$

$$+G^1_{t,1}\left(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}\right) - \exp(-\gamma_0) G^1_{t,1}\left(-\gamma, 0_{\mathbb{R}^N}\right)$$

$$-\exp(-\phi_0) G^1_{t,1}\left(-\phi, 0_{\mathbb{R}^N}\right) + \exp\left(-\left(\gamma_0+\phi_0\right)\right) G^1_{t,1}\left(-\left(\gamma+\phi\right), 0_{\mathbb{R}^N}\right)$$

$$+\sum_{k=2}^{n}\Big[\exp(-(k-1)\gamma_0) G^1_{t,k}\left(-\gamma,\gamma\right) - \exp(-k\gamma_0) G^1_{t,k}(-\gamma, 0_{\mathbb{R}^N})$$

$$-\exp(-(k-1)\gamma_0-\phi_0) G^1_{t,k}(-\gamma,\gamma-\phi) + \exp(-k\gamma_0-\phi_0) G^1_{t,k}(-\gamma,-\phi)\Big]. \quad (25)$$

Proof. See Appendix ■

Proposition 5 *The price at time t of a CDS contract with maturity $t+n$ is*

$$CDS(t,t+n) = \sum_{k=1}^{\frac{n}{p}} -\exp(-kp\gamma_0) G^1_{t,kp}\left(-\gamma, 0_{\mathbb{R}^N}\right) \times S$$

$$+\sum_{k=1}^{\frac{n}{p}} \sum_{i=(k-1)p+1}^{kp} \exp(-(i-1)\gamma_0-\phi_0) G^1_{t,i}\left(-\gamma,\gamma-\phi\right) - \exp(-i\gamma_0-\phi_0) G^1_{t,i}\left(-\gamma,-\phi\right)$$

$$+\sum_{k=1}^{\frac{n}{p}} \sum_{i=(k-1)p+1}^{kp} \left(\exp(-(i-1)\gamma_0) G^1_{t,i}\left(-\gamma,\gamma\right) - \exp(-i\gamma_0) G^1_{t,i}\left(-\gamma, 0_{\mathbb{R}^N}\right)\right) \frac{i}{kp} S. \quad (26)$$

Proof. See Appendix ■

Propositions 4 and 5 show that the closed form expressions are rather simple to compute since they involve a finite summation. Once the Laplace transforms (16) and (19) are known analytically, it is straightforward to calculate expressions (24) and (25). From a practical perspective, it is potentially interesting to note that solving the pricing problem (10), (11) and (13) reduces to the computation of two Laplace transforms. A_1, A_2, B_1 and B_2 need only be computed once and prices of bonds and CDS contracts can then be elicited via these terms. As is the case in continuous time, the number of recursions is linear with the number of time steps. However, the number of terms to be computed does not increase with the number of factors. A multi-factor model is thus still tractable.

Under the assumption of RMV, the price of the risky bond does not admit a closed form expression. However, it can be computed using an efficient Monte Carlo procedure once the dynamics of the state variables are specified. As discussed previously, it is noteworthy that unlike the continuous time pricing solution under RMV, the discrete-time setup allows us to distinguish between the recovery rate and the hazard rate.

The correlation structure between the short rate, the default risk and the recovery rate is captured by the components of the Laplace transform, $A_{i,p}\left(\alpha,\beta\right)$ and $B_{i,p}\left(\alpha,\beta\right)$. This means that the value of the risky debt depends on both the current values of the state variables, namely the short rate, the default risk and the recovery rate, and the correlations among these variables. In other words, the recovery payment may depend on the risk free term structure and the economic conditions. In a recession, when default rates tend to be high, the recovery rate tends to decrease. A positive correlation between the loss given default and the default risk may ensue. The model is able to accommodate flexible structures of correlations among the state variable as long as the conditional Laplace transform (14) is analytically known.

More importantly, the closed form solutions derived previously hold for a large class of discrete-time processes, this empowers the model to provide some new insights about recovery rates implied in defaultable instruments. In what follows, we discuss two econometric specifications of the pricing kernel and the state vector: A Gaussian and a Markov Gamma N-factor model. In each of these examples, the calculations are simple and do not require any numerical approximations. It is also important to emphasize that as is the case in the continuous-time nomenclature of Dai and Singleton (2000), both models are just two members of a rich class of non-linear models that can be constructed as mixtures of the Gaussian and Markov Gamma processes (see Dai, Le and Singleton (2006)).

4.2 A Gaussian Essentially Affine Model

In this example, we consider the discrete-time essentially affine Gaussian model. This model was originally proposed by Duffee (2002) as an extension of the standard Gaussian model. It allows for a market price of risk that is affine in the state variables, a feature which improves the forecasting ability of the model. When used in discrete-time, this model easily accommodates the inclusion of macroeconomic variables (see Ang and Piazzesi (2003) for Treasury yields and Amato and Luisi (2005) for the term structure of credit spreads), which significantly improves its empirical performance.

We assume that the pricing kernel is parametrized as in Ang and Piazzesi (2003)

$$M_{t,t+1} = \exp(-\delta_0 - \delta_1 Y_t - \frac{1}{2}\pi_t' \Sigma \pi_t - \pi_t' \epsilon_{t+1}) \text{ and}$$
$$\pi_t = \pi_0 + \pi_1 Y_t, \tag{27}$$

where π_0 is $N \times 1$ vector and π_1 is a $N \times N$ matrix.

The state vector is assumed to follow a Gaussian VAR(1)

$$Y_{t+1} = \mu + \Phi Y_t + \Sigma \epsilon_t \tag{28}$$

The time varying vector π_t represents the market price of risk and the short rate implied by this specification is an affine function of the state vector

$$r_t = \delta_0 + \delta_1 Y_t$$

The main advantage of this specification is that it its ability to generate time-varying term premia, an important stylized fact of the risk free term structure. However, unlike the standard Gaussian formulation, the model's pricing kernel is not an affine function of (Y_t, Y_{t+1}) and the time varying market price of risk shows up explicitly. Despite these differences, the fact that condition (14) holds allows us to compute the Laplace transforms $G^1_{t,n}$ and $G^2_{t,n}$ and price risky bonds. The conditional Laplace transform (14) is

$$E_t\left[M_{t,t+1}\exp(\alpha Y_{t+1})\right] = \exp\left(\left(-\delta_0 + \alpha\left(\mu + \Sigma\pi_0\right) + \frac{1}{2}\alpha'\Sigma\alpha\right) + \left(\alpha\Phi - \delta_1 + \alpha\Sigma\pi_1\right)Y_t\right).$$

This implies that $a\left(\alpha\right)$ and $b\left(\alpha\right)$ are written as follows

$$a\left(\alpha\right) = \left(\alpha\Phi - \delta_1 + \alpha\Sigma\pi_1\right) \text{ and } b\left(\alpha\right) = -\delta_0 + \alpha\left(\mu + \Sigma\pi_0\right) + \frac{1}{2}\alpha'\Sigma\alpha$$

Using the recursions (16) and (19) with the pricing solutions in (24) and (26), we get the analytical expressions for prices of risky bonds and CDS contracts under recovery risk. The correlation structure in this model clearly shows up in the terms $B_{1,i}\left(\alpha,\beta\right)$ and $B_{2,i}\left(\alpha,\beta\right)$ through the covariance matrix of the state vector Σ. The Gaussian specification allows the correlation between the state variables to be directly modeled. As is the case for the univariate model, the Gaussian VAR(1) can also be viewed as the discrete-time equivalent of the multivariate Ornstein-Uhlenbeck process when the eigenvalues of Φ are strictly positive. In fact, as pointed out by Gouriéroux, Monfort, and Polimenis (2006), the discrete-time Gaussian process allows for non-positive eigenvalues and therefore nests the continuous-time multivariate Ornstein-Uhlenbeck process. We also note that our model easily accommodates a multifactor set-up which is not the case for continuous time models. The latter admits closed-form expressions but at the cost of increased complexity which greatly, complicates any empirical implementation (see Bakshi, Madan and Zhang (2006)). Here, this additional computational cost is avoided.

4.3 A Markov Gamma model

We assume that the pricing kernel $M_{t,t+1} = M(Y_t, Y_{t+1})$ is an exponential affine function of Y_t and Y_{t+1}

$$M_{t,t+1} = \exp(\gamma_1' Y_{t+1} - \gamma_2' Y_t).$$

We now consider a Markov Gamma process for the state variables. The Markov Gamma process is viewed as the analogue in discrete-time of the continuous time square root process (see Gouriéroux and Jasiak (2002)). Furthermore, this specification was used by Nieto-Barajas and Walker (2002) in order to model hazard rates in discrete-time. Formally, a univariate process $\{u_{t+i}, i = 0, ..., n\}$ follows a Markov Gamma process, if

$$\frac{u_{t+1}}{\eta} \mid \mathcal{G}_t \sim \gamma(\delta + Z_t, 1) \text{ and } Z_t \sim \mathcal{P}(\theta\frac{u_t}{\eta})$$

where $\gamma(\delta + Z_t, 1)$ denotes the Gamma distribution with parameters $\delta + Z_t$ and 1, $\mathcal{P}(\theta \frac{\varepsilon_t}{\eta})$ is the Poisson distribution with intensity equal to θ and Z_{t+i} is independent of $Z_{t+j}, \forall\ i \neq j$.

The Markov Gamma process does not admit an AR(1) representation as is the case for the Gaussian process. However, conditional moments of this process can be derived using the conditional Laplace transform. The latter can be written as follows for any scalar α

$$E_t \left[\exp(\alpha u_{t+1})\right] = E_t \left[E \left[\exp(\alpha u_{t+1}) \mid Z_t\right]\right].$$

Using the Laplace transform of a Gamma distributed random variable (see Johnson et al (1995)), we get

$$E_t \left[\exp(\alpha u_{t+1})\right] = \exp\left(-\delta \log\left(1 - \eta\alpha\right) + \frac{\theta\alpha}{1 - \eta\alpha} u_t\right).$$

Now, following Gouriéroux and Jasiak (2002), we build a N-dimensional Markov process by assuming that each component of the state vector Y_t follows a Markov Gamma process and that they are mutually independent

$$\frac{Y_{k,t+1}}{\eta_k} \mid \mathcal{G}_t \sim \gamma(\delta_k + Z_{k,t}, 1) \text{ and } Z_{k,t} \sim \mathcal{P}(\theta_k \frac{Y_{k,t}}{\eta_k}), \text{ for any } k = \{1, .., n\}.$$

It is important to notice that the assumption of independence between the state variables constrains the correlation structure, as opposed to the Gaussian case where the correlation among the state variable is explicitly modeled. However, the conditional variance in the Gamma Markov model depends on each $Y_{k,t}$.

