Risk and Return in Behavioral SDF-based Asset Pricing Models[1]

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Behavioral finance and neoclassical finance have very different implications about asset pricing in general and the relationship between risk and return in particular.

I survey a selection of recent works that taken together generate unifying insights about the behavioral character of asset pricing. The context for these insights is the pricing kernel-based asset pricing framework described in Cochrane (2005). The strength of this framework is its integrated approach, in which specific asset pricing models such as the CAPM, Fama-French multi-factor model, and the Black-Scholes option pricing formula, are all derived as special cases. At the heart of the framework is the concept of a stochastic discount factor (SDF) used to price any asset as the expected value of its expected discounted future cash flow stream.

Proponents of behavioral asset pricing emphasize that asset prices reflect sentiment, broadly understood to mean erroneous beliefs about future cash flows and risks; see Baker and Wurgler (2007). In this survey my main objective is to describe recent contributions which shed light on how sentiment manifests itself within the stochastic discount factor. It is sentiment which potentially impacts the prices of all assets, and in consequence drives the difference between what behavioral finance and neoclassical finance tell us about the relationship between risk and return.

1. Defining Sentiment in the SDF-based Framework

A unified behavioral approach to asset pricing requires a general definition of sentiment that is well-defined, measurable, and whose impact can be traced on market prices and risk premiums. Because the behavioral asset pricing literature developed in order to address a series of specific issues, the treatment of sentiment in the behavioral asset pricing literature has tended to be eclectic. Definitions of sentiment range from loosely worded statements about investors' mistakes to errors which are model specific, but vary from model to model.[2] Despite the fact that the SDF lies at the heart of the modern approach to asset pricing, most behavioral asset pricing models are not SDF-based.

In order to provide a definition of sentiment that is well suited to an SDF-based asset pricing framework, consider the fundamental SDF-based asset pricing equation

(1) $$p = E(mx)$$

Equation (1) states that the price p of an asset with random payoff x is the expected value of its discounted payoff, where m is the discount factor used to capture the effects of both

[2] This range can be discerned in the past behavioral surveys by Barberis and Thaler (2003), Baker and Wurgler (2007), Hong and Stein (2007), along with the collection by Shleifer (2000). Baker and Wurgler (2007) have an excellent discussion about different proxies for sentiment, and the construction of a what they call a "top down" measure. In contrast, the present paper focuses on the development of a "bottoms up measure."

time value of money and risk. In equation (1), both m and x are random variables. That is, the discount factor m typically varies across payoff levels in order to reflect that risk is priced differently across payoff levels.[3]

Sentiment pertains to erroneous beliefs. In this regard, think of x as having a pdf which is objectively correct, but about which individual investors only possess subjective beliefs. The beliefs of an investor whose subjective beliefs are correct are said to feature zero sentiment. The beliefs of an investor whose subjective beliefs are incorrect are said to feature nonzero sentiment.

In the neoclassical SDF framework, investors' beliefs refer to an underlying state variable such as aggregate consumption growth. In this case, the sentiment of an individual investor can be described as the difference between two probability density functions: the objective pdf and the individual investor's subjective pdf.

A central issue in behavioral asset pricing theory is how in equilibrium the market aggregates the probability density functions of the individual investors to arrive at a probability density function for the market as a whole. It is this "market pdf" which underlies equation (1). And of course, the same notion of sentiment that applies to the probability density functions of individual investors also applies to the probability density function associated with the market pdf.

The study of the aggregation question dates back at least as far as Lintner (1969). Shefrin (2005) presents an equilibrium aggregation result for a complete markets model featuring power utility.[4] He demonstrates that the equilibrium market pdf, designated as P_M, is a generalized weighted Hölder average of the individual investors' pdfs, with the weights reflecting relative wealth or consumption, and the exponents being coefficients of relative risk aversion.[5]

The following example from Shefrin (2005) illustrates the general issues. Consider a complete market model featuring two investors, whose initial wealth levels are equal. Suppose that both investors have log-utility functions.[6] Assume that the log of consumption growth is normally distributed, and that both investors believe it to be normally distributed. However, let the two investors disagree about the first moment. Assume that one investor is excessively bullish about mean consumption growth, while the second investor is excessively bearish about mean consumption growth. For now, let both investors hold correct beliefs about the second moment.[7]

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[3] In the neoclassical framework, the variation in m across states of the world stem from varying marginal rates of substitution, which reflects the value of marginal consumption in those states.

[4] Power utility has the form $u(c) = c^{1-\gamma} / (1-\gamma)$ and exhibits constant relative risk aversion (CRRA), where the degree of relative risk aversion is given by γ. Intertemporal expected utility involves multiplying u(c) by a time discount factor δ^t and summing over t.

[5] A Hölder average has the form $(\Sigma_i\, w_i\, q^{\gamma})^{1/\gamma}$ while the generalized Hölder average allows γ to be subscripted by i within the summation, defines the γ used in the exponent $1/\gamma$ as a harmonic average, and allows for some additional terms within the summation.

[6] $\gamma = 1$

[7] Here the second moment refers to the normal distribution for the log-consumption rate of growth.

Figure 1 demonstrates how in equilibrium the market aggregates the pdfs of the two investors. The figure displays four probability density functions over aggregate consumption growth. The two extreme pdfs belong to the two investors. The middle pdf, labeled Objective Density, is indeed the objective pdf, and is denoted by the symbol Π. By assumption, the mean of Π in this example is set equal to the simple average of the two investors' first moments.[8] The fourth pdf is the market pdf P_M, the equilibrium density used in equation (1) to price assets.

As is easily seen, the market pdf does not simply aggregate moments. Instead, P_M aggregates pdfs. For the log-utility case, the coefficient of relative risk aversion is unity, so the generalized wealth weighted Hölder average of the investors' pdfs is the simple wealth-weighted average.

