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Issues of Aggregation Over Time of Conditional Heteroscedastic Volatility Models: What Kind of Diffusion Do We Recover?

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Abstract

Continuous-time models play a central role in the theory of finance whereas empirical finance makes use of discrete-time models. This article investigates the connection between the two classes of models, particularly between conditional heteroscedastic and diffusion processes. As was advocated earlier by Stroock and Varadhan (1979), under some sets of conditions ARCH-type models weakly (in distribution) converge to diffusion processes as the time interval shrinks to zero. We provide the required set of conditions that ensures such a convergence and focus on the kind of the diffusion limit recovered. In the general setting, the diffusion is bivariate and driven by two possibly correlated Brownian motions. We illustrate this result for particular GARCH(1,1) specifications, the augmented GARCH (1,1) and a non-linear specification CEV-ARCH. By imposing an alternate set of conditions regarding the speed of convergence of parameters, a degenerate case is obtained. In the latter, the diffusion limit is governed by a single Brownian motion characterizing the price process while the volatility process becomes deterministic. Finally, we propose a discrete-time heteroscedastic model which shares various properties with ARCH-type models and converges to the complete model with stochastic volatility (CMSV) introduced by Hobson and Rogers (1998) for which the price and the volatility processes are driven by the same Brownian motion. Our analysis bears directly on the market completeness and unicity of asset prices issues.

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1 Introduction

Continuous-time models play a central role in the modern theoretical finance literature, while discrete-time models are often used in the empirical finance literature. Besides, theoretical finance has been mainly treaded in terms of stochastic differential equations (SDE), whereas in practice observations are generally made in discrete time, despite the availability of the high frequencies data.

The two classes have often been brought into conflict as two competitive views of the appropriate variance concept. Nevertheless, through the convergence of discrete time Markov sequences towards continuous time diffusion process, we may legitimate mutual complementarities and possible inter-changeability between them in various financial applications such as parameters estimation, forecasting or options pricing.

Depending on econometricians' needs, two directions of approximations are possible. On one hand, discrete time models may interestingly be considered as diffusions approximation. The advantage of this approximation lies essentially in estimating and forecasting. From a theoretical standpoint, GARCH models are able to give a consistent estimate of the volatility of a continuous time stochastic process as the sample frequency gets larger and larger, even in the presence of serious misspecifications [see Nelson and Foster (1994), and Nelson (1992)]. Thus rather than estimating and forecasting with a diffusion model observed at discrete time points, it may be much easier to use a discrete model directly. On the other hand, approximating discrete time models by diffusion processes is a way to simplify the analysis of discrete models and give insights into their workings, thanks to the tractable treatments in continuous time permitted by Itô calculus. For instance, the probabilistic and statistic properties of discrete time models such as stationary, moment finiteness, consistency and asymptotic normality of maximum likelihood functions are difficult to compute with the non-linear discrete time stochastic difference equations (ARCH). In addition, often distributional results are available for the diffusion limit of discrete processes while being intractable for the discrete models themselves.

Continuous and discrete time asset pricing represent an appealing field to exploit the convergence result we point out in this paper. In the last decade, a discrete time pricing theory has been developed as a counterpart to the continuous time one. The former included mainly exponential and affine Stochastic Discount Factor based approaches [see Gourieroux, Monfort and Polimenis (2003), Gourieroux and Monfort (2003)]. Another pricing approach is based on state variables relying on compound autoregressive processes [see Darolles, Gourieroux and Jasiak (2002)].

We try to confront the two classes of modeling and discuss the underlying connection between them. Since discrete time processes weakly converge to diffusions, we aim to shed light on the kind of diffusion recovered after the approximation is made. The later result is obtained if the discrete time step shrinks to zero and under conditions of the convergence theorem, Stroock and Varadhan (1978). Basically, we point out the existence of (i) bivariate diffusion governed by two possibly correlated sources of noise generating a stochastic volatility and therefore an incomplete market [see Nelson (1990), Duan (1997), Fornari and Mele (2004)] (ii) bivariate diffusion in which price process is stochastic whereas volatility is deterministic inducing a complete market [see Corradi (2000)], and finally (iii) bivariate diffusion, in which both price and volatility processes are stochastic but driven by the same Brownian motion, leading to a complete model with stochastic volatility [see Jeantheau (2004) and Hobson and Rogers (1998)].

The paper is organized as follows. In the second section, we present the general results of the weak convergence of discrete time GARCH models towards bivariate diffusions. First, we provide conditions ensuring the weak convergence of a discrete time Markov chain towards a diffusion. Then, we illustrate the convergence through various GARCH specifications, namely the GARCH-in-mean (GARCH-M), the augmented GARCH and the non-linear CEV-ARCH models. Finally, we present the degenerate diffusion induced by a reparameterization of the convergence conditions. In the third section, we focus on the complete model with stochastic volatility (CMSV) which allows for market completeness while volatility remains stochastic and underlines its link with ARCH-type models. The fourth section concludes.

2 Convergence towards bivariate diffusion

2.1 Convergence conditions

We provide the general set of conditions ensuring the convergence of discrete time models towards diffusions processes initially introduced by Strook and Varadhan (1979), Ethier and Kurtz (1986) and later simplified by Nelson

(1990).
For our subsequent development, we require three kinds of processes:
(a) A sequence of discrete time processes $\{Y_{kh}^{(h)}\}$ that depends on h and the discrete time index kh, $k = 0, 1, \ldots$
h is a strictly positive integer that provides the time separating two consecutive observations. Thus, $Y_0^{(h)}, Y_1^{(h)}, \ldots, Y_{kh}^{(h)}$ is a Markov chain indexed by h.
(b) The Markov chain may be extended to a continuous time process $Y_t^{(h)}$, by assuming that it does not change between two successive observations. The continuous time processes thus recovered are formed as step functions from the discrete time process (a):
$Y_t^{(h)} = Y_{kh}^{(h)} \quad \text{if} \quad kh \leq t < (k+1)h$
$Y_t^{(h)}$ depends on both h and the continuous time index t > 0.
This notation paves the way to study the convergence of *càdlàg* processes $Y_t^{(h)}$ towards Itô processes Y_t. Possible approaches include the study of the convergence of finite-dimension laws of processes[1].
(c) A limiting continuous time diffusion process Y_t to which the discrete time Markov chain converges under conditions hereafter presented.

The following theorem ensures the weak convergence of a Markov chain to a diffusion.

Theorem 1. *(The convergence theorem)*
Let $(Y_0^{(h)}, Y_h^{(h)}, \ldots, Y_{kh}^{(h)})$ *be a h-indexed Markov chain. If there exist a continuous mapping* $a(y,t)$ *from* $R^n \times [0, \infty)$ *into the space of* $n \times n$ *nonnegative definite symmetric matrices and a mapping* $b(y,t)$ *from* $R^n \times [0,\infty)$ *into* R^n *such that for all* $r > 0$*:*

$$(i) \lim_{h \to 0} \sup_{\|y\| \leq r} \left| h^{-1} E\left[Y_h^{(h)} - Y_0^{(h)} \middle| Y_0^{(h)} = y \right] - b(y) \right| = 0 \tag{1}$$

$$(ii) \lim_{h \to 0} \sup_{\|y\| \leq r} \left| h^{-1} Var\left[Y_h^{(h)} - Y_0^{(h)} \middle| Y_0^{(h)} = y \right] - a(y) \right| = 0 \tag{2}$$

$$(iii) \lim_{h \to 0} \sup_{\|y\| \leq r} \left| h^{-1-\delta/2} E\left[\left| Y_h^{(h)} - Y_0^{(h)} \right|^{2+\delta} \middle| Y_0^{(h)} = y \right] \right| \quad \textit{is finite for all } \delta > 0 \tag{3}$$

[1] It is also possible to investigate the convergence of $Y_t^{(h)}$ processes' laws. Processes $Y_t^{(h)}$ should be considered on [0,T] and *càdlàg* function's space should be equipped by the Skorokhod topology, see Billingslay (1986).

