Valuation of Derivatives Based on Single-Factor Interest Rate Models

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† This research has been carried out within the NCCR FINRISK project on " Interest Rate and Volatility Risk"

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Abstract

The CKLS (1992) short-term risk-free interest rate process leads to valuation model for both default free bonds and contingent claims that can only be solved numerically for the general case. Valuation equations of this nature in the past have been solved using the Crank Nicolson scheme. In this paper, we introduce a new numerical scheme – the Box method, and compare it with the traditional Crank Nicolson scheme. We find that in specific cases of the CKLS process where analytical prices are available, the new scheme leads to more accurate results than the Crank Nicolson scheme.

Key Words: CKLS Interest rate model, Box method, Crank Nicolson scheme.

1. Introduction

A feature distinguishing interest rate models from equity models is the need for interest rate models to exhibit mean reversion and for the volatility to be dependent on the interest rate. Due to these complexities and to the fact that interest rates cannot be traded like stock options, two groups of pricing methodology have been developed for the valuation of fixed income derivative securities.

The first approach prices the option and the underlying bond as function of the spot interest rate. The second prices the option as a function of the forward rate and restricts its behaviour to ensure that the observed market prices of zero coupon bonds are respected by the model. The first group comprises of models proposed by Vasicek (1977), Cox, Ingersoll and Ross (1985), Dothan (1978) amongst many others. The second group comprises of models proposed by Ho and Lee (1986), HJM (1992) amongst others. To value derivatives based on the first group, we may use Monte-Carlo simulation, multinomial lattice or the partial differential equation approach. Generally for derivative valuation based on the second group we are restricted to the Monte Carlo simulation and the multinomial lattice approach.
Based on the spot rate approach, the partial differential equation approach leads to analytical solutions for specific models such as the Vasicek and Cox, Ingersoll and Ross. The existence of analytical solutions leads to quick valuation of the bond, option prices, and the hedge parameters such as delta for risk management purposes.

The first group, are summarized by the Chan, Karolyi, Longstaff and Sanders CKLS (1992) model. Unlike many of the specific short-term interest rate models no analytical solutions are available for the bond and option prices based on the CKLS (1992) model. A numerical approach is therefore required. In option pricing literature, the standard numerical approach to value bonds and options on bonds is the Crank Nicolson scheme (see for example Courtadon, (1982)). The basic idea behind the Crank Nicolson scheme is that the numerical solution should converge to the true solution. However, under the Crank

Nicolson scheme, convergence may be guaranteed, but the convergence to the true solution is not.

The contribution of the present paper is to introduce a new numerical method to finance from engineering and the physical sciences called the Box method. The objective of this paper is to compare the bond and option prices derived using the Crank Nicolson scheme and the Box method.

Section II discusses the CKLS model in depth. In Section III we develop the Box method in depth. Section IV compares bond and option prices calculated using both the Crank Nicolson scheme and the Box method. Section V contains summary and conclusion.

II. CKLS Model

CKLS (1992) used the following stochastic differential equation to specify the general form of the short- term interest rate, nesting a range of different term structure models.

$$dr = k(\theta - r) + \sigma r^{\gamma} dZ \tag{1}$$

where $k, \theta, \sigma, \gamma$ are unknown parameters. As in the CKLS (1992) paper, alternative term structure models given below by imposing restrictions on the parameters $k, \theta, \sigma, \gamma$:

1. Merton (1973) $dr = k\theta + \sigma dZ$
2. Vasicek (1977) $dr = k(\theta - r) + \sigma dZ$
3. Cox, Ingersoll, and Ross (1985) $dr = k(\theta - r) + \sigma r^{\frac{1}{2}} dZ$
4. Dothan (1978) $dr = \sigma r dZ$
5. Geometric Brownian Motion $dr = -kr + \sigma dZ$
6. Brennan and Schwartz (1980) $dr = k(\theta - r) + \sigma r dZ$

7. Cox, Ingersoll, and Ross (1980) $dr = \sigma r^{\frac{3}{2}} dZ$

8. Constant Elasticity of Variance $dr = krdt + \sigma r^{\gamma} dZ$

CKLS (1992) found that the value of γ is the most important feature distinguishing interest rate models. In particular they found that for the U.S. $\gamma \geq 1$ captures the dynamics of the short-term interest rate because the volatility of the process is highly sensitive to the value of r.