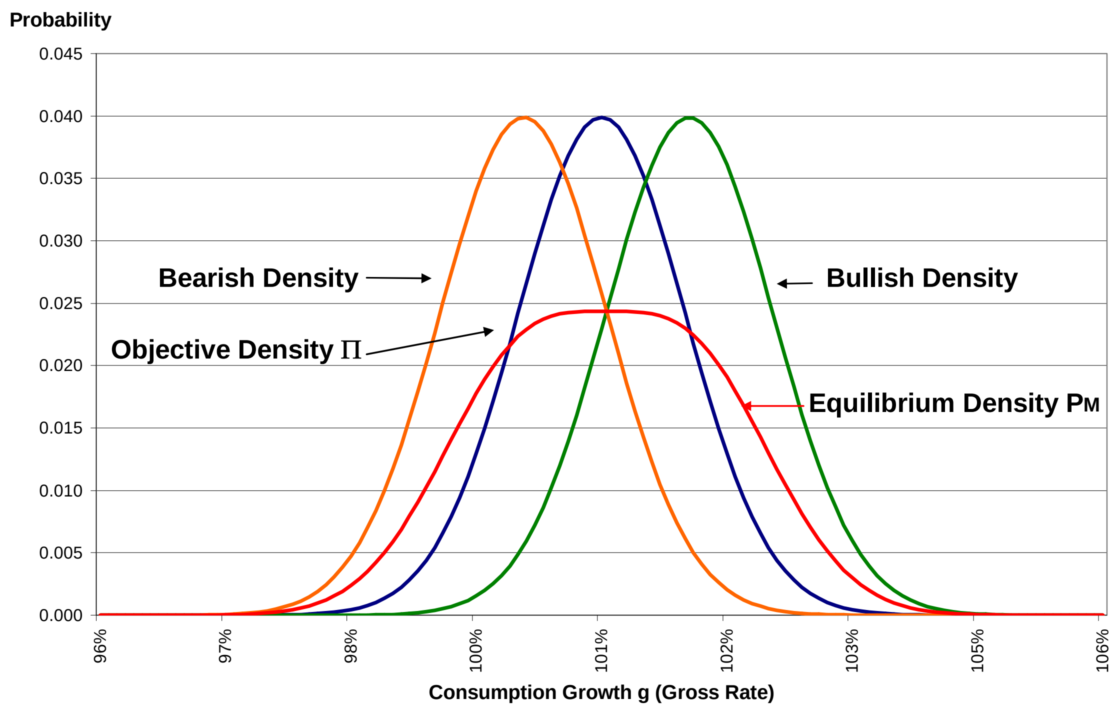

Figure 1. Underlying probability density functions. The bullish probability density function (pdf) and bearish pdf describe investor beliefs in a two-investor complete market example where both investors have erroneous beliefs about the growth rate of gross consumption, the source of fundamental uncertainty in the model. The true pdf is the objective density Π. In equilibrium, prices are set as if there were a representative investor whose beliefs are given by the pdf P_M. When all investors have power utility, P_M is a Hölder average. In the example, all investors have log-utility functions, in which case the Hölder average is a wealth-weighted average of the bullish pdf and bearish pdf.

[8] Moments refer to the moments of the underlying normal distribution for the log-consumption rate of growth.

By assumption, the beliefs of both the bullish investor and bearish investor are wrong. That is, neither the bullish pdf nor the bearish pdf coincides with the objective pdf. In addition, the market pdf, used in equation (1) to price assets, will also exhibit nonzero sentiment. This is because the equilibrium pdf differs from the objective pdf.

One way to formally define a sentiment function Λ is through the function $\ln(P_M/\Pi)$, the log of the ratio of the two pdf values at each point in the domain. $\ln(P_M/\Pi)$ measures the percentage by which, in equilibrium, the market pdf exceeds the objective pdf.[9]

Figure 2 provides a graphical illustration of the sentiment function in the example. The inset in the middle of Figure 2 is a reproduction of Figure 1. The U-shaped function is the sentiment function. In the region of the domain where the market pdf lies above the objective pdf, market sentiment is positive. In the region of the domain where the market pdf lies below the objective pdf, market sentiment is negative. Sentiment must have both positive and negative regions, except for the case when it is the zero function.

The downward sloping portion at the left of the sentiment function in Figure 2 reflects the pessimism of the bearish investor. The upward sloping portion at the right reflects the optimism of the bullish investor. In this example, market sentiment is neither uniformly optimistic nor uniformly pessimistic.

There are other possible shapes for market sentiment. If the sentiment function were monotone increasing, then market sentiment would be uniformly optimistic. If the sentiment function were monotone decreasing, then market sentiment would be uniformly pessimistic. Figure 3 displays the sentiment function in a model with several investors who disagree not only about first moments but second moments as well. That is, the disagreement is about both expected returns and volatility.

What drives the shape of Figure 3 is that bullish investors both overestimate expected consumption growth and overconfidently underestimate volatility, while bearish investors both underestimate expected consumption growth and underconfidently overestimate volatility. Relative to the sentiment function in Figure 2, the bearish investor's overestimate of volatility accentuates the steepness at the left side of the graph, while the bullish investor's underestimate of volatility pulls the function down at the right side, actually rendering its values and slope negative.

Nonzero investor sentiment is necessary, but not sufficient for market sentiment to be nonzero. When all investors have correct beliefs, their shared pdf is the same as the objective pdf. In this case, the market pdf is also the same as the objective pdf, so that the sentiment function is zero. When sentiment is zero, the market will price assets using the objectively correct pdf, and prices will be efficient. That is, prices will accurately reflect the true underlying source of uncertainty in the market. As a general matter, investors

[9] P_M/Π is a Radon-Nikodym derivative.

need not have zero sentiment in order that market sentiment be zero. Investors' individual sentiments can be self-cancelling.

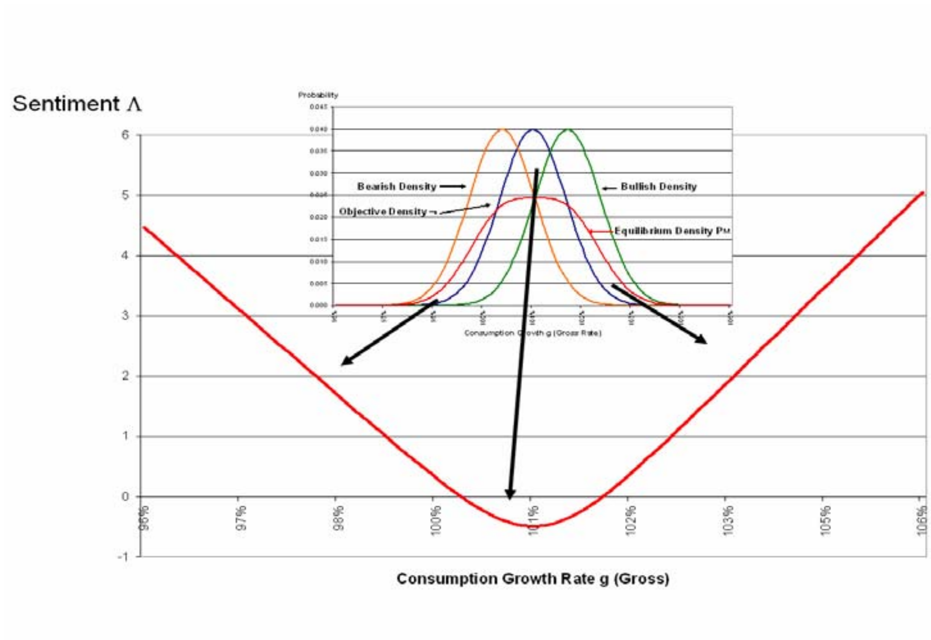

Figure 2. Illustration of sentiment function Λ. For the example described in section 1, $\Lambda = \ln(P_M/\Pi)$. The inset in Figure 2 is Figure 1. The arrows in Figure 2 indicate the three regions of the sentiment function. In the middle region, sentiment is negative because in the inset, the red density P_M lies below the blue density Π. In the left and right regions, sentiment is positive because in the inset, the red density P_M lies above the blue density Π.

Effectively, market efficiency and zero market sentiment go hand in hand. When market sentiment is zero, the market pdf is objectively correct, and all assets are priced efficiently. However, if market sentiment is nonzero, then there must be some asset that is mispriced. From this point forward, the term sentiment is to be understood as market sentiment, unless indicated to the contrary.