In addition, if there exists a continuous mapping $\sigma(y,t)$ from $R^n \times [0,\infty)$ into the space of $n \times n$ nonnegative definite symmetric matrices, such that: $a(y,t) = \sigma(y,t)\sigma(y,t)'$.
Consider the SDE: $dY(t) = b(Y(t))dt + \sigma(Y(t))dW_t$ and $Y(0) = y_0$.

If it has a unique solution (in law)[2] *and if $Y_0^{(h)}$ converges in law towards y_0, then the finite-dimensional laws of processes $Y_t^{(h)} \quad (= Y_{kh}^{(h)}$ if $kh \leq t < (k+1)h)$ converge to those of $Y(t)$, solution of the previous SDE.*

2.2 Application to ARCH-type models

In this paragraph we present some examples of the convergence of GARCH specifications towards SDE, under the conditions of theorem 1. The general model considered is:

$$Y_t = Y_{t-1} + f(\sigma_{t-1}^2) + \sigma_{t-1} Z_t \tag{4}$$

where f is a continuous function[3] the conditional variance of σ_t^2.

2.2.1 GARCH-M model

We provide the explicit details for the GARCH-M model. The approach may be then straightforwardly iterated for other GARCH specifications such as EGARCH, TGARCH models.

The GARCH-M model is a variety of the standard GARCH models introduced by Engel, Lilien and Robbins (1987) for a better consideration of the notion of risk affecting financial products. It allows the conditional variance to appear in the mean equation as an explicative variable.

As far as we are concerned by diffusion limit model, we are interested by Markovian models. The couple (Y_t, σ_{t+1}^2) is Markovian if and only if ϵ_t is a GARCH(1,1).

[2]To ensure local existence and uniqueness of the solution, we need to assume the two conditions: (i) the functions b and σ are Lipchitzian (ii) $\sigma(y)$ is a strictly positive for all y.

[3]Nelson (1990) proposes and explicit parameterization of the function $f(\sigma_t^2) : f(\sigma_t^2) = c\sigma_t^2$. We immediately recover the GARCH-M (GARCH-in-mean) model as the conditional variance appears in the mean equation. In our setting, we rather keep $f(\sigma_t^2)$ as a general continuous function and do not specify a particular parameterization.

Consider the following GARCH-M (1,1) model:

$$\begin{aligned} Y_t &= Y_{t-1} + f(\sigma_{t-1}^2) + \sigma_{t-1} Z_t \\ \sigma_{t+1}^2 &= \omega + \alpha(\sigma_t Z_{t+1})^2 + \beta\sigma_t^2 \end{aligned} \tag{5}$$

An Euler approximation of this model leads to series of h-indexed models:

$$\begin{aligned} Y_{(k+1)h} &= Y_{kh} + f(\sigma_{kh}^2)h + \sqrt{h}\sigma_{kh \cdot h} Z_{k+1} \\ \sigma_{(k+1)h}^2 &= \omega_h + \sigma_{kh}^2(\alpha_{h \cdot h} Z_{(k+1)}^2 + \beta_h) \end{aligned} \tag{6}$$

$Z_k^{(h)}$ are standard variables $\sim D(0,1)$.

Considering conditions: $Y_t^{(h)} = Y_{kh}^{(h)}$ and ${}_h\sigma_t^2 =_h \sigma_{kh}^2$ for $kh \leq t < (k+1)h$, we create the continuous time process and recover the couple (Y_t, σ_t^2).

We allowed coefficients ω, α and β to depend on h because our ultimate objective is to find the proper sequence $\{\omega_h, \alpha_h, \beta_h\}$ for which conditions of theorem 1 are valid that in turn makes the processes[4] $\{Y_t^{(h)},\ {}_h\sigma_t^2\}$ converge in distribution towards an Itô process as far as h vanishes to zero.

The method consists in computing the conditional moments of the approximating process [5] given by equations of the system (6).

Let's start by the conditional mean:

$$\begin{aligned} h^{-1}E\left[Y_h - Y_0 \mid Y_0 = y, \sigma_0^2 = x\right] &= f(x) \\ h^{-1}E\left[\sigma_h^2 - \sigma_0^2 \mid Y_0 = y, \sigma_0^2 = x\right] &= h^{-1}\omega_h + h^{-1}x(\beta_h + \alpha_h - 1) \end{aligned} \tag{7}$$

This sequence converges if:

$$\lim_{h \to 0} h^{-1}\omega_h = \omega \tag{8}$$

$$\lim_{h \to 0} h^{-1}(\beta_h + \alpha_h - 1) = -\theta \tag{9}$$

The second order moments are given by:

$$\begin{aligned} h^{-1}Var\left[Y_h - Y_0 \mid Y_0 = y, \sigma_0^2 = x\right] &= x \\ h^{-1}Var\left[\sigma_h^2 - \sigma_0^2 \mid Y_0 = y, \sigma_0^2 = x\right] &= h^{-1}x^2\alpha_h^2 Var_h Z_1^2 \end{aligned}$$

This sequence converges if [6]:

$$\lim_{h \to 0} h^{-1}\alpha_h^2 = \alpha \tag{10}$$

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[4] For notation reasons, we prefer to write $\sigma_t^{(h),2}$ as ${}_h\sigma_t^2$. It stands for the continuous time process, depending on h, of the conditional variance.

[5] For ease of computations, we consider moments between time 0 and h, which explains why the conditioning is taken over Y_0 and σ_0^2

[6] We should keep in mind this crucial condition for obtaining of bivariate diffusion limit. When relaxing this condition, Corradi (2000) proves that a degenerate diffusion limit driven by only one noise source (one Brownian motion) is possible. This will be discussed in the subsequent paragraph.

Finally, the cross term

$$h^{-1}Cov\left[(Y_h - Y_0), (\sigma_h^2 - \sigma_0^2) \mid Y_0 = y, \sigma_0^2 = x\right] \quad = \quad h^{-1}\alpha_h x^{3/2} E\left({}_hZ_1^3\right)$$

converges under condition (10).

Under conditions (8) et (9), we identify the mappings:

$$b(x,y) = \begin{pmatrix} f(x) \\ \omega - \theta x \end{pmatrix}$$

$$a(x,y) = \sigma(x,y)\sigma'(x,y) = \begin{pmatrix} x & \sqrt{\alpha}x^{3/2}S \\ \sqrt{\alpha}x^{3/2}S & x^2\alpha(K-1) \end{pmatrix}$$

S and K are respectively third and fourth order moment of ${}_hZ_1$.
$a(y,x)$ is a non negative matrix. It is possible to find its root.

$$\sigma(x,y) = \begin{pmatrix} \sqrt{x} & 0 \\ \sqrt{\alpha}xS & x\sqrt{\alpha}\sqrt{K-1-S^2} \end{pmatrix}$$

Proposition 1.