The conditional Laplace transform (14) is

$$E_t \left[M_{t,t+1} \exp(\alpha' Y_{t+1})\right] = \exp\left(\sum_{k=1}^{n} a_k\left(\alpha_k\right) Y_{k,t} + b_k\left(\alpha_k\right)\right),$$

where $a_k(\alpha)$ and $b_k(\alpha)$ are computed follows:

$$a_k(\alpha) = \frac{\theta_k\left(\alpha_k + \gamma_{1k}\right)}{1 - \eta_k\left(\alpha_k + \gamma_{1k}\right)} - \gamma_{2k} \text{ and } b_k(\alpha) = -\delta_k \log\left(1 - \eta_k\left(\alpha_k + \gamma_{1k}\right)\right), \text{ for any } k = \{1, ..., n\}.$$

Assembling these building blocks, the Laplace transforms (16) and (19) can be computed as follows

$$\begin{aligned} G^1_{t,p}(\alpha, \beta) &= \exp\left(\sum_{k=1}^{N} A^k_{1,p}(\alpha, \beta) Y_{k,t} + B^k_{1,p}(\alpha, \beta)\right) \text{ and} \\ G^2_{t,p}(\alpha, \beta) &= \exp\left(\sum_{k=1}^{N} A^k_{2,p}(\alpha, \beta) Y_{k,t} + B^k_{2,p}(\alpha, \beta)\right), \end{aligned}$$

using the recursions derived in proposition 3 for any $k = \{1, ..., N\}$.

The implied risk free short rate is then given by

$$r_t = \sum_{k=1}^{n} \delta_k \log\left(1 + \eta\gamma_{1k}\right) - \left(\frac{\eta_k\gamma_{1k}}{\eta_k\gamma_{1k} + 1} - \gamma_{2k}\right) Y_{k,t}.$$

Combining these Laplace transforms with the pricing solutions in (24) and (25), it becomes possible to derive the analytical expressions for prices of risky bonds under the RT and RFV assumptions for the multifactor factor Markov Gamma model.

4.4 Recovery Rates and the Cross-Section of Bond Spreads

It has been demonstrated in several places in the literature (see Houweling and Vorst (2003) for instance) that it is difficult to back out the unconditional level of the recovery rate using bonds in the constant recovery case. The basic intuition behind this result is that any variation in the level of the recovery is automatically compensated by an opposite variation in the intensity of default, and as a result the goodness of fit the model is not affected. Pan and Singleton (2005) show nonetheless that this identification problem disappears when CDS spreads are used in the estimation.

In this section, we show how a model can be parametrized in a way that makes the recovery rate impact on the cross-section of bond spreads. Our goal is to see if the use of a stochastic recovery rate can potentially improve the goodness of fit of the model. We use a simplified version of the Gaussian model described earlier, where the market price of risk is constant. We compute the spread implied by a four-factor Gaussian model under the RT and RFV assumptions.

The model is specified as follows

$$M_{t,t+1} = \exp(\psi_1 Y_{t+1} - \psi_2 Y_t),$$

and

$$Y_{t+1} = \mu + \Phi Y_t + \Sigma \epsilon_t.$$

The parameters of the model are chosen as follows

$$\Phi = \begin{bmatrix} 0.050 & 0 & 0 & 0 \\ 0 & 0.015 & 0 & 0 \\ 0 & 0 & 0.095 & 0 \\ 0 & 0 & 0 & 0.448 \end{bmatrix}, \begin{bmatrix} \mu_1 \\ \mu_3 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} 0.050 \\ 0.050 \\ 0.17 \\ 0.67 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} 0.2 & 0.15 & -0.15 & -0.1 \\ 0.15 & 0.1 & 0.07 & -0.015 \\ -0.15 & 0.07 & 0.05 & 0.15 \\ -0.1 & -0.015 & 0.15 & 0.15 \end{bmatrix},$$

$$\psi_1 = \psi_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \gamma \\ \phi \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} \gamma_0 \\ \phi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{29}$$

Figure 1 depicts the term structure of credit spreads for varying values of the state variable that control the level of the recovery rate, Y_4. The initial values of the three first components of the state vector are

$$\left(\; Y_1 = 0.15 \quad Y_2 = 0.051 \quad Y_3 = 0.09 \; \right), \tag{30}$$

and the initial value of the fourth state variable Y_4 is set equal to $0.1, 0.5$ and 0.9.

Both recovery assumptions (RT and RFV) result in a term structure that is decreasing in the current value of the recovery rate. As the value of the fourth state variable increases from 0.1 to 0.9, the current value of the loss given default decreases. As a result, the credit spreads significantly decrease. In general, the magnitude of the drop depends on the bond rating. Note that the credit spreads generated by this particular choice of parameters are rather high, which indicates a high yield bond. For this bond, the impact of the initial level of the recovery rate is significant. As the level of the fourth state variable increases from 0.1 to 0.9, the current value of the loss given default decreases and the maximum level of the credit spreads drops from approximately 13% to 11% for the assumption of RFV and from 10% to 8% for the assumption of RT. This is in line with the observation of Altman (2001) that recovery rates have a significant impact on high-yield bonds. Another interesting observation is that the model generates a humped shaped yield spread curve under the RFV assumption and a monotone decreasing curve under the RT assumption. Although the shape of the term structure depends on the particular parameterization of the model, it is important to notice that the model is able to generate both patterns.

In summary, this very simple numerical example shows that a stochastic recovery rate model has the potential to make the recovery rate impact on the cross-section of bond spreads. Building on this observation, we now proceed with an empirical investigation of the impact of the recovery rate on the cross-sectional and time series properties of bonds' spreads under the assumption of RT.

5 Data and Econometric Methodology

Before analyzing the empirical results, we first examine the data in section 5.1, we then present the model and the estimation technique in sections 5.2 and 5.3.

5.1 The Data

We use weekly BBB and B Standard and Poor's yield indices for the sample period August 6, 1996 to September 11, 2001. The maturities of theses indices are of 1, 5 and 10 years. Our choice of BBB and B rated indices is essentially motivated by the fact that they are representative of the investment grade and high yield markets. They also have very different sensitivities towards default and recovery risks. For the purpose of modeling the risk-free term structure, we use weekly zero-coupon Treasury bond yields with maturities of 3 months, 6 months, 1 year, 2 years, 5 years and 10 years that are extracted using the unsmoothed Fama and Bliss method. [9]

The time series plots of spreads and yields are provided in Figure 2. Table 1 presents the summary statistics for the spreads and yields. The spread is calculated as the difference between the yield index and the Treasury yield. B rated yields and spreads are more volatile than their BBB counterpart. For both ratings, the term structure of unconditional volatility is decreasing on average except for BBB spreads which exhibit an increasing unconditional term structure. While yields exhibit significant positive skewness for both BBB and B rated indices, the BBB spread display small skewness for maturities of 1 and 5 years and small negative skewness for the 10-year index. Finally, there does not seem to be strong evidence of excess kurtosis in yields and spreads regardless of their ratings.

5.2 Econometric Model

We assume that the uncertainty is reflected by a five-dimensional Gaussian state vector Y_t that we decompose into a systematic and a specific components. The first three factors are systematic and affect the pricing kernel, the default intensity and the loss given default. The fourth and the fifth factors are specific to the intensity and recovery risks. Furthermore, the specific factors are assumed to be independent from the systematic factors. The transition equation of state vector can therefore be written as follows

$$\begin{aligned} \begin{bmatrix} Y_t^r \\ Y_t^j \end{bmatrix} &= \begin{bmatrix} \mu^r \\ \mu^j \end{bmatrix} + \begin{bmatrix} \Phi^r & 0_{2\times 3} \\ 0_{3\times 2} & \Phi^j \end{bmatrix} \begin{bmatrix} Y_{t-1}^r \\ Y_{t-1}^j \end{bmatrix} + \begin{bmatrix} \Sigma^r & 0_{2\times 3} \\ 0_{3\times 2} & \Sigma^j \end{bmatrix} \begin{bmatrix} \varepsilon_t^r \\ \varepsilon_t^j \end{bmatrix} \\ &= \mu + \Phi Y_{t-1} + \Sigma\varepsilon_t, \end{aligned} \tag{31}$$

where Y_t^r represents the systematic component and Y_t^j corresponds to the specific component for each each yield index j.

The pricing kernel is parametrized as

$$M_{t,t+1} = \exp(-r_t - \frac{1}{2}\pi_t'\Sigma^r p_t - \pi_t'\varepsilon_{t+1}^r), \tag{32}$$

where

$$\pi_t = \pi_0 + \pi_1 Y_t^r \text{ and } r_t = \delta_0 + \delta_1 Y_t^r.$$

[9] A number of studies have questioned the choice of US Treasury yields as a benchmark for the risk-free term structure and suggested the use of swap rates instead (see Longstaff, Mithal and Neis (2004) and Houweling and Vorst (2003)). Since our sample covers the pre-2001 period, we use the US Treasury yield curve for modeling the risk-free term structure.

For each yield index j, the integrated default intensity and the loss given default are parametrized as follows

$$\Lambda_t^j = \gamma_0^j + \sum_{i=1}^{3} \gamma_1^j \left(Y_{it}^r - \overline{Y}_i^r \right) + \gamma_4^j Y_{4t}^j, \tag{33}$$

$$\mathcal{L}_t^j = \phi_0^j + \sum_{i=1}^{3} \phi_i^j \left(Y_{it}^r - \overline{Y}_i^r \right) + \sum_{i=4}^{5} \phi_i^j Y_{it}^j, \tag{34}$$

where $\overline{Y}_i^r$, $i = 1, 2, 3$, are the sample means of the smoothed estimates of the systematic state variables.