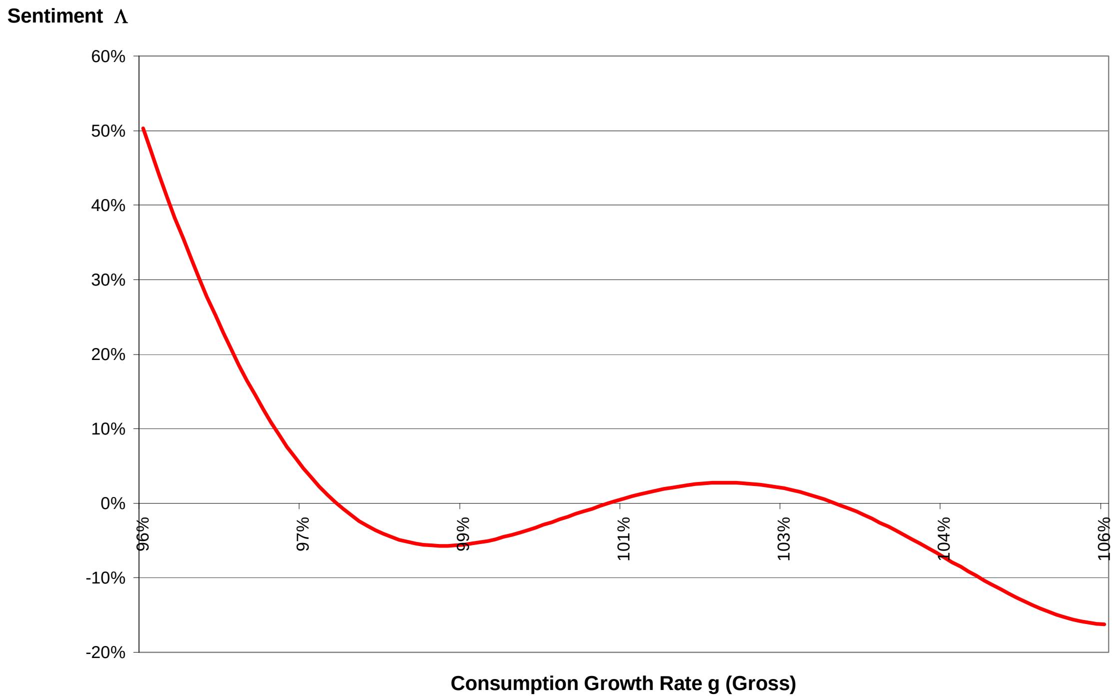

Figure 3. Sentiment Function Λ When Investors Disagree About Volatility. The sentiment function depicted in this figure is generated in an example when the market is dominated by two investors, an overconfident bullish investor and an underconfident bearish investor. The overconfident investor underestimates the volatility of the consumption growth rate while the underconfident investor overestimates volatility.

2. Connection Between Sentiment and Behavioral SDF

What makes an SDF behavioral is that it reflects market sentiment as well as fundamentals. This section describes the connection between sentiment and the SDF, both theoretically and empirically.

Shefrin (2005) established that when all investors have power utility, the log-SDF can be expressed as the sum of a term reflecting market fundamentals and sentiment Λ. The fundamental variables entering the SDF are aggregate consumption growth g, the market coefficient of relative risk aversion γ_M,[10] and the market time discount factor δ_M. Formally, the equation relating the log-SDF and sentiment has the form:

[10] When all investors share the same coefficient of relative risk aversion and time discount factor, the market parameters γ_M and δ_M are equal to these respective shared values. When there is heterogeneity in respect to coefficients of relative risk aversion and time discount factors, the market values are equilibrium aggregates. Lintner (1969) showed that market risk tolerance is a convex combination of the risk tolerances

(2) $\ln(m) = \ln(\delta_M) - \gamma_M \ln(g) + \Lambda$

In the traditional neoclassical framework, market sentiment is zero, and ln(m) is just $\ln(\delta_M) - \gamma_M \ln(g)$.[11]

Figure 4 contrasts a traditional neoclassical SDF m based on fundamentals alone with a behavioral SDF that reflects the sentiment function displayed in Figure 3. The difference between the two functions is driven by sentiment Λ. In view of the decomposition equation (2) for ln(m), the difference between the two SDF functions reflects $\exp(\Lambda)$, which enters multiplicatively in Figure 4, not additively.

Consider how Λ can be estimated empirically. Using options data, Aït-Sahalia and Lo (2000) and Rosenberg and Engle (2002) estimate the empirical SDF, or more precisely its projection in respect to returns on the S&P 500.[12] Both papers find that the empirical SDF has the behavioral shape portrayed in Figure 4. The Rosenberg-Engle paper is especially interesting, in that the authors estimate the SDF in two ways. First, they restrict the SDF to have the traditional neoclassical shape displayed in Figure 4. Second, they use a free form Chebyshev polynomial procedure that involves no such restriction. This

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of the individual investors. Benninga and Mayshar (2000) develop a model with constant relative risk aversion utility (CRRA), and show that the weights in the convex combination are relative consumption. Formally, let γ_j be the coefficient of relative risk aversion for investor j, $c_j(\omega_t)$ be investor j's rate of consumption in event ω_t at time t, and $\theta_j(\omega_t)$ be investor j's share of aggregate consumption in ω_t. Denote by $1/\gamma_M$ the market's coefficient of risk tolerance. Then

$$1/\gamma_M = \Sigma_j\, \theta_j(\omega_t)\, 1/\gamma_j$$

where $1/\gamma_M$ is implicitly a function of ω_t. The equation for δ_M is more complex, and described in Shefrin (2005). See also Jouini and Napp (forthcoming).

[11] This observation provides an easy way to sketch the derivation of equation (2), including the form of sentiment Λ. The neoclassical framework assumes a single representative investor whose preferences and beliefs set prices. When the representative investor has a power utility function, with parameters δ and γ, the log-SDF has the form $\ln(\delta) - \gamma\ln(g)$; see Cochrane (2005). The SDF m is the ratio of a state price υ to a probability density. In the neoclassical framework, the probability density is Π because the representative investor is assumed to hold correct beliefs. Therefore, υ takes the form $\upsilon_\Pi = \delta_\Pi\, \Pi\, g^{-\gamma}$, where $\gamma = \gamma_M$ is defined using consumption weights associated with neoclassical case. In the behavioral framework, the representative investor holds beliefs P_M which manifests sentiment. Here, υ takes the form $\upsilon_R = \delta_R\, P_R\, g^{-\gamma}$. Consider the ratio $\ln(\upsilon_R/\upsilon_\Pi)$. This ratio is the percentage by which a state price differs from its level in a neoclassical equilibrium. Notice that $\ln(\upsilon_R/\upsilon_\Pi) = \ln(P_M/\Pi) + \ln(\delta_R/\delta_\Pi)$. Define this expression as sentiment Λ.Write $m = \ln(\upsilon_R/\Pi) = \ln(\upsilon_R/\upsilon_\Pi) + \ln(\upsilon_\Pi/\Pi) = \Lambda + \ln(\upsilon_\Pi/\Pi)$, which implies equation (2).

[12] The domain of the SDF being estimated is the return on the S&P 500, not the rate of aggregate consumption growth.

approach can be viewed as an inadvertent test of whether the empirical SDF is behavioral or not. Their empirical findings provide strong support that the SDF is behavioral.[13]