As far as h vanishes to zero, and under conditions (8), (9) and (10), the Markov chain sequence defined in (6) converges towards the following bivariate diffusion process driven by two independent Brownian motions:

$$\begin{aligned} dY_t &= f(\sigma_t^2)dt + \sigma_t dW_{1,t} \\ d\sigma_t^2 &= (\omega - \theta\sigma_t^2)dt + \sigma_t^2\sqrt{\alpha}\, S dW_{1,t} + \sigma_t^2\sqrt{\alpha}\sqrt{K-1-S^2}\, dW_{2,t} \end{aligned}$$

Although only one source of noise appears in the initial GARCH model, the diffusion limit is driven by two independent sources of risk because we considered the Markovian couple (Y_t, σ_{t+1}^2) for carrying out our analysis and because of the Donsker's central limit theorem applied to random processes.

Symmetric GARCH specifications exhibit various drawbacks particularly their failure to capture the leverage effect, widely documented stylized fact of financial time series [see Black(1976), Engle and Ng(1993)]. Consequently, various asymmetric specifications have been developed to capture such an effect. The approach tailored for getting the diffusion limit of the GARCH-M model could be straightforwardly extended to asymmetric specifications particularly EGARCH and TGARCH. In both cases, we obtain a bivariate diffusion limit driven by two independent Brownian motions.

2.2.2 Genralization: the augmented GARCH model

The results of the previous paragraph could be extended to various GARCH specifications. Duan (1997) presented the *Augmented* GARCH model encompassing a family of eight particular GARCH specifications. The model is defined in terms of an auxiliary process interpreted as the Box-Cox transformation of the conditional variance. It considers not only multiplicative shocks, already modeled in previous works [see Higgins and Bera (1992), Ding and al. (1993)] but also additional ones, which enlarges the set of the parametric family of models without invoking additional assumptions for ensuring stationarity. The intention behind tailoring such a model is to study, in one time, the stationary conditions and the diffusion limit of the different specifications embedded in the general model.

Definition

We present the augmented GARCH (1,1). We refer to Duan (1997) for the extension [7] to GARCH (p,q). A time series Y_t is an Augmented GARCH(1,1) process if:

• It is a general first order stochastic volatility process, i.e:

$$Y_t = \mu_t + \sigma_t Z_t \quad \text{with} \quad Z_t|\mathcal{F}_{t-1} \sim D(0,1) \tag{11}$$

$$\Phi_t = \alpha_0 + \Phi_{t-1}\xi^1_{t-1} + \xi^2_{t-1} \tag{12}$$

$$\sigma^2_t = \begin{cases} |\lambda\Phi_t - \lambda + 1|^{1/\lambda} & \text{if} \quad \lambda \neq 0 \\ \exp(\Phi_t - 1) & \text{if} \quad \lambda = 0 \end{cases} \tag{13}$$

• The couple (ξ^1_t, ξ^1_t) satisfies:

$$\xi^1_t = \alpha_1 + \alpha_2 \left|Z_t - c\right|^\delta + \alpha_3\, max(0, c - Z_t)^\delta \tag{14}$$

$$\xi^2_t = \alpha_4 f(\left|Z_t - c\right|, \delta) + \alpha_5 f(max(0, c - Z_t), \delta) \tag{15}$$

with $f(x, \delta) = x^\delta/\delta$ and (ξ^1_t, ξ^2_t) is some $\mathcal{F}_t$-measurable stationary and ergodic sequence of random vectors admitting a continuous distribution.

A negative Lyapunov exponent is a sufficient condition ensuring a stationary model. That condition reads:

$$E\left[\log\left|\alpha_1 + \alpha_2 \left|Z_t - c\right|^\delta + \alpha_3 max(0, c - Z_t)^\delta\right|\right] < 0 \tag{16}$$

Using the Jensen's inequality the condition is simplified to two other sufficient conditions.

[7]The generalization only involves changes in the auxiliary dynamic Φ.

σ_t^2 results from a functional transformation of Φ_t. The transformation in equation (13) is general enough to embed a continuum of models as specials cases. More accurately, the augmented GARCH model includes eight particular GARCH specifications: (1) Linear GARCH of Bollerslev (1986), (2) Multiplicative GARCH of Geweke (1986), (3) EGARCH of Nelson(1991), (4) GJR-GARCH, (5) Non-linear GARCH of Engel and Ng(1993), (6) VGARCH specification proposed by Engle and Ng (1993), (7) TS-GARCH specification proposed by Taylor(1986) and (8) the Threshold GARCH of Zakoian(1990).

Each of these models is obtained for a particular set of parameters constraints. For example, the Augmented GARCH model reduces to the LGARCH model if $\lambda = 0$, $c = 0$, $\delta = 2$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\alpha_5 = 0$, $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. The EGARCH model is a particular augmented GARCH if $\lambda = 0$, $c = 0$, $\delta = 1$, $\alpha_2 = 0$, $\alpha_3 = 0$.

Diffusion limit of the augmented GARCH model

We aim to find the general expression of the diffusion limit of the augmented GARCH model. This result would generalize that of Nelson (1990) confined to EGARCH and GARCH-M models.

Let Z_k, $k = 1, 2, \ldots$ be a sequence of Gaussian [8] i.i.d random variables. We define:

$$\begin{aligned}
M_k^{(2)} &= |Z_k - c|^\delta & q_2 &= E(M_k^{(2)}) \\
M_k^{(3)} &= max(0, c - Z_k)^\delta & q_3 &= E(M_k^{(3)}) \\
M_k^{(4)} &= \alpha_4 f(|Z_k - c|, \delta) + \alpha_5 f(max(0, c - Z_k, \delta) & q_4 &= E(M_k^{(4)})
\end{aligned}$$

The approximating augmented GARCH(1,1) model is given by:

$$\begin{aligned}
Y_{kh}^{(h)} &= [\omega_0(\sigma_{kh}^2) + \omega_1(\sigma_{kh}^2)Y_{(k-1)h}]h + Y_{(k-1)h} + \sqrt{h}\sigma_{kh}Z_k \\
\Phi_{(k+1)h} &= (\alpha_0 + q_4)h + \Phi_{kh}[1 + (\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - 1)h] \\
&\quad + \Phi_{kh}[\alpha_2(M_k^{(2)} - q_2) + \alpha_3(M_k^{(3)} - q_3)]\sqrt{h} + (M_k^{(4)} - q_4)\sqrt{h} \\
\sigma_{kh}^2 &= \begin{cases} |\lambda\Phi_{kh} - \lambda + 1|^{1/\lambda} & if\ \lambda > 0 \\ \exp(\Phi_{kh} - 1) & if\ \lambda = 0 \end{cases} \qquad (17)
\end{aligned}$$

Define the constant variance-covariance matrix Ω:

$$\Omega = Var(Z_t, M_k^{(2)}, M_k^{(3)}, M_k^{(4)}) = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_4^2 \end{pmatrix}$$

[8] The normality assumption is required for ensuring the moments existence of $M_k^{(2)}$, $M_k^{(3)}$, $M_k^{(4)}$ and facilitating the derivation of the specific results for the embedded GARCH models.

then, we have the following theorem [9]:

Theorem 2.