The specifications of the intensity and the loss given default allow for a very flexible correlation structure. The latter is captured via the components of γ^j and ϕ^j and is not forced to be of a particular sign. Another related characteristic of this model is the way default and recovery risks are priced. Because the pricing kernel and the market price of risk only depend on the systematic factors, the conditional distribution of the specific factors is invariant to the change of probability measure. However, the conditional distribution of the hazard rate and the loss given default is not the same under both measures. Put differently, default and recovery risk are priced via the systematic factors even though the specific risk is not priced.

We exploit the discrete-time formulation of the dynamics of the state variables and do not impose any stationarity condition. We also impose some normalizations in order to guarantee identification. Following Dai and Singleton (2000), we adopt the following canonical representation

$$Y_t = \Phi Y_{t-1} + \varepsilon_t, \tag{35}$$

where Φ is a lower triangular matrix.

5.3 Estimation Technique

The empirical behavior of defaultable securities within the reduced-form approach has been investigated in a large number of studies. The estimation methods can be classified in two groups. The first type of implementation uses multiple cross section of bonds or CDS spreads to estimate the parameters of the model (see Houweling and Vorst (2003), Longstaff, Mithal and Neis (2004) and Bakshi, Madan and Zhang (2006) among others). The second type of implementation fully exploits the cross-sectional and time series information contained in market prices. Duffee (1999) and Driessen (2005) implement a two-step procedure using the extended Kalman filter on corporate bonds to estimate a multifactor reduced-form model. Pan and Singleton (2005) use a quasi-maximum likelihood (QML) method where the state variables are inverted from the section of CDS spreads.

We follow a two-step procedure, where the risk-free term structure and the yields indices are estimated separately. In the first step, we extract the estimates and the filtered values of the dynamics of the systematic factors. The latter are then treated as constants in the second step, where the yield index-specific parameters are estimated. Because Treasury yields are an affine function of Gaussian state variables, we use the exact Kalman filter in the first step.[10] In the second step, the estimation poses an additional constraint related to the nonlinear mapping between the state variables and the yields. One possibility is to use the extended Kalman which relies on a first order Taylor expansion of the measurement equation. Since the relationship between the state variables and yields provided by equation (24) can be highly nonlinear, a first order Taylor approximation can potentially induce some important biases. We therefore implement the model in a novel way that preserves the non-linearity of the yield while still filtering the state variables as in the extended Kalman filter. We use the unscented Kalman filter technique (see Julier (2000) and Julier and Uhlmann (2004)), which can be viewed as an extension of the Kalman filter to the case of non-linear systems. Christoffersen, Jacobs, Karoui and Mimouni (2006) show that the unscented Kalman filter significantly outperforms its traditional extended counterpart in the presence of a non-linear mapping between the observations and the state variables.[11] A detailed description of the unscented Kalman filter algorithm is provided in Appendix B.

We use the nonlinear least square technique to minimize the following loss function with respect to the parameters

$$MSE = \frac{1}{N_T} \sum_{t,i} \left(z_{t|t-1}\left(n_i\right) - z_t\left(n_i\right) \right)^2, \tag{36}$$

where $N_T = N \times T$, N is the number of maturities used in the estimation, and T is the number of weekly observations, $z_t\left(n_i\right)$ and $z_{t|t-1}\left(n_i\right)$ are respectively the true and the predicted yields.

6 Empirical Results

In the section, we explore the empirical results. In section 6.1, we briefly discuss the parameter estimates of the risk-free term structure. Section 6.2 is devoted to the analysis of a restricted version of the model described in (31)-(34) where the recovery rate is constant. We first follow the literature and market practice by fixing the level of the recovery rate to 40%. We then discuss the case where the recovery is estimated as a parameter of the model. In section 6.3, we analyze the unrestricted version of the model where the loss given default is a stochastic process.

6.1 Parameter Estimates of the Risk-Free Term Structure

Panel A of Table 2 reports the parameter estimates of the essentially affine 3-factor model of the risk-free term structure. The parameters of the matrix Φ satisfy stationarity conditions and show that the state variables are quite persistent and negatively correlated. As well established now in the literature, the state variables can be seen as proxies for the level, slope and curvature factors.

[10] See Hamilton for a general presentation and Duan and Simonato (1999) and de Jong (2000) for applications to affine term structure models.

[11] See Bakshi, Carr and Wu (2006) for an application to the estimation of equity options models.

The second state variable is very persistent, therefore corresponding to the level factor. The third state variable is slightly less persistent and can be viewed as the equivalent of a slope factor. Finally, the first state variable is the least persistent and plays the role of a curvature factor. Turning to the market price of risk, one can see that the components of the vector π_0 are all negative, which indicates that the mean of the short rate under the equivalent martingale measure is higher than under the actual measure. The estimates of the matrix π_1 show that time variability in the term premia is mainly driven by the first and third state variables.

Panel B of Table 2 reports the root mean squared error for each maturity. The goodness of fit of the model is rather good and comparable to many other studies (see for example Duffee (2002) and Ang and Piazzesi (2003)).

6.2 Analysis of the Constant Recovery Case

In this section, we estimate two constrained versions of the model. In both versions, the recovery is constant and therefore $\phi_i^j = 0$, for $i = 1, \cdots, 5$. The yield on a risky bond under the RT assumption is then

$$z_t^j(n) = -\frac{\log((1 - L^j)B(t, n) + L^j \overline{B}(t, n))}{n},$$

where $B(t, n)$ is the price of a zero coupon Treasury bond, $\overline{B}(t, n)$ is the price of a zero coupon risky bond with zero recovery upon default and $L = \exp(-\phi_0^j)$ is the constant loss given default.

In the first version, we follow the literature and fix the loss given default to 0.6. This is equivalent to fixing ϕ_0^j equal to $-\log(0.6)$. In the second version of the model, we estimate L^j from the data and consider ϕ_0^j as a free positive parameter.

Table 3 reports the estimation results. Both models share many similarities regardless of the yields that are used in the estimation. First, the intensity is not stationary and is negatively correlated with the 3-month Treasury yield. The size of the unconditional correlation is higher for the BBB index. This negative correlation is consistent with many other empirical studies (Duffee (1998, 1999)). Figure 2 provides further intuition on this correlation. This figure shows the implied intensity for both ratings in addition to the 3-month Treasury yield. One can clearly see the negative correlation, particularly at the end of the sample period. Second, the estimate of γ_0^j which captures the unconditional mean of the intensity is always higher for the B yield index. This is consistent with the intuition that a lower rated bond has a higher likelihood of default. Panel B shows that the goodness of fit of the models is much better for the BBB yield index than for the B index. This may reflect the inability of the model to capture the much higher conditional and unconditional volatility of the B index.

In addition to these similarities between the two models, we also notice several striking differences. Even though the model with an estimated constant loss given default achieves slightly lower RMSEs for both ratings as shown by Panel B of Table 3, it implies unconditional levels of the loss given default that are dramatically different from 0.6. For the BBB yield index, the unconditional level of the loss given default is equal to 0.30 which is much lower than 0.6, whereas it reaches the boundary of 1 for the B yield index. This demonstrates that it is quite restrictive to arbitrarily fix the level of the recovery rate to 0.4. We also note that these changes in the unconditional levels of the loss given default are compensated by a higher intensity for the BBB yields and a lower intensity for the B yields. While this confirms to some extent the intuition that a lower loss results in a higher likelihood of default, it nonetheless shows that implied losses are different across ratings. The fact that the boundary of 1 is reached for the B yield index provides another reason to investigate the differences in the time series properties of the implied losses for each rating and underlines the need for a richer model that fully accounts for time variation in the loss given default.

6.3 Analysis of the Stochastic Recovery Case

We now relax the assumption of a constant recovery rate and estimate the unrestricted model. The intensity is still affected by the first four factors but the loss given default is driven by one specific extra factor. The yield on a risky bond under the RT assumption is then

$$z_t^j(n) = -\frac{\log(\overline{B}^{RT}(t,n))}{n},$$

where $\overline{B}^{RT}(t,n)$ is the price of a zero-coupon risky bond under the assumption of RT as given by equation (24).

Table 4 reports the estimation results for the BBB and B yields. Panel A indicates that the intensity is stationary for the B index but not for the BBB index. As for the constant recovery model, the lower the rating of the index, the higher is the unconditional mean of the intensity. We also notice that the negative correlation between the intensity and the 3-month Treasury yield is maintained under the unrestricted model.

More importantly, we notice that allowing for a stochastic recovery rate allows for significant reductions in the RMSEs. The RMSE is reduced by 14% for the BBB index and 20% for the B index. Panel B of Table 4 shows that for both indices, most of the improvement is due to a better fit for the 1-year yield. Because the intensity is affected by the same number of factors as in the constant recovery case, these reductions in RMSEs are in large part induced by the time varying nature of the recovery rate.

In addition to the negative correlation between the likelihood of default and the 3-month Treasury yield, the model is able to generate another well established stylized fact: A positive correlation between the implied loss and the intensity. This positive correlation is consistent with the findings of Altman, Brady, Resti and Sironi (2005). The flexible correlation structure of the five-factor model therefore allows to capture two of the most important stylized facts of the term structure of defaultable bonds. It is worth noting that this correlation is captured via γ_1^j, γ_4^j, ϕ_1^j and ϕ_4^j which are not forced to be of a particular sign. Figure 3 depicts the implied loss as a function of the intensity for both ratings. While this figure confirms the positive correlation between the implied loss and the intensity, it also shows large differences across ratings. The implied loss varies around 27% for the BBB index and arround 75% for the B index. An other striking feature of the implied losses is their very low volatility, especially the BBB index. Several potential reasons could explain this result. First the homoskedastic nature of the model does not allow to capture time variability in the conditional second moment. A more sophisticated model could resolve this issue. Second, we estimate the model using indices, which may reduce the volatility of the implied loss given default.

In summary, this section provides evidence that adding a time varying recovery rate improves the performance of the model and also captures the conditional correlation between the risk-free term structure, the default likelihood and the recovery risk.

7 Conclusion and Directions for Future Research

Recent empirical evidence illustrates the need to model the risk associated with the recovery rate jointly with the probability of default. However, while significant progress has been made in modeling defaultable securities, relatively few studies have modeled the time varying nature of the recovery rate. This paper proposes a methodology for modeling defaultable securities that jointly captures the recovery rate and the default probability. While most of the credit risk literature focuses on continuous time processes for modeling the term structure of defaultable bonds, this paper models a discrete-time economy. Interestingly, part of the motivation for continuous time models is usually their analytical tractability, this paper demonstrates that when the recovery rate is allowed to vary stochastically, a discrete-time setup is more tractable than its continuous time counterpart.