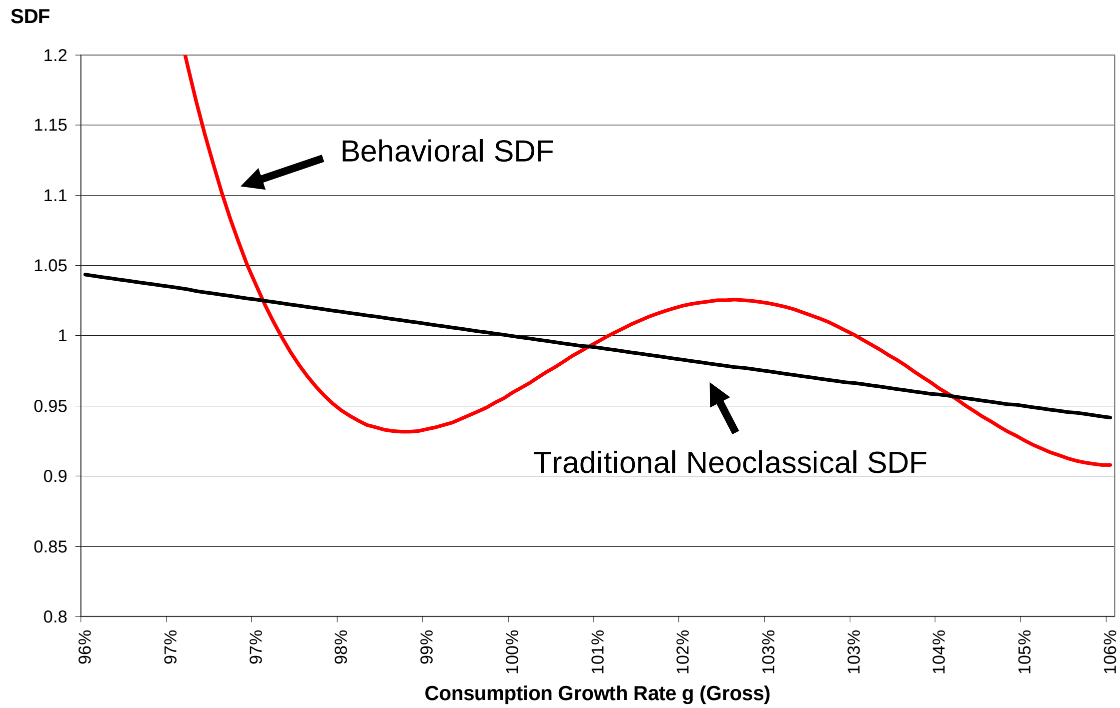

Figure 4. Comparison of Behavioral SDF and Traditional Neoclassical SDF. The behavioral SDF is a graphical portrayal of equation (2). The neoclassical SDF only reflects the fundamental component of risk, whereas the behavioral SDF reflects both the fundamental component and the sentiment component. In this respect, the sentiment function from Figure 3 is the difference between the logarithms of the behavioral SDF and the traditional neoclassical SDF.

The difference between the two Rosenberg-Engle estimated SDFs, the neoclassical restricted and the free form, provides an estimate of the sentiment function Λ. For example, Rosenberg-Engle find that the value for the SDF corresponding to a market return of -9.8% is 2.5 for the neoclassical restricted SDF estimate but 5.0 for the free form SDF estimate featuring the behavioral shape. The log difference between the two values is 80% (= log(5.0/2.5) = 1.61 – 0.81). If we interpret the neoclassical estimate of the SDF as the fundamental component in equation (2), then the value of market

[13] Shefrin (2005) discusses survey evidence that suggests clusters of optimists and clusters of pessimists in investors' expectations. Such clustering is consistent with market sentiment functions consistent with the empirical evidence for the shape of the empirical SDF.

sentiment associated with a market return of -9.8% is 80%. In rough terms, the market overestimates the probability density associated with a return of -9.8% by 80%.[14]

Call an investor informed if that investor's subjective belief is objectively correct. The "limits to arbitrage" is a concept that explains why informed investors do not exploit the biases of less informed investors to the point of driving market sentiment to zero. The general point is that the risks associated with doing so are sufficiently significant to discourage infinitely large positions. Therefore, informed trades generally mitigate uniformed investor sentiment in the short run, but do not eliminate it.

In the example from section 1, there is a formal condition that describes when sentiment will be zero. If the mean investor error is zero, so that errors are unsystematic across the investor population, and the error-wealth covariance is zero so that errors are uniformly distributed across the investor population, then market sentiment will be zero. That condition effectively describes when the market will be efficient in the short run.

The condition under which informed investors are able to drive the market to efficiency in the long run, by exploiting less informed traders, involves the interaction between an entropy measure and relative time preference. An investor's entropy measures the difference between his pdf and the objective pdf. Informed investors have zero entropy, while noise traders have positive entropy. Ceteris paribus, informed investors will exploit their entropy advantage in the long run and drive prices to their efficient levels. But that statement assumes that informed investors are sufficiently patient. Because the long run might be very long, impatient informed investors might consume at too rapid a rate to exploit their entropy advantage over time.

3. Behavioral Mean-variance Portfolios, Coskewness, and Risk Premiums

The present section describes how sentiment impacts the nature of risk premiums across securities. A mean-variance (MV) portfolio is a portfolio of assets that maximizes expected return for a specified return variance. In neoclassical asset pricing, where sentiment is zero, risk premiums for every security is based on its return covariance with the return to any risky mean-variance portfolio. For example, in the CAPM the market portfolio is mean-variance efficient, and this is why risk premiums are based on the covariance between the security's returns and the returns to the market portfolio.

How does sentiment impact the relationship between risk and return? The short answer is in the same way as in neoclassical asset pricing. Because risk premiums for all securities are based on return covariance with MV portfolios, even when sentiment is nonzero, the key lies in understanding how sentiment impacts the nature of the return distributions for both MV portfolios and individual assets. This means appreciating that risk has both a

[14] This statement needs to be tempered a bit, because sentiment is actually composed of both a direct effect and an indirect effect stemming from the sentiment of investors. In equation (11), $\Lambda = \ln(P_M/\Pi) + \ln(\delta_R/\delta_\Pi)$. The first term captures the direct effect of investor errors, while the second term captures an indirect effect associated with the Hölder average: see footnote (10).

fundamental component and a sentiment component. The intuition of many readers might be based on neoclassical finance where risk comprises only the fundamental component, and abnormal returns are associated with the portion of returns that are not compensation for bearing fundamental risk. In the behavioral SDF-based framework, sentiment impacts both expected returns and risk, and of course the relationship between them.

The SDF can be used to price all assets, including portfolios. Hence, the SDF also prices all mean-variance efficient portfolios, and therefore the SDF can be used to generate the return distribution for an MV portfolio. In fact, the return to an MV portfolio is a linear function of the SDF, with a negative coefficient. See Cochrane (2005) and Shefrin (2005). This implies that the shape of the MV return patterns are essentially inversions of the SDF functions in Figure 4.

Figure 5 contrasts the return pattern for a behavioral mean-variance portfolio to the return pattern for a neoclassical mean-variance portfolio. Each curve in Figure 5 is a plot of the linear function described in Shefrin (2005) that links the return of an MV portfolio to the consumption growth rate through the SDF. The behavioral MV curve corresponds to the case in which sentiment is given by Figure 4, whereas the neoclassical MV curve corresponds to the case when sentiment is zero. Notice that the shapes in Figure 5 correspond to the inverted shapes in Figure 4.

Figure 5 has three points worthy of note. First, the return to a traditional MV portfolio is approximately the return to a portfolio consisting of the market portfolio and the risk-free security.[15] For the purpose of illustration, the weight attached to the risk-free asset in both portfolios is high.

Second, the return to a behavioral MV portfolio is more volatile than the return to a traditional MV portfolio. This is because the peaks and valleys in the behavioral MV portfolio correspond to exposure from risky arbitrage. This is a true MV portfolio, based on the objective pdf Π. Therefore, maximizing expected return involves the exploitation of nonzero sentiment.

Notice that in Figure 3, sentiment is positive at the extreme left and negative at the extreme right. This means that extreme out-of-the-money put options on the market portfolio are overpriced, while extreme out-of-the-money call options are underpriced. As a result, a true mean-variance portfolio would feature a naked short position in extreme out-of-the-money put options on the market portfolio and a long position in extreme out-of-the-money call options on the market portfolio. Interestingly, Lo (2001) discusses the details of a strategy featuring the sale of naked puts, describing the favorable return pattern which the strategy would have generated in the past.