The approximating augmented GARCH(1,1) model converges towards the unique solution of the diffusion model:

$$\begin{aligned} dY_t &= [\omega_0(\sigma_t^2) + \omega_1(\sigma_t^2)Y_t]dt + \sigma_t dW_{1,t} \\ d\Phi_t &= [\alpha_0 + q_4 + \Phi_t(\alpha_1 + \alpha_2 q_2 + \alpha_3 q_3 - 1)]dt \\ &\quad + \vartheta_t \rho_t dW_{1,t} + \vartheta_t \sqrt{1-\rho_t^2}\, dW_{2,t} \\ \sigma_t^2 &= \begin{cases} |\lambda \Phi_t - \lambda + 1|^{1/\lambda} & if\ \lambda > 0 \\ \exp(\Phi_t - 1) & if\ \lambda = 0 \end{cases} \end{aligned}$$

with $\vartheta = [\sigma_4^2 + 2(\alpha_2\sigma_{24} + \alpha_3\sigma_{34})\Phi_t + (\alpha_2^2\sigma_2^2 + \alpha_3^2\sigma_3^2 + 2\alpha_2\alpha_3\sigma_{23})\Phi_t^2]^{1/2}$
and $\rho_t = \vartheta_t^{-1}[\sigma_{14} + (\alpha_2\sigma_{24} + \alpha_3\sigma_{31})\Phi_t]$
Where W_1 and W_2 are two independent Brownian motions.

The expression of the drift of the diffusion limit, $\omega_0(\sigma_t^2) + \omega_1(\sigma_t^2)Y_t$, is large enough to include various specifications. For instance, in the particular case where $\omega_1(\sigma_t^2) = 0$ and $\omega_0(\sigma_t^2) = c_0 + c_1\sigma_t^2$, we recover a time varying risk premium model. However if $\omega_0(\sigma_t^2) = 0$ and $\omega_1(\sigma_t^2) = -c_1$, we recover a mean reverting model, widely used in interest rate modeling.

If we apply the parameters constraints used for identifying the various GARCH specifications embedded in the augmented GARCH to the general expression of the diffusion limit of the augmented GARCH model, we recover familiar bivariate diffusions such as that of Scott (1987), Hull and White (1987), Stein and Stein (1991), Heston (1993). For example, applying parameter restrictions relevant to the LGARCHJ model, the diffusion limit becomes:

$$d\sigma_t^2 = [\alpha_0 + (\alpha_1 + \alpha_2 - 1)\sigma_t^2]dt + \sqrt{2}\alpha_2\sigma_t^2 dW_{2,t}$$

In the same vein, the diffusion of the EGARCH model reduces to that of Wiggins (1987) and reads:

$$\begin{aligned} d\log\sigma_t^2 &= [\alpha_0 + \sqrt{\frac{2}{\pi}}(\alpha_4 + \frac{1}{2}\alpha_5) - \alpha_4 - \alpha_5 + \alpha_1 - 1 + (\alpha_1 - 1)\log\sigma_t^2]dt \\ &\quad -\frac{1}{2}\alpha_5 dW_{1,t} + \left|\alpha_4 + \frac{1}{2}\alpha_5\right| \sqrt{\frac{\pi-2}{\pi}} dW_{2,t} \end{aligned}$$

[9] We refer to Duan (1997) for proof.

The same reasoning could be iterated for the other remaining specifications embedded in the augmented GARCH model which therefore leads to a familiar bivariate diffusion.

2.2.3 CEV-ARCH model

We extend the analysis of the diffusion limit of ARCH-type models to a new model, CEV-ARCH model, introduced by Fornari and Mele (2004). The CEV-ARCH model approximates any diffusion model in which volatility follows any constant elasticity of variance process. The convergence is ensured under some moment conditions linking the discrete and continuous time parameters.

The CEV-ARCH model allows volatility to react nonlinearly to past shocks. The use of this model is motivated by the general empirical success of the CEV model for assets pricing in finance. We attempt to investigate possibilities of exploiting their performance in both discrete and continuous time. In addition the CEV-ARCH model considered neither constraint the elasticity of volatility to one, nor the volatility equation to be a variance or a standard deviation. This is illustrated by the continuous time CEV model applied to short term interest rate, r :

$$\begin{aligned} dr(t) &= (\lambda - \theta r(t))dt + \sigma(t)\sqrt{r(t)}\, dW_{1,t} \\ d\sigma^{\delta}(t) &= (\omega - \varphi\sigma^{\delta}(t))dt + \psi\sigma^{\delta\eta}(t)\, d\left(\rho W_{1,t} - \sqrt{1-\rho^2}\, W_{2,t}\right) \end{aligned} \tag{18}$$

The parameter δ should be estimated from data[10].

One possible application consists on using the discrete time CEV-ARCH model, that approximate (18), to estimate the vector of parameters $(\lambda, \theta, \delta, \omega, \varphi, \psi, \eta, \rho)$.
The Euler discrete time approximation of (18) leads to:

$$\begin{aligned} {}_h r_{(k+1)h} - {}_h r_{kh} &= (\lambda_h - \theta_h \cdot {}_h r_{kh})h + {}_h\sigma_{kh}\sqrt{{}_h r_{kh}}\; {}_h Z_{1,(k+1)h} \\ {}_h\sigma^{\delta}_{(k+1)h} - {}_h\sigma^{\delta}_{kh} &= (\omega_h - \varphi_h \cdot {}_h\sigma^{\delta}_{kh})h + \psi_h \cdot {}_h\sigma^{\delta\eta}_{kh}\sqrt{h}\; {}_h Z_{2,(k+1)h} \end{aligned} \tag{19}$$

${}_h Z_{1,(k+1)h} \sim D(0,h)$, ${}_h Z_{2,(k+1)h} \sim D(0,h)$ and $Cov({}_h Z^1_{(k+1)h}, {}_h Z^2_{(k+1)h}) = \sqrt{h}\rho$.

[10] Engle and Lee (1996) found $\delta = 2$ when fitting a restricted version of the volatility equation (18) to stock returns, while Fornari and Mele (2004) advocate empirically $\delta = 1$ for short term interest rates.

The convergence result, we underline here, allows an instantaneous correlation between $\{_h r_{kh}\}$ and $\{_h \sigma^\delta_{kh}\}$. As the discretization step h vanishes to zero, model (19) converges weakly or in distribution towards (18). This means that the finite dimensional distributions of $\{_h r_{kh},_h \sigma^\delta_{kh}\}$ converge to those of $\{r(t), \sigma^\delta(t)\}$, $t \geq 0$, as $h \downarrow 0$.
The GARCH(1,1) model is a particular CEV-ARCH model when $\delta = 2, \eta = 1$ and $\rho = 0$. The ARCH models that have been studied in previous sections do not converge in distribution to any unrestricted CEV process. Since they make the variance of volatility proportional to the square of volatility, the elasticity of volatility is restricted to one. To circumvent this drawback, we consider ARCH schemes that does not force the elasticity of variance to unity. By chopping time such as $kh \leq t < (k+1)h$, let's consider the approximating process:

$$\begin{aligned} {}_h r_{(k+1)h} -_h r_{kh} &= \lambda_h - \theta_h \cdot_h r_{kh} +_h \sigma_{(k+1)h} \sqrt{_h r_{kh}}\, _h Z_{kh} \\ {}_h \sigma^\delta_{(k+1)h} -_h \sigma^\delta_{kh} &= \omega_h - [1 - \beta_h - \alpha_h |_h Z_{kh}|^\delta h^{-\delta/2} (1 - \gamma sign(_h Z_{kh})^\delta)] \sigma^\delta_{kh} \end{aligned} \quad (20)$$

It could be proved that the above system (20)converges towards (18):

Proposition 2.