Based on standard arbitrage arguments, we can derive valuation equations for default free bonds and options based on the CKLS (1992) model. The valuation equation for the default free bond will be the same irrespective of the type of option, which is based on it. However, the valuation equation for different types of options will be different, due to the differing boundary conditions associated with each type of options. In this paper we concentrate on the valuation of zero coupon default free bonds and the valuation of call options based on the zero coupon bonds.

As in Courtadon (1982), we take the time expiry of the option as τ, and the time to maturity of the bond as $\tau' = \tau + T$, where T is the time to maturity remaining on the bond when the option expires.

Letting $u(r, \tau')$ be the value of the default-free bond; the valuation equation based on the CKLS process is:

$$\frac{1}{2}\sigma^2 r^{2\gamma} \frac{\partial^2 u}{\partial r^2} + k[\theta - r]\frac{\partial u}{\partial r} - ru = \frac{\partial u}{\partial \tau'} \qquad (2)$$

Subject to: $u(r,0) = 1$

$u(\infty, \tau') = 0$

Similarly the valuation equation for a call option $w(r, \tau)$ based on the CKLS process is:

$$\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 w}{\partial r^2}+k[\theta-r]\frac{\partial w}{\partial r}-rw=\frac{\partial w}{\partial \tau} \qquad (3)$$

Subject to: $w(r,0)=\mathrm{Max}[u(r,T)-E,0]$

$w(\infty,\tau)=0$

where E is the exercise price of the option.

III Numerical Solution for the Valuation of Default Free Bonds and Options

In this section we derive the Box method in depth. To derive the algorithm for the Box method, we focus on equation (2), as exactly the same analysis holds for equation (3). We start by dividing equation (2) by $\frac{\sigma^2 r^{2\gamma}}{2}$ and further we let: $a=\frac{2k\theta}{\sigma^2}, b=\frac{2k}{\sigma^2}, c=\frac{2}{\sigma^2}$.

This results in the following equation:

$$\frac{\partial^2 u}{\partial r^2}+\left[ar^{-2\gamma}-br^{1-2\gamma}\right]\frac{\partial u}{\partial r}-cr^{1-2\gamma}u=cr^{-2\gamma}\frac{\partial u}{\partial \tau'} \qquad (4)$$

We combine the first term and the second term on the left hand side of the above equation by choosing a function $\Psi(a,b,r,\gamma)$ or $\Psi(r)$ abbreviated such that:

$$\frac{1}{\Psi(r)}\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial r^2}+\left[ar^{-2\gamma}-br^{1-2\gamma}\right]\frac{\partial u}{\partial r} \qquad (5)$$

Expansion and simplification of the above formula leads to the following expression.

$$\frac{1}{\Psi(r)}\frac{\partial \Psi}{\partial r}=ar^{-2\gamma}-br^{1-2\gamma} \qquad (6)$$

Integrating the above equation with respect to r gives:

$$\Psi(r) = \exp\left[\frac{ar^{1-2\gamma}}{1-2\gamma} - \frac{br^{2-2\gamma}}{2-2\gamma}\right] \tag{7}$$

Note that with the above expression for $\Psi(r)$ there is singularity at $\gamma = \frac{1}{2}$ and $\gamma = 1$. Thus the above expression for $\Psi(r)$ is not valid at these two specific points. Further if $\gamma \neq 1$ or $\gamma \neq \frac{1}{2}$ but γ is very close to $\gamma = 1$ or $\gamma = \frac{1}{2}$, then the value of $\Psi(r)$ may be excessively large because of the nature of the denominators in equation (7). In such cases we need to use a more complex approach or simply switch to the expression for $\Psi(r)$ when $\gamma = 1$ or $\gamma = \frac{1}{2}$. To derive expression for $\Psi(r)$ when $\gamma = 1$ or $\gamma = \frac{1}{2}$, we substitute, these two values of γ directly into equation (6) and integrate to give:

$$\Psi(r) = \exp\left(\frac{-a}{r}\right) r^{-b} \qquad \text{for } \gamma = 1$$

$$\Psi(r) = \exp(-br)\, r^{a} \qquad \text{for } \gamma = \frac{1}{2}$$

With this choice of $\Psi(r)$, our original equation becomes

$$\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial u}{\partial r}\right) - \Psi(r) r^{1-2\gamma} cu = c\Psi(r) r^{-2\gamma} \frac{\partial u}{\partial \tau'} \tag{8}$$