[15] The approximation mentioned in this sentence refers to the fact that the market portfolio is not mean-variance efficient. In this context, the market portfolio refers to a portfolio that returns the rate of consumption growth. The reason why the market portfolio is not mean-variance efficient is that investors have power utility functions, not quadratic utility functions. Nevertheless, by plotting out the return to an MV portfolio and comparing it to the return to a combination of the risk-free security and the market portfolio, it is possible to see that the return patterns are very similar. If they were identical, the neoclassical MV portfolio depicted in Figure 5 would be linear. However, it is slightly concave but nonlinear.

That a behavioral mean-variance portfolio is more volatile than its traditional counterpart is a natural consequence of a behavioral SDF being more volatile than its traditional counterpart. See Figure 4. The difference in volatility has profound implications for the magnitudes of risk premiums and Sharpe ratios. Sharpe ratios are bounded from above by the coefficient of variation of the SDF. As a result, the more volatile behavioral SDF admits higher risk premiums and Sharpe ratios than does the less volatile neoclassical SDF. As implied by the previous paragraph, achieving these higher risk premiums and Sharpe ratios involves the use of derivatives.[16]

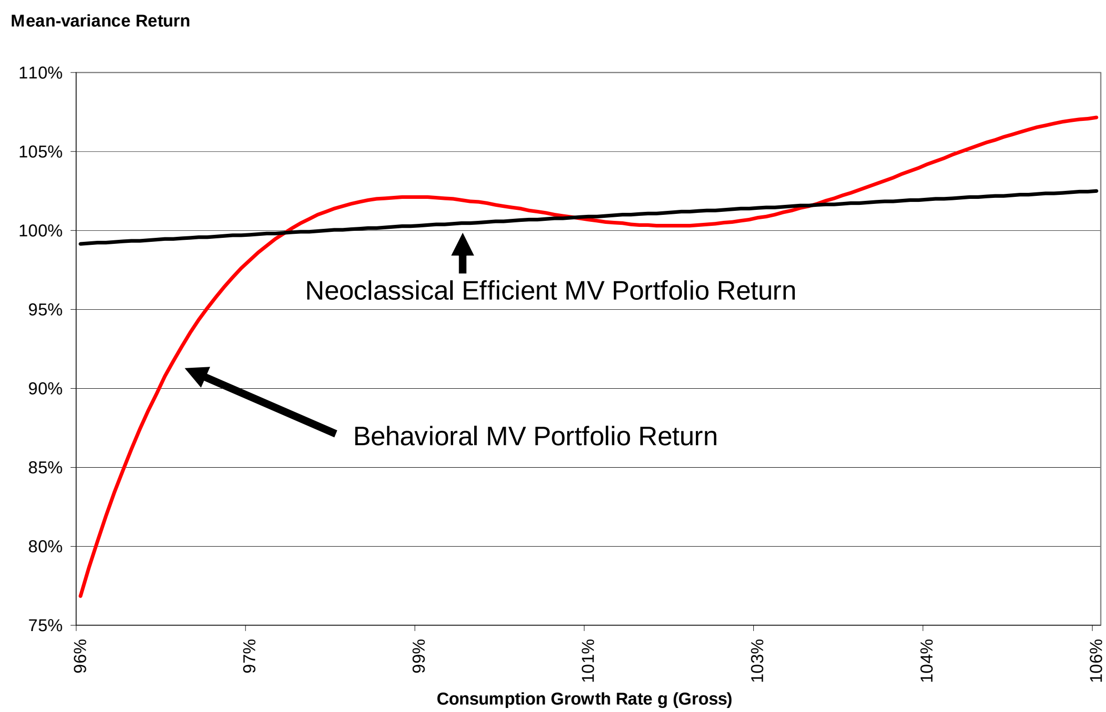

Figure 5. Comparison of Behavioral and Neoclassical Mean-variance Returns. The return to a mean-variance portfolio is a linear function of the SDF, with a negative coefficient. Therefore, the two functions portrayed in Figure 5 are inverted images of their respective SDF functions in Figure 4.

[16] The additional volatility injected by sentiment can also be seen in Figure 3, which contrasts a neoclassical and behavioral SDF. The additional volatility is important for understanding the asset pricing inequality $|E(R^i)-R^f| \leq \sigma(R^i)\, \sigma(m)/E(m)$ which links the risk premium for any security to its return standard deviation and the coefficient of variation of the SDF. The key point here is that sentiment typically increases $\sigma(m)$ and therefore augments the magnitude of risk premiums and Sharpe ratios.

Third, MV-returns are very negatively skewed relative to the market portfolio because returns to a behavioral MV-portfolio are not only extremely low in low growth states, but fall off quickly as the rate of consumption growth declines. Some readers might be surprised that skewness is an issue at all in a discussion about mean-variance portfolios. After all, skewness pertains to the third moment, and mean-variance preferences are neutral in respect to all moments higher than the second. The reason why skewness can be an issue for a behavioral MV portfolio involves the risky arbitrage feature of true MV portfolios. An investor who takes a large naked short position in extreme out-of-the-money put options on the market portfolio will earn a very low return when the return on the market portfolio is very low. An investor who takes a long position in extreme out-of-the-money call options on the market portfolio will earn a high return when the return on the market portfolio is very high.

A classic result in asset pricing theory is that the risk premium to any asset can be expressed as the product of the asset's beta with respect to any risky MV portfolio and the risk premium of the MV portfolio. Of course, this result continues to hold when the MV frontier has a behavioral structure, because it is an equilibrium property, not a property that depends on whether sentiment is zero or nonzero.

To understand the implications that the behavioral MV shape in Figure 5 has for the nature of risk premiums for individual securities, consider what systematic risk entails. Systematic risk is risk associated with the returns to an MV portfolio. Securities whose returns are highly correlated with the returns to an MV portfolio will be associated with high degrees of systematic risk. In this regard, remember that the returns to the behavioral MV portfolio depicted in Figure 5 are negatively skewed relative to the market portfolio. Therefore, securities whose returns are high in systematic risk will feature negative skewness relative to the market portfolio.

Harvey and Siddique (2000) study the cross-section of stock returns using coskewness relative to the market portfolio. They propose several definitions of coskewness. One definition of coskewness is the beta of the security's return relative to the squared market return, controlling for covariation with the market return. With this definition, coskewness measures the extent to which a security's return covaries with the squared market return.

The definition of coskewness is particularly appropriate when the SDF has a quadratic-like U-shape. Recall that the SDF will be U-shaped when the sentiment component has the U-shape depicted in Figure 2, and is large relative to the fundamental component of the SDF. In this case, the shape of the MV function is an inverted U. Observe that this type of MV return function will feature negative coskewness with respect to the market portfolio.[17] In this situation, a security that is high in systematic risk, meaning risk that is

[17] In this case, the return to the MV portfolio is a quadratic function of aggregate consumption growth g, with a positive coefficient associated with g, and a negative coefficient associated with g^2. Hence, risk premiums would be determined by a multi-factor model featuring covariation with the market portfolio (through g) and a coskewness term reflecting covariation with the squared return. Shefrin (2005) presents results showing that equation (2) implies that risk premiums and betas can be decomposed into fundamental

priced, will mimic the MV-return pattern, and therefore also feature negative coskewness. This is because coskewness measures the amount of mean-variance skewness that a security's return adds to an investor's portfolio. Coskewness plays a role analogous to a factor loading in a multifactor asset pricing model. Notably, there is a negative premium to holding stocks whose returns exhibit positive coskewness relative to the market portfolio.