Under the moment conditions:
$\lim_{h\to 0} h^{-1}\omega_h = \omega$; $\lim_{h\to 0} h^{-1}\lambda_h = \lambda$; $\lim_{h\to 0} h^{-1}\theta_h = \theta$;
$\lim_{h\to 0} h^{-1}\varphi_h = \varphi$ $\varphi < \infty$; $\lim_{h\to 0} h^{-1}\psi_h = \psi$ $\psi < \infty$;
$\left\{_h r_{(k-1)h}\ ,\ _h \sigma^\delta_{kh}\right\}$ weakly converges to $\{r_t, \sigma^\delta_t\}$, $t \geq 0$ as $h \downarrow 0$.
Where $\{r_t, \sigma^\delta_t\}$ are solutions of (18) for $\eta = 1$ and:

$$\varphi_h = 1 - n_{\delta,\vartheta}((1-\gamma)^\delta + (1+\gamma)^\delta)\alpha_h - \beta_h$$

$$\psi_h = \alpha_h \sqrt{(m_{\delta,\vartheta} - n^2_{\delta,\vartheta})((1-\gamma)^{2\delta} + (1+\gamma)^{2\delta}) - 2n^2_{\delta,\vartheta}((1-\gamma)^\delta + (1+\gamma)^\delta)}$$

$$\rho = \frac{2^{\frac{\delta-\vartheta+1}{\vartheta}} \nabla^{\delta+1}_\vartheta \Gamma(\frac{\delta+2}{\vartheta})((1-\gamma)^\delta + (1+\gamma)^\delta)}{\Gamma(\vartheta^{-1})\sqrt{(m_{\delta,\vartheta} - n^2_{\delta,\vartheta})((1-\gamma)^{2\delta} + (1+\gamma)^{2\delta}) - 2n^2_{\delta,\vartheta}((1-\gamma)^\delta + (1+\gamma)^\delta)}}$$

and $m_{\delta,\vartheta} = \frac{2^{\frac{2\delta}{\vartheta}-1}\nabla^{2\delta}_\vartheta \Gamma(\frac{2\delta+1}{\vartheta})}{\Gamma(\vartheta^{-1})}$; $n_{\delta,\vartheta} = \frac{2^{\frac{\delta}{\vartheta}-1}\nabla^{\delta}_\vartheta \Gamma(\frac{\delta+1}{\vartheta})}{\Gamma(\vartheta^{-1})}$; $\nabla^2_\vartheta = \frac{\Gamma(\vartheta^{-1})}{2^{\frac{2}{\vartheta}}\Gamma(3\vartheta^{-1})}$
$Z_{kh}/\sqrt{h}$ are considered to be Generalized Error Distributed. The generalization to any η is possible, see Theorem 3.2 in Fornari and Mele (2004).

Other possible extension consists in proposing other ways for modeling asymmetries in volatility. We propose a particular asymmetric ARCH-type model allowing correlation between Brownian motions of its diffusion limit. The discrete time scheme is:

$$
\begin{aligned}
{}_h\sigma^{\delta}_{(k+1)h} - {}_h\sigma^{\delta}_{kh} &= \omega_h - (1-\beta_h).{}_h\sigma^{\delta}_{kh} \\
&\quad + \alpha_h(1-\gamma\, sign\, {}_hZ_{kh})^{\delta\eta}(|{}_hZ_{kh}|^{\delta\eta} - E|{}_hZ_{kh}|^{\delta\eta})h^{\delta\eta/2}{}_h\sigma^{\delta}_{kh}
\end{aligned} \tag{21}
$$

The basic difference between (21) and (20) is the way asymmetries in volatilities are modeled. This specification illustrates a pervasive stylized fact 'volatility reversal' which stipulates that larges negatives shocks introduce more volatility than positive shocks of the same size.

Under the moment conditions:
$\lim_{h\to 0} h^{-1}\omega_h = \omega; \quad \lim_{h\to 0} h^{-1}\lambda_h = \lambda$
$\lim_{h\to 0} h^{-1/2}[(1+\gamma)^{2\delta\eta} + (1-\gamma)^{2\delta\eta}](m_{\delta\eta,\vartheta} - 2n^2_{\delta\eta,\vartheta})\alpha_h = \psi$

Scheme (21) converges in distribution to (18) as the sampling frequency gets higher.
As in the previous paragraphs, the proofs are carried out through the computation of conditional moments, given the filtration $\mathcal{F}_{kh}$ generated by: $\{{}_hr_{(i-1)h}\,,\ {}_h\sigma^{\delta}_{ih}\}_{i=1}^{k}$.

2.3 Degenerate case: diffusion with deterministic volatility

In the previous paragraphs, we have shown that, as the discrete time interval decreases to zero, the continuous time limit of the GARCH model is a bi-dimensional diffusion process with two independent, or even possibly correlated, noise source following the pioneering works of Nelson(1990), Duan (1997) and Fornari and Mele (1996). Corradi (2000) outlines the existence of a degenerate case where, under some particular set of convergence conditions, the volatility equation becomes deterministic. We present the degenerate diffusion limit and the new parameterization of the sampling interval ensuring it, and compare it to the general non degenerate bivariate diffusion.

The approximating discrete time GARCH(1,1) process is given by:

$$Y_{(k+1)h} - Y_{kh} = \sigma_{kh}Z_{(k+1)h} \tag{22}$$

$$\sigma^2_{(k+1)h} - \sigma^2_{kh} = \omega_h + (\beta_h - 1)\sigma^2_{kh} + h^{-1}\alpha_h\sigma^2_{kh}Z^2_{(k+1)h} \tag{23}$$

Computing the conditional moments, the diffusion limit was obtained under conditions:

$$\lim_{h\to 0} h^{-1}\omega_h = \omega \tag{24}$$

$$\lim_{h\to 0} h^{-1}(\alpha_h + \beta_h - 1) = -\theta \qquad (\theta > 0) \tag{25}$$

$$\lim_{h\to 0} 2h^{-1}\alpha_h^2 = \lambda^2 \tag{26}$$

Conditions (24) and (25) are satisfied if [11]:

$$\alpha_h = h^{1/2}\sqrt{\frac{\lambda^2}{2}} + \theta_2 h + o(h) \qquad \text{and} \qquad \beta_h = 1 - h^{1/2}\sqrt{\frac{\lambda^2}{2}} + \theta_1 h + o(h)$$

$$\theta_1 + \theta_2 = \theta$$

Condition (26) guarantees a bivariate diffusion limit governed by two independent Brownian motions, *à la* Nelson (1990). We focus on this condition and propose a reparametrization of it that could switch the behavior of the diffusion limit into a bivariate diffusion driven by only one Brownian motion and characterized by a deterministic volatility[12].
The new condition assumes that α_h is at most of order h and is given by:

$$\lim_{h\to 0} h^{-\delta}\alpha_h = 0 \qquad \forall\delta < 1 \tag{27}$$

The two parameterization (26) and (27) are similar in the particular case where $\delta = 1/2$ and λ is null. The non degenerate diffusion is obtained because when $\lambda \neq 0$, α_h is of order $h^{1/2}$ which implies that the last term of the RHS of equation (23) is of order $h^{1/2}$. Consequently, the second conditional moment, scaled by h^{-1}, does not vanish as $h \to 0$. A bi-dimensional diffusion driven by two independent Brownian motions is therefore obtained.