Similar analysis of the option valuation equation yields:

$$\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial w}{\partial r}\right) - \Psi(r) r^{1-2\gamma} cw = c\Psi(r) r^{-2\gamma} \frac{\partial w}{\partial \tau} \tag{9}$$

We can represent, equations, (6) and (7) as a general equation:

$$\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial v}{\partial r}\right)-\Psi(r)r^{1-2\gamma}cv=c\Psi(r)r^{-2\gamma}\frac{\partial v}{\partial t} \tag{10}$$

where v may represent either $u(r,\tau')$ or $w(r,\tau)$ and t may represent either τ' or τ. We let r take value on the interval $H=[0,R]$ and t take value on the interval $T=[0,T']$. To solve equation (10) we need to fit the space $H\times T$ with a grid. We let Δr represent the grid spacing in the r dimension and Δt represent the grid spacing in the t direction, such that:

$r_n = n\Delta r$, $t_m = m\Delta t$ with $0\le n\le N, 0\le m\le M$ such that $N\Delta r = R$ and $M\Delta t = T'$.

To derive the Box method scheme, we integrate equation (10) from $r_a=\frac{r_n+r_{n-1}}{2}$ to $r_b=\frac{r_{n+1}+r_n}{2}$ yielding the following equation:

$$\int_{r_a}^{r_b}\frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial v}{\partial r}\right)dr-\int_{r_a}^{r_b}\left(\Psi(r)r^{1-2\gamma}cv\right)dr=\int_{r_a}^{r_b}\left(c\Psi(r)r^{-2\gamma}\frac{\partial v}{\partial \tau}\right)dr \tag{11}$$

Equation (11) is solved by numerical integration (full details in the Appendix). v is approximated by V_n^m at the grid points r_n and t_m. The resulting difference equation is:

$$\alpha_n=\chi_n V_{n-1}^m+\eta_n V_n^m+\beta_n V_{n+1}^m \tag{12}$$

$$\alpha_n = \frac{c r_b^{1-2\gamma}}{\Delta t(1-2\gamma)}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right)V_n^{m-1} \qquad \text{if } \gamma\ne\frac{1}{2} \text{ or } \gamma\ne 1$$

$$= -\frac{c}{\Delta t}\ln\left(\frac{r_a}{r_b}\right)V_n^{m-1} \qquad \text{for } \gamma=\frac{1}{2}$$

$$= \frac{c}{\Delta t}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)V_n^{m-1} \qquad \text{for } \gamma=1$$

$$\chi_n = -\frac{1}{\Delta r}\frac{\Psi(r_a)}{\Psi(r_n)}$$

$$\beta_n = -\frac{1}{\Delta r}\frac{\Psi(r_b)}{\Psi(r_n)}$$

$$\eta_n = \frac{1}{\Delta r}\left(\frac{\Psi(r_b)}{\Psi(r_n)} + \frac{\Psi(r_a)}{\Psi(r_n)}\right) + X$$

where:

$$X = \frac{c r_b^{2-2\gamma}}{2-2\gamma}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right) + \frac{c r_b^{1-2\gamma}}{\Delta t(1-2\gamma)}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right) \quad \text{provided } \gamma \neq \frac{1}{2} \text{ or } \gamma \neq 1$$

$$= \qquad\qquad \text{for } \gamma = \frac{1}{2}$$

$$= -c\ln\left(\frac{r_a}{r_b}\right) + \frac{c}{\Delta t}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \qquad \text{for } \gamma = 1$$