Of course, the shape of the behavioral MV return pattern depicted in Figure 5 is not exactly an inverted U. In the right portion of Figure 5, the behavioral MV-return pattern does not have the same feature as an inverted-U. However, in the left portion, the patterns are similar. This suggests that when sentiment has the shape depicted in Figure 3, negative coskewness with respect to the market portfolio will capture some, but not all, aspects of systematic risk.

Harvey and Siddique (2000) study coskewness using a four factor model, where the factors are, respectively the market return, size, book-to-market equity, and momentum. Harvey and Siddique find that the correlation between coskewness and mean returns of portfolios sorted by size, book-to-market equity, and momentum is -0.71. This means that much of the explanatory power of size, book-to-market equity (B/M), and momentum in the returns of individual stocks plausibly derives from coskewness.

Harvey and Siddique point out that the market factor by itself explains only 3.5% of cross-sectional returns. However, the combination of the market factor and coskewness explains 68.1% of cross-sectional returns. This value rivals the 71.8% associated with the use of the market factor, size, and B/M. For momentum, recent winners feature lower coskewness than that of recent losers. All of these findings are consistent with MV portfolio returns having the shape depicted in Figure 5.

Dittmar (2002) discusses the superiority of nonlinear models over linear factor models such as the Fama-French three-factor model. He develops a flexible nonlinear pricing kernel approach is in the tradition of a zero sentiment representative investor model. Notably, he finds that the empirical SDF in his analysis is not monotone decreasing over its range, but instead has a U-shaped pattern. His analysis suggests a sentiment function such as the one displayed in Figure 2. In this regard, Poti (2006) uses a quadratic U-shaped SDF to extend the Harvey-Siddique analysis.

Risk premiums can be decomposed into a fundamental component and a sentiment component which can be loosely interpreted as an abnormal return.[18] Underlying the

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components and sentiment components. In a related vein, Baker and Wurgler (2007) discuss how their composite sentiment index serves to explain stock returns, both in the cross-section and over time. Their cross-sectional analysis makes use of a "sentiment beta."

[18] As discussed in Shefrin (2005), the sentiment component of the risk premium associated with an asset's return distribution r is based on the function i(1-h)/h where i is the gross risk-free rate and

$h = E(\delta \exp(\Lambda)\, g^{-\gamma}\, r) / E(\delta\, g^{-\gamma}\, r)$

determination of behavioral risk premiums is the manner in which sentiment impacts returns to individual assets, and returns to MV portfolios as described above.

The impact of sentiment on individual stock returns can be discerned from the analysis of Blackburn and Ukhov (2006). They explore the preferences for skewed returns using various utility functions that have been studied in the behavioral finance literature, but under the assumption that sentiment is zero. However, Blackburn and Ukhov's data can also be analyzed in a power utility framework that allows for nonzero sentiment.[19] Doing so provides the means to estimate the shape of the sentiment functions associated with different stocks. Most sentiment functions tend to have the behavioral shapes that correspond to the sentiment functions depicted in Figures 2 and 3.

Figure 6 illustrates the shape of sentiment function for Chevron stock, derived from Blackburn and Ukhov (2006). This graph has the same interpretation as Figure 2. Consider the positive region at the left of Figure 6. Positive values in this region mean that the market density P_M overestimated the probability that the return to Chevron stock would be very low. Negative values at the right mean that the market density P_M underestimated the probability that the return to Chevron stock would be very high. As with Figure 3, this pattern suggests a mixture of investor types trading Chevron stock, with significant clusters of both underconfident bears and overconfident bulls.

In addition to sentiment, a preference for positive skewness might also contribute to the low premium associated with positively skewed stocks. The choice models of Lopes (1987) and Lopes and Oden (1999) use probability weighting to capture this preference.[20] That is, investors who favor positively skewed return patterns, such as long shots at the race track or payoffs from lottery tickets, overweight the probability density of high returns as if they were unrealistically optimistic. Kumar (2005) characterizes what he calls lottery stocks. These are stocks that share similar properties as lottery tickets, that is, the small probability of a high reward, but a negative expected payoff. Lottery stocks feature high variance and positive skewness. Kumar finds that both individual investors

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with the expectation in h taken with respect to Π. When the market is efficient, meaning $\Lambda=0$, $\exp(\Lambda)=1$ and therefore h=1. In this case, the abnormal return component $i(1-h)/h = 0$, as it must be.

[19] Blackburn and Ukhov use the equation Jackwerth (2000) developed to estimate absolute risk aversion. That equation is $Q'/Q - P'/P$, where Q denotes the market probability density and P denotes the associated risk-neutral density. In their estimation, Blackburn and Ukhov use an estimate of Π for Q, thereby implicitly treating sentiment as zero. However, market prices are based on the beliefs P_M of the representative investor, which only equals Π when sentiment is zero. To incorporate sentiment into their analysis, write $P_M = \Pi\ (P_M/\ \Pi)$ and substitute this expression for P_M for Q in $Q'/Q - P'/P$. Effectively, this substitution the last two terms on the right-hand-side of the equation in footnote 13 as small. Finally, use power utility to write down the expression for absolute risk aversion. This enables $\Lambda \approx P_M/\ \Pi$ to be solved for in terms of γ. The solution involves an integral equation, which introduces a constant of integration. Figure 6 is based on γ=1, but the general shape is the same for plausible higher values of γ.
[20] Shefrin and Statman (2000) apply the Lopes-Oden (1999) model to portfolio selection. Notably, equation (2) continues to hold when the CRRA-assumption is modified to accommodate the features modeled by Lopes and Oden (1999) and Shefrin and Statman (2000). In this case, Λ will reflect both investor errors and behavioral deviations from expected utility maximizing behavior associated with CRRA.

and institutional investors hold more lottery stocks than chance alone would predict. He also finds that investors who favor lottery stocks underperform other investors by 5.9% a year on a risk adjusted basis.[21]

5. Market Risk Aversion or Sentiment?

Attempts to estimate the market's aversion to risk have produced puzzling results. In this section, I discuss whether the role that sentiment has played in generating the puzzle.

In an intriguing paper, Jackwerth (2000) estimated the coefficient of absolute risk aversion for the market in the case when sentiment is (implicitly) zero. The papers by Aït-Sahalia and Lo (2000) and Rosenberg and Engle (2002) also estimate market risk aversion under the assumption that sentiment is zero. All of these studies find coefficients of risk aversion for the market that are extreme. Reasonable benchmark values for the coefficient of relative risk aversion (CRRA) involve the range 1.5 to 3.5, with a few outliers between 0 and 1 and between 5 and 6.5.