From (27), α_h is at most of order h when $\delta = 1$ which implies that the last term of the RHS of equation (23) is also at most of order h. The conditional second order moment scaled by h^{-1} vanishes as $h \to 0$ leading to a degenerate diffusion i.e one dimensional diffusion driven by a single Brownian motion.

Results are gathered in the following two propositions:

Proposition 3. *Degenerate diffusion*

[11]Note that $\alpha_h + \beta_h = 1 + \theta h$. This is in accordance with temporal aggregation results of Drost and Nijman (1993): $\alpha_h + \beta_h = (\alpha + \beta)^h$. When h shrinks to zero the h-th power of $\alpha + \beta$ is approximated by $1 + \theta h$.

[12]The same result has been derived by Kallsen and Taqqu (1998) by invoking an other approach, assuming that volatility jumps only at integer values of time.

If $(Y_0^{(h)},_h \sigma_0^2)$ converges to (Y_0, σ_0^2) as $h \to 0$, then under the parameterization (24), (25) and (27) as $h \to 0$, $(Y_t^{(h)},_h \sigma_t^2)$ weakly converges to (Y_t, σ_t^2) where (Y_t, σ_t^2) is a diffusion process solution to :

$$dY_t = \sigma_t dW_t \tag{28}$$

$$d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt \tag{29}$$

Proposition 4. *Non-degenerate diffusion*

If $(Y_0^{(h)},_h \sigma_0^2)$ converges to (Y_0, σ_0^2) as $h \to 0$, then under the parameterization (24), (25) and (26) as $h \to 0$, $(Y_t^{(h)},_h \sigma_t^2)$ weakly converges to (Y_t, σ_t^2) where (Y_t, σ_t^2) is a diffusion process solution to :

$$dY_t = \sigma_t dW_{1,t} \tag{30}$$

$$d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt + \lambda\sigma_t^2 dW_{2,t} \tag{31}$$

Diffusion discretization: GARCH or Stochastic Volatility model ? We investigate the exact discretization of both types of diffusions. In the degenerate case, using $dt = (dW_t)^2$ and $\theta = \theta_1 + \theta_2$, the exact Euler discretization of system (28)-(29) gives:

$$\begin{aligned} Y_{(k+1)h} - Y_{kh} &= \sigma_{kh} Z_{(k+1)h} \\ \sigma^2_{(k+1)h} - \sigma^2_{kh} &= h\omega - \theta_1 h\sigma^2_{kh} - \theta_2\sigma^2_{kh} Z^2_{(k+1)h} \end{aligned}$$

by assuming: $\omega_h = h\omega$, $\beta_h - 1 = -\theta_1 h$ and $h^{-1}\alpha_h = -\theta_2$, we recover exactly equations (22) and (23), the approximating process of the GARCH process.

However, the Euler discretization of the non degenerate case leads to:

$$\begin{aligned} Y_{(k+1)h} - Y_{kh} &= \sigma_{kh} Z_{1,(k+1)h} \\ \sigma^2_{(k+1)h} - \sigma^2_{kh} &= \omega_h - \theta h\sigma^2_{kh} + \lambda\sigma^2_{kh} Z^2_{2,(k+1)h} \end{aligned}$$

which is the approximating process of a SV model rather then GARCH one, as the system is governed by two sources of noise, $Z_{1,(k+1)h}$ and $Z_{2,(k+1)h}$.

Thus the exact discretization of the non degenerate diffusion leads to a SV process, however the exact discretization of the degenerate diffusion leads to to a GARCH(1,1) process.

3 Convergence and market completeness

The study of the kind of the diffusion limit of the GARCH model raises straightforwardly the question of market completeness. In the general case of a bivariate diffusion driven by two possibly correlated sources of noise and characterized by stochastic volatility, the market is incomplete. Therefore, there is no longer a unique martingale measure and unique price of contingent claims [see Harrison and Kreps (1979), Harisson and Pliska (1981) and Poncet (1999)]. However, in the degenerate case of bivariate diffusion driven by a single Brownial motion and characterized by a deterministic variance, the market is complete. Computation of options' prices are possible without imposing other assumptions regarding utility function and investors' preferences.

In this paragraph, following Hobson and Rogers (1998) and Jeantheau (2004), we introduce the Complete Model with Stochastic Volatility, CMSV, a model that preserves the market completeness and the stochastic feature of the volatility in the same time. The intuition is to make all SDE depend on the same source of noise or Brownian motion. Through offset functions, we derive the instantaneous volatility in terms of exponentially-weighted moments of historic log-prices. The non constant instantaneous volatility is driven by the same stochastic factor as the price process. Therefore, we conciliate the stochastic feature of the volatility and the market completeness. The CMSV model developped shares many properties with the ARCH-type models and consequently with the Zumbach's (2004) family of models.

3.1 The continuous Time CMSV

Let P_t be the stock price and r the risk free interest rate. The discounted log-price process is given by $Y_t = \log(P_t e^{-rt})$

For λ a positive real, the CMSV relies on the following offset function:

$$S_t^{(m)} = \lambda \int_0^\infty e^{-\lambda u}(Y_t - Y_{t-u})^m du \tag{32}$$

Y_t is assumed to solve the SDE:

$$dY_t = \mu(S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)})dt + \sigma(S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)})dW_t \tag{33}$$

with W_t a standard Brownian motion, and $\mu(.)$ and $\sigma(.)$ Lipschitz functions.

By Itô calculus, the offset function is driven by the SDE:

$$dS_t^{(m)} = -\lambda S_t^{(m)} + mS_t^{(m-1)}dY_t + \frac{1}{2}m(m-1)S_t^{(m-2)}d < Y >_t \tag{34}$$

Hobson and Rogers (1998) confined their analysis to d = 1. In this paper we follow Jeantheau (2004) and extended the analysis to d = 2.

In the SDE (33), the drift $\mu(S_t^{(1)}, S_t^{(2)})$ and the diffusion $\sigma(S_t^{(1)}, S_t^{(2)})$ coefficients depend on the past price changes through the offset functions. The model is therefore considered as a stochastic volatility one. Yet, by contrast to the familiar stochastic volatility models (Hull and White (1987), Heston (1993), Stein and Stein (1991), among others) only one Brownian motion defines the model and no additional random source is needed to capture the stochastic feature of the volatility. The market is complete and a unique preference-free price of contingent claims may be obtained.