We provide general pricing solutions for CDS contracts and risky bonds under three standard assumptions: RT, RFV and RMV. We then focus on the case of an economy with affine state variables, and derive closed form expressions for prices of risky bonds and CDS contracts using the conditional Laplace transform of the state variables. Availability of the conditional Laplace transform of the state vector in closed form is a sufficient condition for pricing risky bonds and CDS contracts analytically. The family of affine discrete-time process for which closed form solutions under the RT and RFV assumption are available allows for potentially flexible correlation structures. We empirically investigate the impact of a time varying loss given default on the time series properties of BBB and B Standard and Poor's yield indices within a five-factor essentially affine Gaussian model. The model significantly outperforms a nested model whith a constant recovery rate. The flexible correlation structure enables the model to simultaneously capture two very important stylized facts of defaultable securities: A negative correlation between the likelihood of default and the risk free interest rate, and a positive correlation between the implied loss and the intensity of default.

Our framework is amenable to the study of several other related issues. For instance, it makes it possible to study the pricing of recovery and default risks embedded in bonds and CDS contracts, and the impact of the specification of the pricing kernel. Another important question is the impact of the recovery assumption on the performance of the model. Using data on defaulted debt, Guha (2002) argues that the assumption of RFV is best supported by the data. However, Bakshi, Madan and Zhang (2006) estimate a single-factor reduced form model on corporate bonds and find that the RT assumption achieves lower pricing errors than the RFV assumption both in and out-of-sample. Our framework can be used to investigate which recovery assumption is best supported by market prices when a rich econometric specification is used.

Appendices

A Proofs

Proof of proposition 1

We commence by recalling how to construct a Cox process. The filtration $\{\mathcal{F}_{t+i}; i = 0, ..., n\}$ reflects the evolution of the set of state variables up to time $t + i$ and the occurrence of default. If we denote by $\{\mathcal{G}_{t+i}, i = 0, ..., n\}$ the basic filtration without the information on the occurrence of default, then $\mathcal{F}_{t+i}$ can be written as

$$\mathcal{F}_{t+i} = \mathcal{G}_{t+i} \vee \mathcal{H}_{t+i},$$

where $\mathcal{H}_{t+i}$ is the sigma field $\sigma(\tau < s, s \leq t + i)$ that holds the information on whether default occurred before $t + i$.

Formally a Cox process, also known as a doubly stochastic process driven by the tribe $\mathcal{G}_{t+i}$, is a counting process that has a Poisson distribution with intensity Λ_{t+1}, conditional on $\mathcal{G}_{t+1} \vee \mathcal{H}_t, \forall$ $i \geq 1$, where

$$\Lambda_{t+i} = \int_{t+i-1}^{t+i} \lambda(u)\, du < \infty$$

The default time τ is said to be doubly stochastic driven by $\{\mathcal{F}_{t+i}; i = 0, ..., n\}$ if the underlying counting process whose first jump time is τ is doubly stochastic.

For the simple case of zero recovery in the event of default, the double stochasticity of the default time τ may be utilized to show that

$$E\left[\mathbf{1}_{(\tau>t+i)} \mid \mathcal{G}_{t+k} \vee \mathcal{H}_t\right] = \mathbf{1}_{(\tau>t)} \exp\left(-\sum_{i=1}^{k} \Lambda_{t+i}\right).$$

The price of the risky bond under the zero-recovery assumption is then

$$\begin{aligned}
\overline{B}(t, t+n) &= E\left[M_{t,t+n}\mathbf{1}_{(\tau>t+n)} \mid \mathcal{F}_t\right] \\
&= E\left[M_{t,t+n}E\left[\mathbf{1}_{(\tau>t+n)} \mid \mathcal{G}_{t+n} \vee \mathcal{H}_t\right] \mid \mathcal{F}_t\right] \\
&= \mathbf{1}_{(\tau>t)}E\left[M_{t,t+n}\exp\left(-\sum_{i=1}^{n} \Lambda_{t+i}\right) \mid \mathcal{F}_t\right]
\end{aligned}$$

Now, recall that $\{M_{t,t+i}, i = 0, ..., n\}$ and $\{\Lambda_{t+i}, i = 0, 1, ..., n\}$ depend only on the set of state variables. Since the state variables are $\mathcal{G}_t$-measurable processes, and thus independent of $\mathcal{H}_t$, one can replace the conditioning on $\mathcal{F}_t$ by a conditioning on $\mathcal{G}_t$,which implies that

$$\begin{aligned}
\overline{B}(t, t+n) &= \mathbf{1}_{(\tau>t)}E\left[M_{t,t+n}\exp\left(-\sum_{i=1}^{n} \Lambda_{t+i}\right) \mid \mathcal{G}_t\right] \\
&= \mathbf{1}_{(\tau>t)}E_t\left[M_{t,t+n}\exp\left(-\sum_{i=1}^{n} \Lambda_{t+i}\right)\right] \qquad (37)
\end{aligned}$$

The argument of the double stochasticity of τ is also used to derive the pricing solutions under the RT, RFV and RMV assumptions:

1. Under the RT assumption, the value of the risky bond verifies the following recursive equation

$$\overline{B}^{RT}(t,t+n) = E_t\left[M_{t,t+1}\left(\overline{B}^{RT}(t+1,t+n)\,\mathbf{1}_{(\tau>t+1)} + (1-L_{t+1})\,B(t+1,t+n)\mathbf{1}_{(\tau<t+1)}\right)\right]$$

and

$$\overline{B}^{RT}(t+n-1,t+n) = E_t\left[M_{t+n-1,t+n}\left(1-L_{t+n}\left(\mathbf{1}_{(\tau<t+n)}\right)\right)\right].$$

The same argument as in (37) yields

$$\begin{aligned}\overline{B}^{RT}(t,t+n) &= E_t\left[M_{t,t+n}(1-L_{t+1})\left(1-\exp(-\Lambda_{t+1})\right)\right]\\ &\quad\underbrace{+E_t\left[M_{t,t+1}\exp\left(-\Lambda_{t+1}\right)\overline{B}^{RT}(t+1,t+n)\right]}_{K_1}.\end{aligned} \tag{38}$$

Using the law of iterated expectations, K_1 can be written as follows

$$\begin{aligned}K_1 &= E_t\left[M_{t,t+2}\exp(-\sum_{i=1}^{2}\Lambda_{t+i})\overline{B}^{RT}(t+2,t+n)\right]\\ &\quad+E_t\left[M_{t,t+n}\left(1-\exp(-\Lambda_{t+2})\right)\exp(-\Lambda_{t+1})(1-L_{t+2})\right].\end{aligned}$$

Equation (38) implies that

$$\overline{B}^{RT}(t,t+n) = \underbrace{E_t\left[M_{t,t+2}\exp(-\sum_{i=1}^{2}\Lambda_{t+i})\overline{B}^{RT}(t+2,t+n)\right]}_{K_2}$$

$$+E_t\left[M_{t,t+n}\exp(-\Lambda_{t+1})\left(L_{t+1}-L_{t+2}\right)+(1-L_{t+1})-\exp(-\sum_{i=1}^{2}\Lambda_{t+i})\left((1-L_{t+2})\right)\right] \tag{39}$$

K_2 can be written as follows

$$\begin{aligned}K_2 &= E_t\Bigg[M_{t,t+3}\exp(-\sum_{i=1}^{3}\Lambda_{t+i})\overline{B}^{RT}(t+3,t+n)\\ &\quad+M_{t,t+n}\left((1-\exp(-\Lambda_{t+i})\right)\exp(-\sum_{i=1}^{2}\Lambda_{t+i})(1-L_{t+3})\Bigg].\end{aligned} \tag{40}$$

Plugging (40) into (39) yields

$$\begin{aligned}\overline{B}^{RT}(t,t+n) &= E_t\left[M_{t,t+3}\exp(-\sum_{i=1}^{3}\Lambda_{t+i})\overline{B}^{RT}(t+2,t+n)\right]+E_t\left[M_{t,t+n}(1-L_{t+1})\right]\\ &\quad+E_t\Bigg[M_{t,t+n}\Bigg(\sum_{k=1}^{2}\exp(-\sum_{i=1}^{k-1}\Lambda_{t+i})\left(L_{t+p}-L_{t+p+1}\right)\\ &\quad-\exp(-\sum_{i=1}^{3}\Lambda_{t+i})\left((1-L_{t+3})\right)\Bigg)\Bigg].\end{aligned} \tag{41}$$

Continuing the recursion, adding the coupons and taking into account the boundary condition gives

$$\overline{B}^{RT}(t,t+n) = E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right] + \sum_{k=1}^{\frac{n}{p}} E_t\left[M_{t,t+kp}\exp(-\sum_{i=1}^{kp}\Lambda_{t+i})\times C\right]$$
$$+ E_t\left[M_{t,t+n}\left(\sum_{k=0}^{n}\exp(-\sum_{i=0}^{k}\Lambda_{t+i}\mathbf{1}_{(k>0)})((L_{t+k}\mathbf{1}_{(k>0)}+\mathbf{1}_{(k=0)})-(L_{t+k+1}\mathbf{1}_{(k<n)}+\mathbf{1}_{(k=n)}))\right)\right].$$

2. Under the RFV assumption, the value of the risky bond verifies the following recursive equation

$$\overline{B}^{RFV}(t,t+n) = E_t\left[M_{t,t+1}\left(\overline{B}^{RFV}(t+1,t+n)\mathbf{1}_{(\tau>t+1)} + (1-L_{t+1})\,\mathbf{1}_{(\tau<t+1)}\right)\right]$$

and

$$\overline{B}^{RFV}(t+n-1,t+n) = E_{t+n-1}\left[M_{t+n-1,t+n}\left(1-L_{t+n}\left(\mathbf{1}_{(\tau<t+n)}\right)\right)\right].$$

Using the same argument as in proposition 1, we obtain

$$\begin{aligned}\overline{B}^{RFV}(t,t+n) &= \underbrace{E_t\left[M_{t,t+1}\exp\left(-\Lambda_{t+1}\right)\overline{B}^{RFV}(t+1,n)\right]}_{K_1}\\ &\quad + E_t\left[M_{t,t+1}\left(1-\exp(-\Lambda_{t+1})\right)\left(1-L_{t+1}\right)\right].\end{aligned} \quad (42)$$