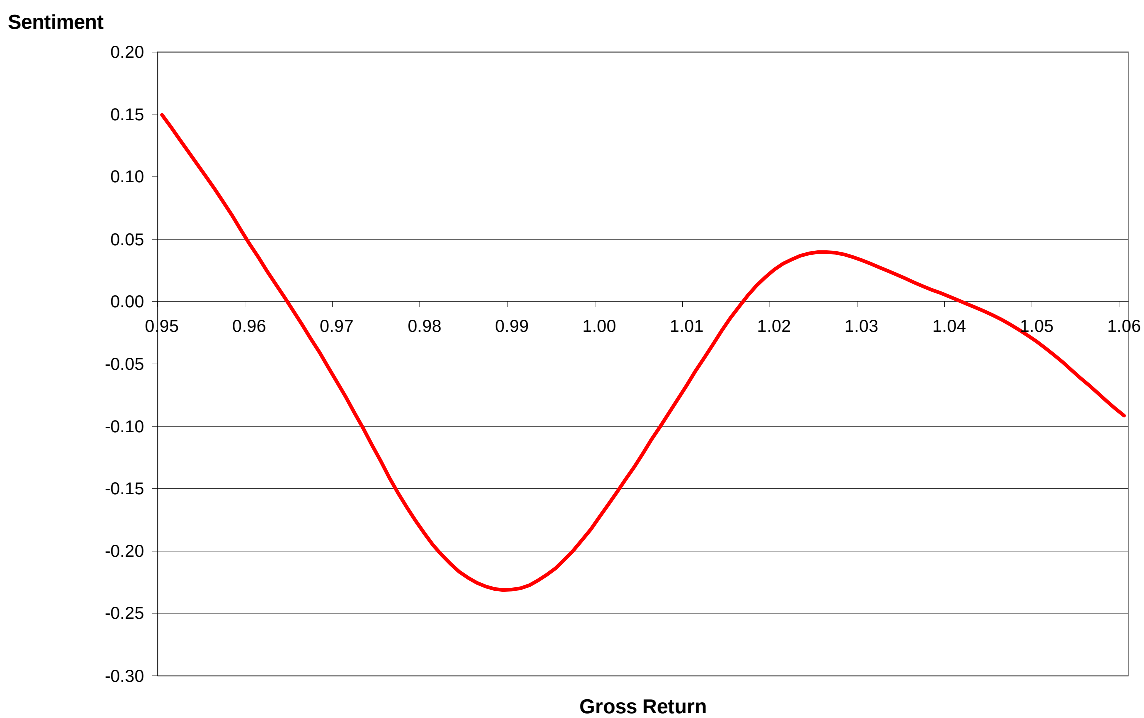

Figure 6. Shape of Sentiment Function Projection for Chevron Stock. The sentiment function for Chevron stock is derived from the analysis in Blackburn and Ukhov (2006).

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[21] Post and Levy (2005) analyze the asset pricing implications of risk seeking preferences.

The Aït-Sahalia-Lo range for CRRA is 1 to 60. The Rosenberg-Engle estimated range for the CRRA is 2 to 12, with a mean of 7.6. Moreover, they find that market risk aversion is highly variable over time. Jackwerth (2000) estimates imply that the market's coefficient of CRRA ranges between -14 and 27. Negative values, which indicate risk seeking, are especially surprising. The recent paper by Blackburn and Ukhov (2006), which extends Jackwerth's analysis from the market to individual stocks, also features highly variable rates of risk aversion that take on exotic values.

However, negative values for CRRA and values over 6 are at odds with survey data. Jackwerth points out that the shape of the absolute risk aversion function which he estimates for the market is not monotone declining and time invariant, as traditional theory would suggest. Instead, it mostly tends to be U-shaped and time varying.

The culprit in these risk aversion estimation exercises is the assumption that sentiment is zero. If sentiment is nonzero in practice, then assuming it to be zero forces the models' risk aversion parameters to pick up the effects of sentiment in the market; see Shefrin (2005). Substituting P_M for Π in Jackwerth's framework[22] suggests nonzero sentiment functions for the 1986-1995 time period he studied.[23] These are mostly upward sloping sentiment shapes, which conforms to unrealistic optimism.

6. Related Asset Pricing Models Featuring Behavioral Elements

Although most SDF-based asset pricing models implicitly assume sentiment to be zero, there are notable exceptions. Cecchetti, Lam and Mark (2000) and Abel (2002) develop models that seek to explain the equity premium puzzle in terms of sentiment in the form of time varying pessimism.[24] Shefrin (2005) discusses evidence which supports the contention that historically investors have predominantly been pessimistic, and that their pessimism has been time varying.[25]

In contrast to the sentiment functions displayed in Figures 2 and 3, which have both upward and downward sloping segments, sentiment functions that feature uniform pessimism are monotone decreasing. This is because pure pessimism overweights the probabilities attached to unfavorable consumption growth rates and underweights the probabilities attached to favorable consumption growth rates. However, the evidence

[22] See footnote 19.

[23] In addition, the second term in the right-hand-side of the equation $\Lambda = \ln(P_M/\Pi) + \ln(\delta_R/\delta_\Pi)$ in footnote 14 is treated as small.

[24] Behavioral asset pricing theorists have focused on explaining the equity premium puzzle through preferences rather than erroneous beliefs. A complete explanation may well involve both erroneous beliefs and behavioral preferences.

[25] A related issue is that investors overestimate the covariance between economic fundamentals and equity returns. The latter issue surfaces because a key aspect of the traditional equity premium puzzle is the low covariance between aggregate consumption growth and equity returns.

discussed in section 2 suggests that market sentiment is neither monotone increasing nor monotone decreasing. Rather, sentiment simultaneously exhibits both optimism and pessimism.

Dumas, Kurshev, and Uppal (2006) develop a continuous time behavioral model in which individual investors have power utility functions, but differ in their beliefs. As in Shefrin (2005), the market density P_M is determined in accordance with a Hölder average. One advantage of using a continuous time model is that it yields closed form solutions that permit an in depth treatment of the manner in which investor sentiment impacts security prices over time, especially as regards the role of entropy.

Dumas, Kurshev, and Uppal (2006) build on the work of Scheinkman and Xiong (2003), who assume that the aggregate consumption growth rate follows a mean-reverting diffusion process with a time varying stochastic drift. Investors do not observe the drift directly. Instead they receive a noisy signal whose change is equal to the sum of drift plus a noise term that is uncorrelated with the change in drift. All investors make forecasts of the change in drift. However, some investors overreact to their forecast errors for the rate of change in the signal. In this respect, they read more into their signal forecast errors than is appropriate, because they believe that the noisy signal features a positive correlation with the change in drift. As a result, they overweight the contribution of their signal forecast errors when predicting how the drift term will subsequently change. This overweighting leads them to underestimate the drift volatility, thereby leading to overconfidence on their part.

Anderson, Ghysels, and Juergens (2005) use an SDF-based model with power utility that focuses on the impact of heterogeneity on asset pricing. The empirical portion of their paper finds that the degree of dispersion can be treated as a risk factor, alongside the Fama-French factors and momentum. Anderson, Ghysels, and Juergens (2005) sidestep the behavioral aspects of their framework, meaning whether the contribution of heterogeneity to asset prices corresponds to sentiment.

Jouini and Napp (forthcoming) investigate the implications of heterogeneity on asset pricing in a continuous time framework. Their model also features the Hölder average property for P_M in the case when investors have power utility functions. As in Shefrin (2005), they emphasize that heterogeneity introduces a bias into asset pricing through the time discount factor associated with utility. However, their analysis demonstrates that the extent of this bias depends on the degree of dispersion among investors' beliefs.

Although Jouini and Napp (forthcoming) concentrate on heterogeneity rather than sentiment, Jouini and Napp (2006) is more explicit about investor errors such as pessimism and overconfidence. In this regard, Jouini and Napp (2006) extend the representative investor models of Cecchetti, Lam and Mark (2000) and Abel (2002) by demonstrating how pessimistic sentiment emerges as an aggregate of the pessimistic beliefs of individual investors. They demonstrate that relative to zero sentiment, pessimistic sentiment causes a higher market price of risk and a lower risk free rate. They also demonstrate that a positive correlation between risk tolerance and pessimism induces a higher market price of risk. In complementary work, Jouini and Napp (2007) study the correlation between risk tolerance and pessimism, while Ben Mansour, Jouini, and Napp (2006) report experimental evidence which supports pessimistic beliefs in an i.i.d. setting.[26]

Bakshi and Wu (2006) examine an important behavioral issue using a neoclassical SDF-based approach. The issue involves the extent to which the Nasdaq bubble of 1999-2000 featured nonzero sentiment, particularly irrational exuberance. Bakshi and Wu estimated the SDF by combining return distribution estimates based on historical data with risk neutral density information inferred from option prices. They then studied whether, from a neoclassical perspective, the relationship between risk and return was the same during the bubble period as it had been outside the bubble period. In their analysis, risk is decomposed into a time varying volatility component, a diffusion component, and jump components.