3.2 Discrete time model with offset functions

The appealing feature of the offset functions stems on the possibility of expressing them in discrete time. We try to construct the discrete time process Y_k based on that offset functions, which writes:

Let define the following offset function:

$$S_k^{(m)} = (1-\beta)\sum_{i=1}^{\infty}\beta^{i-1}(Y_k - Y_{k-i})^m \tag{35}$$

The discrete time analogue of (33) is:

$$\begin{aligned} Y_{k+1} - Y_k &= \mu(S_k^{(1)}, \ldots, S_k^{(d)}) + \sigma(S_k^{(1)}, \ldots, S_k^{(d)})Z_{k+1} \\ Y_0 &= y_0 \end{aligned} \tag{36}$$

where Z_k is defined as in GARCH setting in the previous paragraphs: $Z_k \sim i.i.d(0,1)$.

In the special case where d=2, Jeantheau (2004) suggests, as a counterpart of the SDE (34), the following relation between ($S_k^{(1)}$ and $S_k^{(2)}$):

$$\begin{aligned} S_{k+1}^{(1)} &= \beta S_k^{(1)} + (Y_{k+1} - Y_k) \\ S_{k+1}^{(2)} &= \beta S_k^{(2)} + (Y_{k+1} - Y_k)^2 + 2\beta S_k^{(1)}(Y_{k+1} - Y_k) \\ (S_0^{(1)}, S_0^{(2)}) &= (s_0^{(1)}, s_0^{(2)}) \end{aligned} \tag{37}$$

As far as we are concerned by checking possible links between the CMSV and the ARCH-type models, we have to deal with Markov processes.

$(Y_t, S_t^{(1)}, S_t^{(2)})$ is a Markov process and by Cauchy Schwartz inequality, we have the relation:

$$(S_t^{(1)})^2 \leq S_t^{(2)} \tag{38}$$

More accurately the process $(S_k^{(1)}, S_k^{(2)})$ is also a Markov chain [13].

We put forward now the conditional heteroscedastic property of the process Y_k. First and second conditional moment derived from (36) lead to:

$$\begin{aligned} E(Y_{k+1} - Y_k|\mathcal{F}_k) &= \mu(S_k^{(1)}, S_k^{(2)}) \\ Var(Y_{k+1} - Y_k|\mathcal{F}_k) &= \sigma^2(S_k^{(1)}, S_k^{(2)}) \end{aligned}$$

where $\mathcal{F}_k$ is the σ-field generated by $\{Y_m, 0 \leq m \leq k, k \in N\}$.

ARCH-type features clearly appear in this context. Through the offset function, the conditional mean and the conditional variance depend on the changes of price (current price and the past price) raised to a certain power, however ARCH model depends only on past values of perturbations.

Similarly to the GARCH case, we try to find the diffusion limit of $(Y_k, S_k^{(1)}, S_k^{(2)})$ defined upon offset functions. We assume:

$$(Y_t, S_t^{(1)}, S_t^{(2)})^{(h)} = (Y_{kh}, S_{kh}^{(1)}, S_{kh}^{(2)}) \quad \text{for} \quad kh \leq t < (k+1)h$$

The triplet $(Y_{kh}, S_{kh}^{(1)}, S_{kh}^{(2)})$ is presented by the system:

$$\begin{aligned} Y_{(k+1)h} &= Y_{kh} + h\mu(S_{kh}^{(1)}, S_{kh}^{(2)}) + \sqrt{h}\sigma(S_{kh}^{(1)}, S_{kh}^{(2)})Z_k \\ S_{(k+1)h}^{(1)} &= \beta_h S_{kh}^{(1)} + (Y_{(k+1)h} - Y_{kh}) \\ S_{(k+1)h}^{(2)} &= \beta_h S_{kh}^{(2)} + (Y_{(k+1)h} - Y_{kh})^2 + 2\beta_h S_{kh}^{(1)}(Y_{(k+1)h} - Y_{kh}) \\ (Y_0, S_0^{(1)}, S_0^{(2)}) &= (y_0, s_0^{(1)}, s_0^{(2)}) \end{aligned}$$

We compute conditional moments[14].

[13] It suffices to prove that its state space representation is given by: $\xi = \{x_1 \in R, x_2 \in R^+/x_1^2 < x_2\}$ and the sequence $U_k = S_k^{(2)} - (S_k^{(1)})^2$ remains positive, for $0 < \beta < 1$ and initial term $U_0 \geq 0$, since it could be expressed recurrently using (37) by: $U_{k+1} = \beta U_k + \beta(1-\beta)(S_k^{(1)})^2$
$S_k^{(2)}$ is therefore positive and we have the inequality (38). We conclude that the Markov chain $(S_k^{(1)}, S_k^{(2)})$ has ξ as a state space.

[14] For ease of computation we consider the case of k=0 and therefore I_0 is the conditional information set: $I_0 \equiv I_0(Y_0 = y_0, S_0^{(1)} = s_0^{(1)}, S_0^{(2)} = s_0^{(2)})$;

The conditional mean reads:

$$\begin{aligned} h^{-1}E\left[Y_h - Y_0 \mid I_0\right] &= \mu(S_0^{(1)}, S_0^{(2)}) \\ h^{-1}E\left[S_h^{(1)} - S_0^{(1)} \mid I_0\right] &= h^{-1}(\beta_h - 1)s_0^{(1)} + \mu(S_0^{(1)}, S_0^{(2)}) \\ h^{-1}E\left[S_h^{(2)} - S_0^{(2)} \mid I_0\right] &= h^{-1}(\beta_h - 1)s_0^{(1)} + h\mu^2(S_0^{(1)}, S_0^{(2)}) + \sigma^2(S_0^{(1)}, S_0^{(2)}) \\ &\quad + \ 2\beta_h s_0^{(1)}\mu(S_0^{(1)}, S_0^{(2)}) \end{aligned}$$

As for the conditional variance, we have:

$$\begin{aligned} h^{-1}Var\left[Y_h - Y_0 \mid I_0\right] &= \sigma^2(S_0^{(1)}, S_0^{(2)}) \\ h^{-1}Var\left[S_h^{(1)} - S_0^{(1)} \mid I_0\right] &= \sigma^2(S_0^{(1)}, S_0^{(2)}) \\ h^{-1}Var\left[S_h^{(2)} - S_0^{(2)} \mid I_0\right] &= h^2\mu^2(S_0^{(1)}, S_0^{(2)})\sigma^2(S_0^{(1)}, S_0^{(2)}) \\ &\quad + \ h\sigma^2(S_0^{(1)}, S_0^{(2)})Var(Z_1^2) \\ &\quad + (2\beta_h S_0^{(1)}\sigma^2(S_0^{(1)}, S_0^{(2)}))^2 \end{aligned}$$

As concerning the conditional covariance, we have:

$$\begin{aligned} h^{-1}Cov\left[(Y_h - Y_0), (S_h^{(1)} - S_0^{(1)}) \mid I_0\right] &= \sigma^2(S_0^{(1)}, S_0^{(2)}) \\ \lim_{h\to 0} h^{-1}Cov\left[(Y_h - Y_0), (S_h^{(2)} - S_0^{(2)}) \mid I_0\right] &= 2s_0^{(1)}\sigma^2(S_0^{(1)}, S_0^{(2)}) \\ \lim_{h\to 0} h^{-1}Cov\left[(S_h^{(1)} - S_0^{(1)}), (S_h^{(2)} - S_0^{(2)}) \mid I_0\right] &= 2s_0^{(1)}\sigma^2(S_0^{(1)}, S_0^{(2)}) \end{aligned}$$

One basic difference with Nelson's framework is that only one condition regarding the rate of convergence of the parameters is required for enduring convergence, as h shrinks to zero:

$$\lim_{h\to 0}(\beta_h - 1)/h = -\theta$$

We assume moreover that there exists $\delta > 0$ such that $E[Z_k^{4+2\delta}]$ is finite, in order to make $Var(Z_1^2)$ finite.