K_1 can be written as follows

$$\begin{aligned}K_1 &= E_t\left[M_{t,t+2}\exp(-\sum_{i=1}^{2}\Lambda_{t+i})\overline{B}^{RFV}(t+2,t+n)\right]\\ &\quad + E_t\left[M_{t,t+2}\exp(-\Lambda_{t+1})\left(1-\exp(-\Lambda_{t+2})\left(1-L_{t+2}\right)\right)\right].\end{aligned}$$

Equation (42) implies that

$$\begin{aligned}\overline{B}^{RFV}(t,t+n) &= E_t\left[M_{t,t+2}\exp(-\sum_{i=1}^{2}(\Lambda_{t+i}))\overline{B}^{RFV}(t+2,t+n)\right]\\ &\quad + E_t\left[M_{t,t+1}\left(1-\exp(-\Lambda_{t+1})\right)\left(1-L_{t+1}\right)\right]\\ &\quad + E_t\left[M_{t,t+2}\exp(-\Lambda_{t+1})\left(1-\exp(-\Lambda_{t+2})\right)\left(1-L_{t+2}\right)\right].\end{aligned}$$

Continuing the recursion and taking into account the boundary condition gives

$$\overline{B}^{RFV}(t,t+n) = E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right] + \sum_{k=1}^{\frac{n}{p}} E_t\left[M_{t,t+kp}\exp(-\sum_{i=1}^{kp}\Lambda_{t+i})\times C\right]$$
$$+\sum_{k=1}^{n} E_t\left[M_{t,t+k}\exp\left(-\sum_{i=1}^{k-1}\Lambda_{t+i}\mathbf{1}_{(k>1)}\right)\left(1-\exp(-\Lambda_{t+k})\right)\left(1-L_{t+k}\right)\right]. \quad (43)$$

3. Under the RMV assumption, using the same arguments as previously, we find

$$\begin{aligned}\overline{B}^{RMV}(t,t+n) &= E_t\left[\prod_{i=1}^{n} M_{t+i-1,t+i}\left[1-L_{t+i}\left(1-\exp(-\Lambda_{t+i})\right)\right]\right] \\ &\quad+\sum_{k=1}^{\frac{n}{p}} E_t\left[\prod_{i=1}^{k} M_{t+i-1,t+i}\left(1-L_{t+i}\left(1-\exp(-\Lambda_{t+i})\right)\right)\times C\right]. \end{aligned} \tag{44}$$

Proof of proposition 2

Recall that the discount payoff, at time t, of a CDS between two consecutive coupon-date $t+(k-1)p+1$ and $t+kp$ is

$$-M_{t,t+kp}\mathbf{1}_{(\tau>t+kp)}\times S+\sum_{i=(k-1)p+1}^{kp} M_{t,t+i}\mathbf{1}_{(t+i-1<\tau<t+i)}\left(L_{t+i}-\frac{i}{kp}S\right).$$

The price of the CDS is then

$$\begin{aligned}CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -E\left[M_{t,t+kp}\mathbf{1}_{(\tau>t+kp)}\mid\mathcal{F}_t\right]\times S \\ &\quad+\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[M_{t,t+i}\mathbf{1}_{(t+i-1<\tau<t+i)}\left(L_{t+i}-\frac{i}{kp}S\right)\mid\mathcal{F}_t\right], \end{aligned} \tag{45}$$

Using the double stochasticity of the default time, we get

$$\begin{aligned}CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -E_t\left[M_{t,t+kp}\exp\left(-\sum_{j=1}^{kp}\Lambda_{t+i}\right)\right]\times S \\ &\quad+\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[\xi_i\left(L_{t+i}-\frac{i}{kp}S\right)\right], \end{aligned} \tag{46}$$

where

$$\xi_i = M_{t,t+i}(\exp(-\mathbf{1}_{(j>0)}\sum_{j=1}^{i-1}\Lambda_{t+j})-\exp(-\sum_{j=1}^{i}\Lambda_{t+j})).$$

The fair spread can be inverted by setting the price equal to 0.

Proof of proposition 3

The proof of this proposition proceeds by induction, we first verify the formula for $t+1$, assume that it holds for $t+n-1$ and prove that it is also true for $t+n$.

$$\begin{aligned} G^1_{t,1}(\alpha,\beta) &\equiv E_t\left[M_{t,t+1}\exp\left((\alpha+\beta)' Y_{t+1}\right)\right] \\ &= \exp(a'(\alpha+\beta)Y_t + b(\alpha+\beta)). \end{aligned}$$

Now for $t+1$, we have

$$\begin{aligned} G^1_{t,p}(\alpha,\beta) &\equiv E_t\left[M_{t,t+1}\exp\left(\alpha' Y_{t+1}\right)E_{t+1}\left[M_{t+1,t+p}\exp(\alpha'\sum_{i=1}^{p-1}Y_{t+1+i}+\beta' Y_{t+p})\right]\right] \\ &= E_t\left[M_{t,t+1}\exp\left((A_{1,p-1}+\alpha)' Y_{t+1}+B_{1,p-1}\right)\right] \\ &= \exp\left(a'(\alpha+A_{1,p-1})Y_t + B_{1,p-1} + b(\alpha+A_{1,p-1})\right). \end{aligned} \tag{47}$$

Using the same reasoning, one can derive $G^2_{t,p}(\alpha,\beta)$.

Proof of proposition 4

1. Under the assumption of RT, the price of a risky bond with maturity $t+n$ is

$$\overline{B}^{RT}(t,t+n) = E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right] + \sum_{k=1}^{n/p}E_t\left[M_{t,t+kp}\exp(-\sum_{i=1}^{kp}\Lambda_{t+i})\times C\right]$$
$$+E_t\left[M_{t,t+n}\left(\sum_{k=0}^{n}\exp(-\sum_{i=0}^{k}\Lambda_{t+i}\mathbf{1}_{(k>0)})((L_{t+k}\mathbf{1}_{(k>0)}+\mathbf{1}_{(k=0)})-(L_{t+k+1}\mathbf{1}_{(k<0)}+\mathbf{1}_{(k=n)}))\right)\right]$$

Equivalently, we can rewrite it as

$$\overline{B}^{RT}(t,t+n) = E_t[M_{t,t+n}] - E_t[M_{t,t+n}L_{t+1}] + E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})L_{t+n}\right]$$
$$+\sum_{k=1}^{n/p}E_t\left[M_{t,t+kp}\exp(-\sum_{i=1}^{kp}\Lambda_{t+i})\times C\right] + E_t\left[M_{t,t+n}\left(\sum_{k=1}^{n-1}\exp(-\sum_{i=0}^{k}\Lambda_{t+i})(L_{t+k}-L_{t+k+1})\right)\right]$$

The first three terms can be computed using the Laplace transform in (16) :

$$E_t[M_{t,t+n}] - E_t[M_{t,t+n}L_{t+1}] + E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})L_{t+n}\right]$$
$$= G^1_{t,n}(0_{\mathbb{R}^N},0_{\mathbb{R}^N}) - \exp(-\phi_0)G^2_{t,n}(0_{\mathbb{R}^N},-\phi) + \exp(-\phi_0-n\gamma_0)G^1_{t,n}(-\gamma,-\phi)$$

The last term can be written as follows

$$\begin{aligned} &E_t\left[M_{t,t+n}\left(\sum_{k=1}^{n-1}\exp(-\sum_{i=0}^{k}\Lambda_{t+i})(L_{t+k}-L_{t+k+1})\right)\right] \\ &= \sum_{k=1}^{n-1}J_k - K_k. \end{aligned}$$

J_k can be decomposed as follows

$$\begin{aligned} J_k &= E_t\left[M_{t,t+k}\exp(-\sum_{i=1}^{k}\Lambda_{t+i})L_{t+k}E_{t+k}\left[M_{t+k,t+n}\right]\right] \\ &= \exp\left(B_{1,n-k}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-\phi_0-k\gamma_0\right)E_t\left[M_{t,t+k}\exp\left(-\gamma\sum_{i=1}^{k}Y_{t+i}+\left(A_{1,n-k}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-\phi\right)Y_{t+k}\right)\right] \\ &= \exp\left(B_{1,n-k}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-\phi_0-k\gamma_0\right)G^1_{t,k}\left(-\gamma,\left(A_{1,n-k}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-\phi\right)\right). \end{aligned}$$

K_k can be written as

$$\begin{aligned} K_k &= E_t\left[M_{t,t+k}\exp\left(-\sum_{i=1}^{k}\Lambda_{t+i}\right)E_{t+k}\left[M_{t+k,t+n}L_{t+k+1}\right]\right] \\ &= \exp\left(B_{2,n-k}\left(0_{\mathbb{R}^N},-\phi\right)-\phi_0-k\gamma_0\right)E_t\left[M_{t,t+k}\exp\left(-\gamma\sum_{i=1}^{k}Y_{t+i}+A'_{2,n-k}\left(0_{\mathbb{R}^N},-\phi\right)Y_{t+k}\right)\right] \\ &= \exp\left(B_{2,n-k}\left(0_{\mathbb{R}^N},-\phi\right)-\phi_0-k\gamma_0\right)G^1_{t,k}(-\gamma,A_{2,n-k}\left(0_{\mathbb{R}^N},-\phi\right)). \end{aligned}$$

This implies that

$$\begin{aligned} \overline{B}^{RT}(t,t+n) = G^1_{t,n}(0_{\mathbb{R}^N},0_{\mathbb{R}^N}) - \exp(-\phi_0)G^2_{t,n}(0_{\mathbb{R}^N},-\phi) + \exp(-\phi_0-n\gamma_0)G^1_{t,n}(-\gamma,-\phi) \\ +\sum_{k=1}^{n/p}\exp(-kp\gamma_0)G^1_{t,kp}(-\gamma,0_{\mathbb{R}^N})\times C \\ +\sum_{k=1}^{n-1}\left[\exp\left(B_{1,n-k}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-\phi_0-k\gamma_0\right)G^1_{t,k}\left(-\gamma,\left(A_{1,n-k}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-\phi\right)\right)\right. \\ \left.-\exp\left(B_{2,n-k}\left(0_{\mathbb{R}^N},-\phi\right)-\phi_0-k\gamma_0\right)G^1_{t,k}(-\gamma,A_{2,n-k}\left(0_{\mathbb{R}^N},-\phi\right))\right]. \end{aligned}$$