The key findings from the Bakshi-Wu study pertain to the fact that the risk-return relationship was different during the bubble period than it had been outside the bubble period. Outside the bubble period, all risk premium components were positive. During the bubble period, the risk premium components for both time varying volatility and diffusion became negative. The neoclassical interpretation of a negative risk premium is risk-seeking preferences. The risk premium component associated with jump risk suggests that out-of-the-money index put options became very expensive, especially relative to out-of-the-money index call options. Data on open interest and trading volume for index options suggests that institutional investors became increasingly concerned about a crash during the bubble period.

Bakshi and Wu do not estimate sentiment directly. Instead, they reject the neoclassical null hypothesis that the risk-return relationship was the same during the bubble period as outside the bubble period. In rejecting the null hypothesis, they suggest an alternative hypothesis, namely irrational exuberance, which strikes them as more reasonable.

Although Bakshi and Wu do not investigate how the sentiment function might have changed during the bubble period, one can certainly ask what type of change would support their findings. Figure 7 is a hypothetical illustration that is generally consistent with the story they tell. The black sentiment function is the same function depicted in Figure 3. For sake of argument, assume that it relates to the period outside the bubble period. The red sentiment function pertains to the bubble period itself. Notice that the red function lies above the black function at both right and left extremes, and lies below it in the middle section. At the right, irrational exuberance among bullish investors pushes up the probability that the market (meaning P_M) attaches to very favorable events (continuation of a strong bull market). At the left, bearish institutional investors concerned about a crash push up the probability that the market attaches to very unfavorable events (bursting of bubble/crash). The increased absolute values of the slopes

[26] The term i.i.d. stands for independent and identically distributed.

at the extremes of Figure 7 connote the strength of unrealistic optimism by bulls and unrealistic pessimism (by bears).[27]

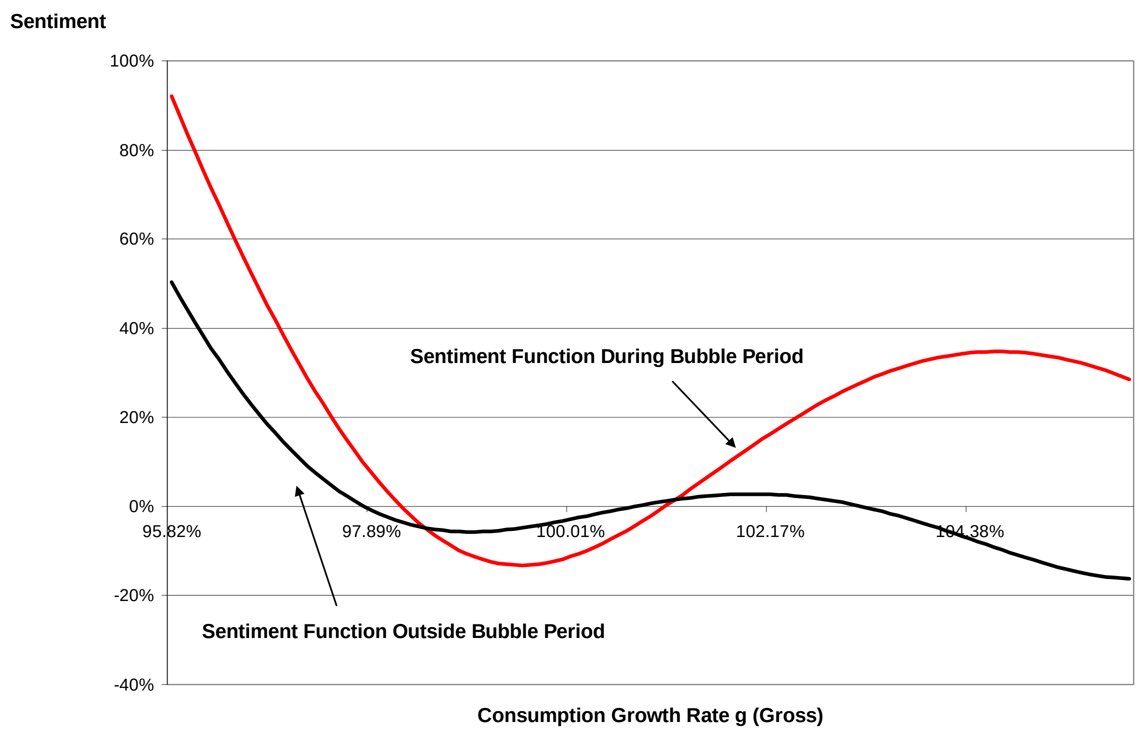

Figure 7. Illustration of the Sentiment Functions During and Outside of the Nasdaq Bubble Period. The two sentiment functions depicted in Figure 7 are hypothetical constructions whose shapes are consistent with the analysis of Bakshi and Wu (2006) about how the price of risk changed during the bubble period.

7. Summary and Directions for the Future

Modern asset pricing theory is built around the concept of a stochastic discount factor. A well-defined notion of sentiment lies at the heart of a behavioral version of this theory. The definition of sentiment is based on the percentage error in probability density, both at the level of the individual investor and the level of the market.

A proxy for market sentiment is the difference between two SDFs, one neoclassical and the other freely fit. Sentiment manifests itself in the pricing of all assets through the SDF.

[27] Given that the bubble burst, the pessimism of the bears might have been warranted. In that case, the prices of index put options might well have been efficient, in which case sentiment would actually have moved towards zero at the left extreme, not away from zero as Figure 7 depicts.

The log-SDF is the sum of market sentiment and a fundamental component that serves as the neoclassical SDF.

An indication of the strength of sentiment can be obtained by measuring market risk aversion under the assumption that sentiment is zero. The resulting coefficients of risk aversion are extreme and highly unrealistic.

Behavioral asset pricing theory features mean-variance portfolios whose returns are negatively skewed relative to the squared return of the market portfolio. The theory offers a parsimonious explanation for why assets exhibiting positive coskewness are associated with lower risk premiums than assets exhibiting negative coskewness.

Investors' pdfs and relative wealth typically vary over time. Therefore the determinants of market sentiment Λ vary over time, and with it the SDF.

In future work, the analysis of traditional topics in behavioral asset pricing, such as underreaction and overreaction, will be placed into a more general asset pricing context. Future work will develop the dynamic evolution of the market sentiment function, such as is displayed in Figure 3, the sentiment function associated with individual securities, such as is displayed in Figure 6, the SDF, such as displayed in Figure 4, and mean-variance returns, such as displayed in Figure 5. Hypotheses will be tested about how the sentiment function will change over time. An example would involve Figure 7, and whether the sentiment function changed during the Nasdaq bubble in the way suggested by that figure.

Perhaps the widest application of the behavioral SDF-based approach will be to the derivatives market, which is both very large and growing quickly. To see why, think about the sentiment function for Chevron stock in Figure 6. Overall, Chevron stock might feature a nonzero sentiment function regardless of whether or not the stock itself is correctly priced. Derivatives are the natural vehicles to exploit the various pockets of inefficiency associated with the projections of nonzero sentiment functions onto individual assets.

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