The tightness of the components of the process, i.e. the third condition of the convergence theorem, is verified:

$$\lim_{h\to 0}\sup_{|y|\le r}\left|h^{-1-\delta/2}E\left[\left|Y_h - Y_0\right|^{2+\delta} \mid I_0\right]\right| \quad \text{still to be bounded for all} \quad \delta > 0$$

$$\lim_{h\to 0}\sup_{|y|\le r}\left|h^{-1-\delta/2}E\left[\left|S_h^{(m)} - S_0^{(m)}\right|^{2+\delta} \mid I_0\right]\right|$$

still to be bounded for all $\delta > 0; m \in \{1,2\}$

Finally, noting that we can write:

$$\begin{pmatrix} 1 & 1 & 2s_0^{(1)} \\ 1 & 1 & 2s_0^{(1)} \\ 2s_0^{(1)} & 2s_0^{(1)} & (2s_0^{(1)})^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2s_0^{(1)} & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 2s_0^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the discrete time process $(Y_t, S_t^{(1)}, S_t^{(2)})^{(h)}$ converges in distribution to the CMSV:

$$\begin{aligned} dY_t &= \mu(S_t^{(1)}, S_t^{(2)})dt + \sigma(S_t^{(1)}, S_t^{(2)})dW_t \\ dS_t^{(1)} &= [\mu(S_t^{(1)}, S_t^{(2)}) - \theta S_t^{(1)}]dt + \sigma(S_t^{(1)}, S_t^{(2)})dW_t \\ dS_t^{(2)} &= [2S_t^{(1)}\mu(S_t^{(1)}, S_t^{(2)}) + \sigma^2(S_t^{(1)}, S_t^{(2)}) - \theta S_t^{(2)}]dt \\ & \quad + \, 2S_t^{(1)}\sigma(S_t^{(1)}, S_t^{(2)})dW_t \\ (Y_0, S_0^{(1)}, S_0^{(2)}) &= (y_0, s_0^{(1)}, s_0^{(2)}) \end{aligned} \tag{39}$$

3.3 Links with ARCH-type models

The CMSV exhibits various common features with ARCH-type models particularly in matter of stationary conditions, moments or estimation methods. The similarity could also be checked between the CMSV and the class of long memory (LM) volatility models introduced by Zumbach (2004), in their affine or linear versions. LM processes are equivalent to GARCH(1,1) and IGARCH(1,1) under some restrictions. They provide good descriptions of the empirical data and behave well when largest time horizon are included in the process.

Let's consider the following model, obtained under a particular choice of the function $\sigma(.)$ by making the conditional volatility depend on only offset functions of order 2:

$$\begin{aligned} Y_{k+1} - Y_k &= \sigma(S_k^{(2)})Z_{k+1} \\ \sigma(S_k^{(2)}) &= \omega + \alpha S_k^{(2)} \qquad \omega > 0, \alpha \geq 0 \end{aligned} \tag{40}$$

The third equation of the diffusion limit system (39) becomes:

$$dS_t^{(2)} = \left[\omega - (\theta - \alpha)S_t^{(2)}\right] dt + 2S_t^{(1)}\sqrt{\omega + \alpha S_t^{(2)}}dW_t$$

The mean reverting feature of this diffusion is another aspect of the connection between model (40) and the ARCH models. Under restriction defining (40), the system (37) is rewritten:

$$
\begin{aligned}
S_{k+1}^{(1)} &= \beta S_k^{(1)} + \sqrt{\omega + \alpha S_k^{(2)}} Z_{k+1} \\
S_{k+1}^{(2)} &= \beta S_k^{(2)} + (\omega + \alpha S_k^{(2)}) Z_{k+1}^2 + 2\beta S_k^{(1)} \sqrt{\omega + \alpha S_k^{(2)}} Z_{k+1} \qquad (41)
\end{aligned}
$$

The system (41) admits a unique stationary and positive recurrent solution, with $E[S_k^{(2)}] < \infty$, if and only if $\alpha + \beta < 1$, and we have:

$$
E\left[\sigma^2(S_k^{(2)})\right] = \frac{\omega(1-\beta)}{1-(\alpha+\beta)}
$$

Moreover, if $E[ln(\beta + \alpha Z_k^2)] < 0$, not only the system (41) has a unique stationary solution but also there exists $\gamma \in (0, 1]$ such that [15]:

$$
E\left[\left(S_k^{(2)}\right)^\gamma\right] < \infty
$$

The stationary conditions are similar to those of GARCH model and implicitly with those of the Zumbach's family of models. This reinforce the connection between the CMSV and the ARCH-type model. Others aspects of connection deal with estimations methods. Similarly to ARCH models, conditional likelihood method are also suitable for the CMSV.

4 Conclusion

We intented to examine the relationship between long time conflicting discrete time and continuous time models. We emphasized the convergence of conditional heteroscedastic discrete time models towards continuous time diffusion processes and therefore possible substitutability and complementarity between the two classes.

We illustrated the connection through several GARCH specifications. For all of them, as the time step shrinks to zero and under the conditions advocated by the Markov chain theorem, the diffusion limit recovered is bivariate. Both price and volatility processes are then stochastic and are governed by two (possibly correlated) sources of noise. By imposing a slightly

[15] We refer to Jeantheau (2004) for detailed proofs of the stationary results.

different set of conditions regarding the speed of convergence of parameters, we pointed out the existence of a degenerate diffusion limit driven by a single Brownian motion which characterizes the price process, while the volatility process remains deterministic. In contrast with the first general case, the model is complete and therefore a unique price is obtained for each asset without additional assumptions regarding investors' preferences.

A third possible diffusion limit is inspired from the complete model with stochastic volatility, for which price and volatility processes are governed by the same Brownian motion. We find a discrete-time heteroscedastic model admitting the CMSV as a diffusion limit. The discrete model is an ARCH-type one as it shares various ARCH properties such as stationarity conditions or statistical inference.

As a possible extension of this work, we propose to apply the convergence results to option pricing or statistical inference and filtering. Moreover, it is interesting to investigate stronger convergence notions and the statistical equivalence or nonequivalence between GARCH and their diffusion limit. Wang (2002) proves statistically the asymptotic nonequivalence of GARCH and diffusions in terms of Le Cam's deficiency distance. The two models are equivalent only under the obvious case of deterministic volatility. By contrast, under stochastic volatility, because of the difference between the structure of noise propagation in the conditional variances, the likelihood processes in the two model behave asymptotically quite differently and thus are nonequivalent asymptotically. This discredits the general belief of asymptotic equivalence and warns against the common practice of applying without care statistical inferences derived under GARCH model to their diffusion limit. Investigation of the statistical equivalence between GARCH and their continuous time diffusion is left to further research.

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