2. Under the assumption of RFV, the price at t of a risky bond with maturity $t+n$ is

$$\begin{aligned} \overline{B}^{RFV}(t,t+n) = E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right]+\sum_{k=1}^{n/p}\exp\left(-k\gamma_0\right)E_t\left[M_{t,t+kp}\exp(-\sum_{i=1}^{kp}\Lambda_{t+i})\times C\right] \\ +\sum_{k=1}^{n}E_t\left[M_{t,t+k}\exp\left(-\sum_{i=1}^{k-1}\Lambda_{t+i}\mathbf{1}_{(k>1)}\right)\ \left(1-\exp(-\Lambda_{t+k})\right)\left(1-L_{t+k}\right)\right]. \end{aligned} \tag{48}$$

As for the RT assumption, the first two terms can be written as

$$\begin{aligned} E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right]+\sum_{k=1}^{n/p}E_t\left[M_{t,t+kp}\exp(-\sum_{i=1}^{kp}\Lambda_{t+i})\times C\right] \\ =\exp(-n\gamma_0)G^1_{t,n}(-\gamma,0_{\mathbb{R}^N})+\sum_{k=1}^{n/p}\exp(-kp\gamma_0)G^1_{t,kp}(-\gamma,0_{\mathbb{R}^N})\times C. \end{aligned}$$

Using the Laplace transforms in (16) and (19), the third term can be decomposed as follows

$$\sum_{k=1}^{n} E_t\left[M_{t,t+k}\exp\left(-\sum_{i=1}^{k-1}\Lambda_{t+i}\mathbf{1}_{(k>1)}\right)\ (1-\exp(-\Lambda_{t+k}))\,(1-L_{t+k})\right]$$

$$\begin{aligned}
&= G^1_{t,1}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right) - \exp(-\gamma_0)G^1_{t,1}\left(-\gamma,0_{\mathbb{R}^N}\right)\\
&- \exp(-\phi_0)G^1_{t,1}\left(-\phi,0_{\mathbb{R}^N}\right) + \exp\left(-\left(\gamma_0+\phi_0\right)\right)G^1_{t,1}\left(-\left(\gamma+\phi\right),0_{\mathbb{R}^N}\right)\\
&+\sum_{k=2}^{n}\left[\exp(-(k-1)\gamma_0)G^1_{t,k}\left(-\gamma,\gamma\right) - \exp(-k\gamma_0)G^1_{t,k}(-\gamma,0_{\mathbb{R}^N})\right.\\
&\left.- \exp(-(k-1)\gamma_0-\phi_0)G^1_{t,k}(-\gamma,\gamma-\phi) + \exp(-k\gamma_0)G^1_{t,k}(-\gamma,-\phi)\right].
\end{aligned}$$

It therefore follows that

$$\begin{aligned}
\overline{B}^{RFV}(t,t+n) &= G^1_{t,n}\left(-\gamma,0_{\mathbb{R}^N}\right) + \sum_{k=1}^{n/p} G^1_{t,kp}(-\gamma,0_{\mathbb{R}^N})\times C\\
&+G^1_{t,1}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right) - \exp(-\gamma_0)G^1_{t,1}\left(-\gamma,0_{\mathbb{R}^N}\right)\\
&- \exp(-\phi_0)G^1_{t,1}\left(-\phi,0_{\mathbb{R}^N}\right) + \exp\left(-\left(\gamma_0+\phi_0\right)\right)G^1_{t,1}\left(-\left(\gamma+\phi\right),0_{\mathbb{R}^N}\right)\\
&+\sum_{k=2}^{n}\left[\exp(-(k-1)\gamma_0)G^1_{t,k}\left(-\gamma,\gamma\right) - \exp(-k\gamma_0)G^1_{t,k}(-\gamma,0_{\mathbb{R}^N})\right.\\
&\left.- \exp(-(k-1)\gamma_0-\phi_0)G^1_{t,k}(-\gamma,\gamma-\phi) + \exp(-k\gamma_0-\phi_0)G^1_{t,k}(-\gamma,-\phi)\right]. \qquad (49)
\end{aligned}$$

Proof of proposition 5

The price, at time t, of a CDS contract with maturity $t+n$ is

$$\begin{aligned}
CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -E_t\left[M_{t,t+kp}\exp\left(-\sum_{j=1}^{kp}\Lambda_{t+i}\right)\right]\times S\\
&+\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[\xi_i\left(L_{t+i}-\frac{i}{kp}S\right)\right], \qquad (50)
\end{aligned}$$

where

$$\xi_i = M_{t,t+i}(\exp(-\mathbf{1}_{(j>0)}\sum_{j=1}^{i-1}\Lambda_{t+j}) - \exp(-\sum_{j=1}^{i}\Lambda_{t+j})).$$

Once again, using the Laplace transform in (16), $CDS(t,t+n)$ can be rewritten as

$$\begin{aligned}
CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -\exp(-kp\gamma_0)G^1_{t,kp}\left(-\gamma,0_{\mathbb{R}^N}\right)\times S\\
&+\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[M_{t,t+i}(\exp(-\sum_{j=1}^{i}\Lambda_{t+j}+\Lambda_{t+i})\right.\\
&\left.-\exp(-\sum_{j=1}^{i}\Lambda_{t+j}))\left(L_{t+i}-\frac{i}{kp}S\right)\right]. \qquad (51)
\end{aligned}$$

It therefore follows that

$$CDS(t,t+n) = \sum_{k=1}^{\frac{n}{p}} - \exp(-kp\gamma_0) G^1_{t,kp}\left(-\gamma, 0_{\mathbb{R}^N}\right) \times S$$

$$+ \sum_{k=1}^{\frac{n}{p}} \sum_{i=(k-1)p+1}^{kp} \exp(-(i-1)\gamma_0 - \phi_0) G^1_{t,i}\left(-\gamma, \gamma - \phi\right) - \exp(-i\gamma_0 - \phi_0) G^1_{t,i}\left(-\gamma, -\phi\right)$$

$$+ \sum_{k=1}^{\frac{n}{p}} \sum_{i=(k-1)p+1}^{kp} \left(\exp(-(i-1)\gamma_0) G^1_{t,i}\left(-\gamma, \gamma\right) - \exp(-i\gamma_0) G^1_{t,i}\left(-\gamma, 0_{\mathbb{R}^N}\right)\right) \frac{i}{kp} S. \tag{52}$$

B The Unscented Kalman Filter Algorithm

The structure of the Kalman filter relies on a state-space representation of a dynamic system. Assume that the uncertainty in the economy is reflected by a d dimensional state vector process $\{Y_t, t \leq T\}$ and that at each date, an observed N dimensional vector of yields z_t is available. The state-space representation involves the following two equations

$$Y_{t+1} = \mu + \Phi Y_t + \Sigma \epsilon_t, \tag{53}$$

and

$$z_t = G(Y_t) + u_t, \tag{54}$$

where ϵ_{t+1} is the state noise and u_t is the observation noise that has zero mean and a diagonal covariance matrix denoted by R.

The transition equation (53) reflects the discrete time evolution of the state variables, whereas the measurement equation expresses the relationship between the unobserved state vector and the observed variables. As is the case of the extended Kalman filter, the unscented Kalman filter is minimum mean square error estimator that relies on the following linear update rule at each date t

$$Y_{t|t} = Y_{t|t-1} + K_t \left(z_t - z_{t|t-1}\right), \tag{55}$$

and

$$P_{yy(t|t)} = P_{yy(t|t-1)} - K_t P_{yy(t|t-1)} K_t', \tag{56}$$

where

$$Y_{t|t-1} = \mu + \Phi y_{t-1|t-1},$$

$$P_{yy(t|t-1)} = \Phi P'_{yy(t-1|t-1)} \Phi + \Sigma$$

$$K_t = P_{yz(t|t-1)} P^{-1}_{zz(t|t-1)},$$

and

$$z_{t|t-1} = E\left[G(y_{t|t-1}, u_t)\right].$$

Rather than approximating $G(y)$, the unscented Kalman filter approximates the conditional distribution of the z_{t+1} using the scaled unscented transformation (see Julier (2000) for more details), which can be defined as a method for computing the statistic of a non-linear transformation of a random variable. Consider a random variable y with mean μ_y and covariance matrix P_y and a non-linear transformation $z = g(y)$. The basic idea behind the scaled unscented transformation is to generate a set of points, called sigma points, so that their sample two first moments equate μ and P_x. The non-linear transformation is then applied to y at each sigma point. To be more precise, the n_y-dimensional random variable is approximated by a set of $2n_y + 1$ weighted points given by

$$\mathcal{Y}_0 = \mu,\ W_0^m = \frac{\lambda}{(n_y + \lambda)}, W_0^c = \frac{\lambda}{(n_y + \lambda)} + \left(1 - \alpha^2 + \beta\right)$$

$$\mathcal{Y}_i = \mu + \left(\sqrt{(n_y + \lambda) P_x}\right)_i,\ \text{for } i = 1, \cdots, n_y \tag{57}$$

$$\mathcal{Y}_i = \mu - \left(\sqrt{(n_y + \lambda) P_x}\right)_i,\ \text{for } i = n_y, \cdots, 2n_y \tag{58}$$

and

$$W_i^m = W_i^c = \frac{1}{2(n_y + \lambda)},\ \text{for } i = 1, \cdots, 2ny,$$

where $\lambda = \alpha^2 (n_y + \kappa) - n_x$, $\left(\sqrt{(n_y + \lambda) P_y}\right)_i$ is the ith column of the Cholesky decomposition of $(n_y + \lambda) P_y$. α is a positive caling parameters that minimizes higher order effects and can be made arbitrarily small. κ is a positive parameter that guarantees the positivity of the covariance matrix. β is a non-negative parameter that can be used to capture higher order moments of the state distribution. It is equal to 2 for Gaussian distributions.