Equation (12) represents the individual element of system of equations covering the whole grid. The entire system can be summarized by the following matrix equation:

$$\begin{pmatrix} \alpha_1 - \chi_1 V_0^m \\ \alpha_2 V_2^{m-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_{N-1} - \beta_{N-1} V_N^m \end{pmatrix} = \begin{pmatrix} \eta_1 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \chi_2 & \eta_2 & \beta_2 & 0 & 0 & \cdots & 0 \\ 0 & \chi_3 & \eta_3 & \beta_3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \chi_{N-3} & \eta_{N-3} & \beta_{N-3} & 0 \\ \vdots & \ddots & \ddots & 0 & \chi_{N-1} & \eta_{N-2} & \beta_{N-2} \\ 0 & \cdots & \cdots & 0 & 0 & \chi_{N-1} & \eta_{N-1} \end{pmatrix} \begin{pmatrix} V_1^m \\ V_2^m \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ V_{N-1}^m \end{pmatrix} \qquad (13)$$

We use the iterative SOR (Successive Over Relaxation) method to solve the matrix equation. The first step of the SOR process involves forming an intermediate quantity z_n^m for a point n on the grid. Based on this intermediate quantity, a trial solution V_n^m is formed. This trial solution is iterated until certain accuracy is achieved between successive iterations. Having achieved this accuracy we move onto n+1 point on the grid at a particular time step.

$$z_n^m = \frac{1}{\eta_n}\left(\alpha_n V_n^{m-1} - \chi_n V_{n-1}^{m} - \beta_n V_{n+1}^{m-1}\right) \tag{14}$$

$$V_n^m = {}^4\omega z_n^m + (1-\omega)V_n^{m-1} \tag{15}$$

for n=1,......,N-1, and $\omega \in (1,2]]$

IV. Analysis of Results

In this section, we investigate the two numerical methods. Each method is implemented to value zero coupon bond and call option prices when the short-term interest rate follows the CKLS (1992) stochastic process.

Initially we check each of the numerical methods using assumed parameter values. We then use the parameters for the U.S. as estimated in the CKLS paper. As in Tian (1994), we define a quantity $\alpha_1 = (4k\theta - \sigma^2)/8$ for the assumed parameter values. $\alpha_1 > 0$ corresponds to low volatility and high mean reversion rate. For $\alpha_1 < 0$ the converse condition holds. We consider the specific case $\gamma = 0.5$. The maturities of the bonds are 5 and 15 years. The face value of the zero coupon bond is $100. Short -term interest rates of 5% and 11% are considered. For $\alpha_1 > 0, k = 0.5, \sigma = 0.1, \theta = 0.08$, and for $\alpha_1 < 0, k = 0.1, \sigma = 0.5, \theta = 0.08$. Table 1 – Table 2, and Table 3 – Table 4 contain the bond and call prices respectively calculated using each of the suggested numerical methods for different combinations of α_1. Table 5 and Table 6; contain the bond and call option prices based on the CKLS parameters respectively. For the sake of brevity, following notation will be used in all of the tables:

BMS: prices calculated using the Box method

CNS: prices calculated using the Crank Nicolson method

Table 1 – Table 2 both have the same format and contain zero coupon bond prices. In each of these tables, we alter the annual number of time steps from 20 to 1000. This variation serves as a check as to the convergence of each of the numerical schemes. Examination of Table 1 – Table 2 leads to the following observations:

From Table 1 we see that both combinations converge to produce bond prices which are in agreement with the analytical prices. As an example, consider a 5 year bond at 5% interest rate and 50 annual time steps. The Box bond price is $71.0754, compared to the Crank Nicolson bond price of $71.6853. We also find that Box bond prices are always lower than Crank Nicolson, and that Box bond prices are closer to the analytical prices than Crank Nicolson prices. For example, we see that a 5-year bond at 5% interest is valued at 71.0379. The same bond at twenty annual time step is valued at $71.1006 using Box and $71.6853 using Crank Nicolson.

Table 2 contains bond prices for $a_1 < 0$. Overall bond price accuracies are similar to those found in Table 1.