The linear transformation is applied to the sigma points (57) – (58) as follows

$$\mathcal{Z}_i = g(\mathcal{Y}_i),\ \text{for } i = 0, \cdots, 2n_y.$$

The first conditional moments of y are computed using the weights

$$\mu_z = \sum_{i=0}^{2n_x+1} W_i^m \mathcal{Z}_i \tag{59}$$

and

$$P_z = \sum_{i=0}^{2n_x+1} W_i^c (\mathcal{Z}_i - \mu_z)(\mathcal{Z}_i - \mu_z)' \tag{60}$$

The unscented Kalman filter applies unscaled unscented transformation to the augmented vector

$$Y_t^a = \begin{bmatrix} Y_t \\ \epsilon_t \\ u_t \end{bmatrix},$$

and works as follows:

1. Start with

$$Y_{0|0} = E\left[Y_t\right]$$

$$P_{0|0} = var\left[Y_t\right]$$

$$Y^a_{0|0} = \begin{bmatrix} Y_{0|0} \\ 0_{2\times 1} \\ 0 \end{bmatrix}$$

$$P^a_{0|0} = \begin{bmatrix} P_{0|0} & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & R \end{bmatrix}.$$

2. For $t = 1, \cdots, T$. Generate the sigma points of the augmented vector $\mathcal{Y}^a_{t-1}$ as in (57) – (58), then perform the prediction step

$$\mathcal{Y}^y_{i,t|t-1} = \mu + \Phi\mathcal{Y}^y_{t-1} + \Sigma\mathcal{Y}^\epsilon_{t-1}$$

$$Y_{t|t-1} = \sum_{i=0}^{2n_x} W^m_i \mathcal{Y}^y_{i,t|t-1}$$

$$P_{yy(t|t-1)} = \sum_{i=0}^{2n_x} W^c_i \left(\mathcal{Y}^y_{i,t|t-1} - Y_{t|t-1}\right)\left(\mathcal{Y}^y_{i,t|t-1} - Y_{t|t-1}\right)'$$

$$\mathcal{Z}_{t|t-1} = h(\mathcal{Y}^y_{t|t-1}, \mathcal{Y}^\epsilon_{t-1})$$

$$z_{t|t-1} = \sum_{i=0}^{2n_x} W^m_i \mathcal{Z}_{i,t|t-1}$$

$$P_{zz(t|t-1)} = \sum_{i=0}^{2n_x} W^c_i \left(\mathcal{Z}_{i,t|t-1} - z_{t|t-1}\right)\left(\mathcal{Z}_{i,t|t-1} - z_{t|t-1}\right)'$$

$$P_{zy(t|t-1)} = \sum_{i=0}^{2n_x} W^c_i \left(\mathcal{Z}_{i,t|t-1} - z_{t|t-1}\right)\left(\mathcal{Y}^y_{i,t|t-1} - Y_{t|t-1}\right)'$$

3. The update equations are obtained via (55) – (56).

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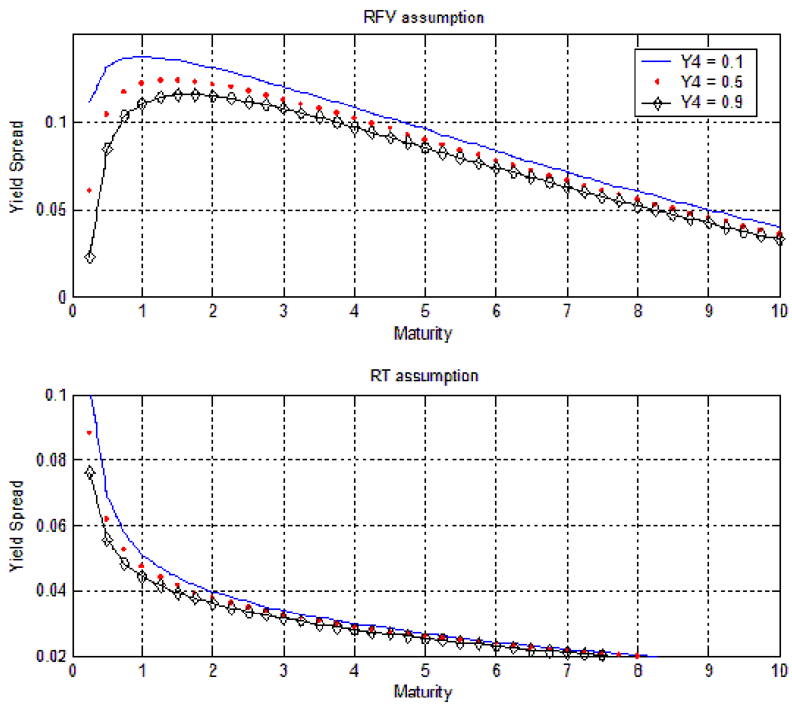
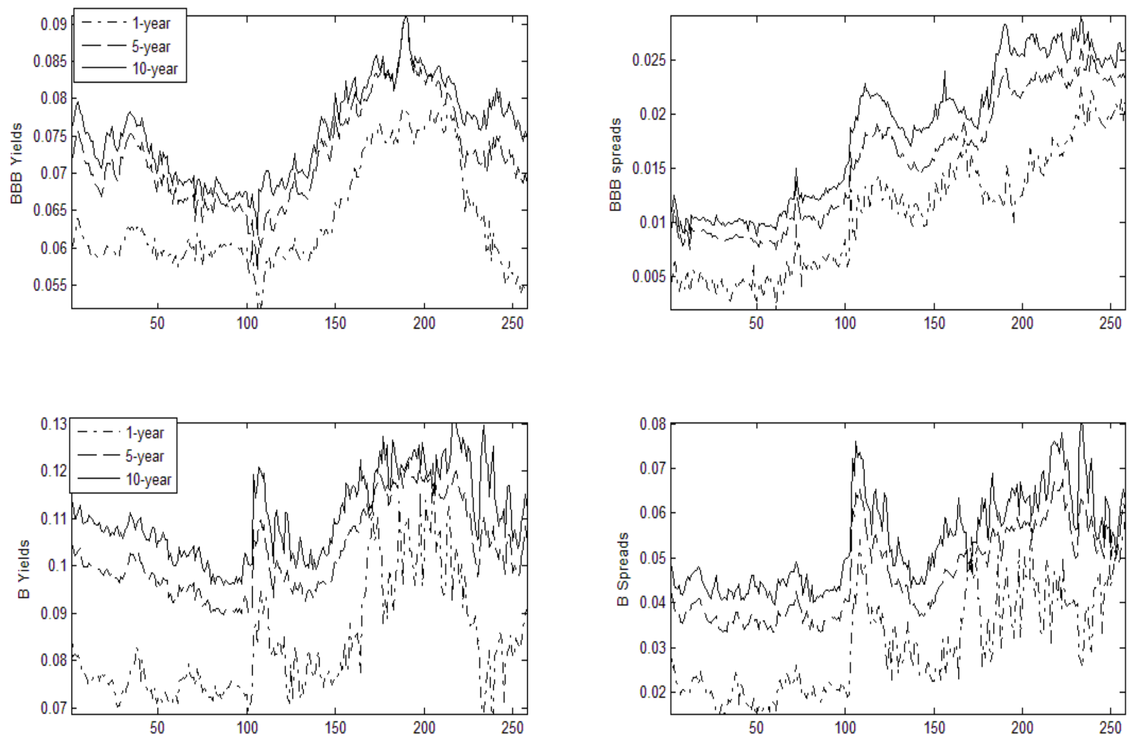
Figure 1: Term structure of bond spreads for a Four-Factor Gaussian Model



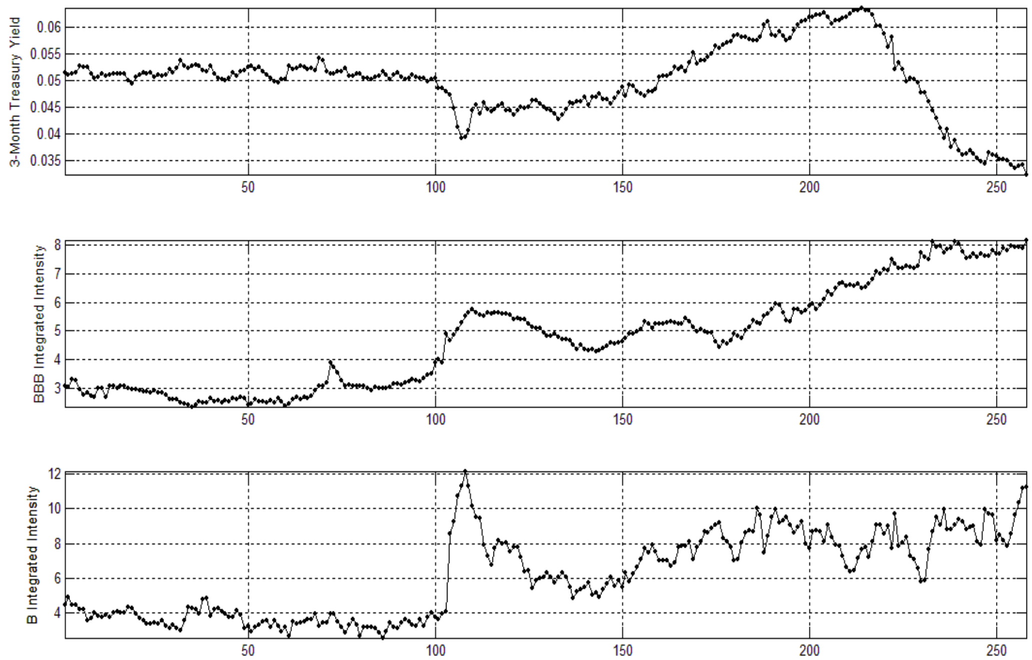
Notes to Figure: We compute the term structure of the bond spreads, implied by a four-factor Gaussian model, under the RFV and RT assumptions. The intensity is affected by the first three factors and the the loss given default has a specific fourth factor. We plot the term structure of bond spreads for various level of the loss given default specific state variable.

Figure 2: Standard and Poor's yields and spreads for various maturities



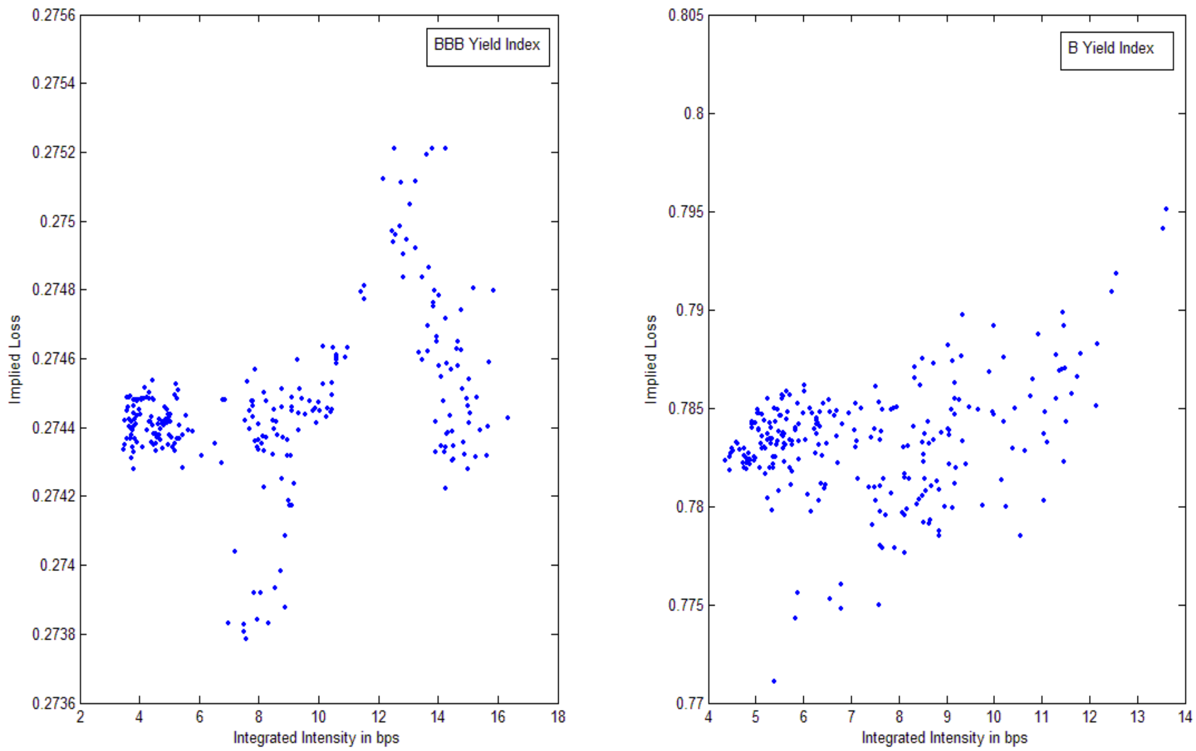
Notes to Figure: This figure shows BBB and B weekly yields and spreads. The spread is computed as the difference between the index yield and the corresponding Treasury yield. The sample period is from august 1996 to September 2001.

Figure 3: Intensity implied by a model with a fixed loss given default and the 3-month Treasury yield



Notes to Figure: The top panel plots the 3-month Treasury yield. The bottom two panels show respectively the BBB and B implied intensity paths when the recovery rate is fixed at 40%.

Figure 4: Implied loss and intensity



Notes to figure: We plot the model-implied loss given default for BBB and B yield indices as a function of the implied intensity given in bps. The right column corresponds to the BBB yield index and the left column shows the B yield index.

Table 1: Summary Statistics of the Standard and Poor's Yield Indices

Panel A: BBB Index

Maturity	Yields			Spreads		
	1-year	5-year	10-year	1-year	5-year	10-year
Mean	0.0638	0.0724	0.0755	0.0109	0.0159	0.0183
Standard Deviation (in %)	0.7094	0.6705	0.6138	0.5465	0.5845	0.6455
Skewness	0.8500	0.4980	0.1680	0.2013	0.0608	-0.0552
Kurtosis	2.3697	2.5433	2.1188	1.8825	1.5440	1.5080

Panel B: B Index

Maturity	Yields			Spreads		
	1-year	5-year	10-year	1-year	5-year	10-year
Mean	0.0834	0.1028	0.1106	0.0305	0.0463	0.0535
Standard Deviation (in %)	1.1845	0.9042	0.8871	1.1129	0.9719	1.0379
Skewness	1.0768	0.5204	0.3188	0.5006	0.3489	0.5704
Kurtosis	3.1819	2.0626	2.0937	2.0818	1.7735	2.2497

Notes: The sample contains weekly data, from August 1996 to September 2001, on Standard and Poor's BBB and B yield indices with maturities of 1, 5 and 10 years. The spread is computed as the difference between the yield index and the corresponding Treasury yield.

Table 2: Parameter Estimates for the Risk-Free Term Structure

Panel A: Transition Equation Estimates

Parameter	**Factor** 1	2	3
δ_0	0.00096 (0.0000953)		
δ_{1i}	0.00077 (0.0000004)	-0.00011 (0.0000003)	-0.00080 (0.0000266)
Φ^r_{1i}	0.97368 (0.0000004)		
Φ^r_{2i}	-0.00710 (0.0000016)	0.99990 (0.0000003)	
Φ^r_{3i}	-0.01160 (0.0000003)	-0.00261 (0.0000004)	0.99074 (0.0000004)
π_{0i}	-0.16944 (0.0047744)	-0.32432 (0.0126886)	-0.11485 (0.0126886)
π_{1i}	-0.00121 (0.0000004)	-0.00028 (0.0000004)	0.00047 (0.0000004)
π_{2i}	0.05602 (0.0000003)	-0.00006 (0.0000003)	0.00412 (0.0000004)
π_{3i}	-0.00058 (0.0000004)	-0.00019 (0.0000004)	0.00539 (0.0000004)
Total RMSE (in bps)		11.86	

Panel B: RMSE's (in bps) For Each Maturity

Maturity	**3-month**	**6-month**	**1-year**	**2-year**	**5-year**	**10-year**
RMSE	10.89	9.92	12.15	11.80	12.88	13.21

Notes: We estimate the model using the Kalman filter on a sample of weekly data from August 1996 to September 2001. Zero-coupon yields are interpolated using the unsmoothed Fama and Bliss method. Standard errors are given in parentheses.

Table 3: Parameter Estimates for the Yield Indices in the Constant Recovery Case

Panel A: Parameter Estimates

Parameter	L=60%		Estimated L	
	BBB	B	BBB	B
γ^j_0	0.00031 (0.00002)	0.00017 (0.00014)	(0.00046) (0.00006)	0.00054 (0.00010)
γ^j_1	-0.00019 (0.00000)	-0.00057 (0.00003)	-0.00064 (0.00002)	-0.00013 (0.00005)
γ^j_2	0.00001 (0.00001)	0.00008 (0.00001)	0.00007 (0.00001)	0.00001 (0.00001)
γ^j_3	0.00012 (0.00001)	0.00075 (0.00009)	0.00058 (0.00005)	0.00018 (0.00004)
γ^j_4	0.00001 (0.000007)	0.00001 (0.000022)	-0.000003 (0.00005)	-0.00001 (0.00013)
Φ^j_1	1.00124 (0.00007)	1.00227 (0.00017)	1.00211 (0.00016)	0.99944 (0.00049)
R_{11}	0.00105 (0.00064)	0.00038 (0.00091)	0.00011 (0.00153)	0.00018 (0.00436)
R_{22}	0.00102 (0.00062)	0.00032 (0.00076)	0.00010 (0.00144)	0.00012 (0.00283)
R_{33}	0.00091 (0.00055)	0.00036 (0.00086)	0.00010 (0.00139)	0.00018 (0.00447)
L	0.6	0.6	0.30525 (0.01893)	1 (0.16346)
ρ	-0.27048	-0.15810	-0.31820	-0.05242
Total RMSE (in bps)	10.56	43.73	10.55	40.17

Panel B: RMSE (in bps) by Maturity

Maturity	L=60%			Estimated L		
	1-year	5-year	10-year	1-year	5-year	10-year
BBB	12.96	8.43	9.79	12.63	8.43	10.17
B	55.07	27.43	44.19	51.88	23.10	40.19

Notes: In Panel A , we estimate the model using the unscented Kalman filter on a sample of weekly BBB and B rated yield indices from August 1996 to September 2001. In the second and third column, we consider a fixed recovery rate equal to 0.4. In the fourth and fifth column, we estimate the model and consider the recovery rate as an additional parameter. In both Panels, ρ denotes the sample correlation between the fitted integrated intensity and the 3-month Treasury yield and the standard errors are given in parentheses. In Panel C, we report the RMSE on the yield indices for each maturity.

Table 4: Parameter Estimates for the Yield Indices in the Stochastic Recovery Case

Panel A: Parameter Estimates

Parameter	**BBB**	**B**
γ^j_0	0.00048 (0.000234)	0.0007 (0.00026)
γ^j_1	-0.00025 (0.000140)	-0.00026 (0.0001)
γ^j_2	0.00001 (0.000014)	0.00002 (0.0000)
γ^j_3	0.00011 (0.000102)	0.00023 (0.0002)
γ^j_4	-0.00001 (0.000115)	-0.00002 (0.0003)
ϕ^j_0	1.29318 (0.541737)	0.24446 (0.46540)
ϕ^j_1	0.00007 (0.000083)	0.00005 (0.00015)
ϕ^j_2	0	0
ϕ^j_3	0	0
ϕ^j_4	0.00000 (0.000021)	0.00009 (0.00152)
ϕ^j_5	-0.00006 (0.000761)	0.00057 (0.00717)
Φ^j_{11}	1.00105 (0.000797)	0.99888 (0.00186)
Φ^j_{12}	0.00031 (0.017583)	-0.03046 (0.283753)
Φ^j_{22}	0.94748 (0.018943)	0.82586 (0.03915)
R_{11}	0.00015 (0.002168)	0.00063 (0.00824)
R_{22}	0.00011 (0.001577)	0.00019 (0.00246)
R_{33}	0.00012 (0.001624)	0.00041 (0.00539)
ρ_1	-0.13194	-0.15801
ρ_2	0.37515	0.32319
Total RMSE (in bps)	8.49	35.62

Panel B: RMSE's For Each Maturity

Maturity	**1-year**	**5-year**	**10-year**
BBB	9.58	7.63	8.15
B	44.71	23.81	35.25

Notes: We estimate the unconstained model using the unscented Kalman filter on a sample of weekly BBB and B rated yield indices from August 1996 to September 2001. ρ_1 and ρ_2 denote the sample correlation between the fitted integrated intensity and respectively the 3-month Treasury yield and the implied loss given default. Standard errors are given in parentheses.