Table 3 – Table 4 all have the same format and comprise of call options based on zero coupon bond prices for various expiry dates and exercise prices. In Table 3 – Table 4 the first column indicates the range of exercise prices and the first row indicates the different expiry dates of the option ranging from 1 year to 5 years. All the call options are based on a 10 year zero coupon bond. Further the third column entitled, "Bond Price", indicates the price of a 10 year zero coupon bond based on each of the possible combinations. For example, turning to Table 3's, third column, we find that the price of a 10 year zero coupon bond calculated using the Box method is $45.5000, whereas the same bond is priced at $45.8809 using the Crank Nicolson method. Examination of Table 3 – Table 4 leads to the following observations:

Box prices are closer to the analytical prices than Crank Nicolson call prices. For example, from Table 3, consider a 5-year call option, exercised at $35. The analytical call price is $21.8802; Box pricing is $21.9445 and the Crank Nicolson price again using $22.1132.

As with bonds, Box prices are always lower than the corresponding call prices calculated using Crank Nicolson. However, unlike bonds, the differences are significant when $\alpha_1 < 0$. To illustrate the differences in call prices between the Box and the Crank Nicolson, we consider an example from Table 4. In particular, consider a 5-year option, exercise at $60, the analytical call price is $23.9008, the Box price is $23.9476, and the Crank Nicolson price is $32.2997.

Table 5 contains bond prices based on the parameter values calculated by CKLS. We find that both combinations converge to produce similar prices. As with Table 1 and Table 2, we find that Box prices are slightly lower than the corresponding Crank Nicolson prices.

Table 6 contains call option prices based on the parameter values calculated by CKLS. We again observe the same the same trend as with Table 3 and Table 4.

V. Conclusions

We have introduced a new numerical method from engineering and the physical sciences to finance – the Box method. We have compared it with the existing scheme in finance – the Crank Nicolson. By, first assuming parameter values for the CKLS process, and then using historical parameters calculated by CKLS; we were able to test each of the numerical schemes both for the case of, extreme parameters values and historical parameter values.

We found that for assumed parameter values, where there was high mean reversion rate and low volatility parameter both the numerical schemes yielded bond and call option prices which were close to the analytical prices. For the reverse case, bond prices using both Crank Nicolson and Box were close to analytical prices, but only the Box option prices were close to the analytical option prices. In summary we found that Box prices were always closer to the analytical bond and call option prices than the Crank Nicolson bond and call option prices.

Finally, we find that when using the historical parameter values from the CKLS parameter values both numerical schemes yield bond and call option prices, which are close to each other, with Box prices being lower again than the corresponding prices calculated using the Crank Nicolson scheme.

Table 1. Bond Prices calculated analytically (CIR), using the Box and the Crank Nicolson method.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$k = 0.5,\ \theta = 0.08,\ s = 0.1,\ \Delta r = 0.5\%,\ g = 0.5$$

			Annual number of time steps (n)					
Maturity (years)	**Model**	**r(%)**	**20**	**50**	**100**	**300**	**500**	**1000**
5	CIR	5	71.0379	71.0379	71.0379	71.0379	71.0379	71.0379
	BMS		71.1006	71.0754	71.0670	71.0614	71.0603	71.0595
	CNS		71.6853	71.6853	71.6858	71.6914	71.6853	71.6854
5	CIR	11	63.7161	63.7161	63.7161	63.7161	63.7161	63.7161
	BMS		63.7850	63.7475	63.7349	63.7266	63.7249	63.7237
	CNS		64.3129	64.3130	64.3134	64.3188	64.3130	64.3131
15	CIR	5	32.5442	32.5442	32.5442	32.5442	32.5442	32.5442
	BMS		32.6428	32.5979	32.5829	32.5728	32.5711	32.5689
	CNS		32.8647	32.8648	32.8648	32.8648	32.8646	32.8657
15	CIR	11	28.9322	28.9322	28.9322	28.9322	28.9322	28.9322
	BMS		29.0135	28.9735	28.9601	28.9511	28.9496	28.9476
	CNS		29.2251	29.2251	29.2251	29.2251	29.2250	29.2259

Table 2. Bond Prices calculated analytically (CIR), using the Box and the Crank Nicolson methods.

$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$ $k = 0.1,\ \theta = 0.08,\ s = 0.5,\ \Delta r = 0.5\%,\ g = 0.5$								
			Annual number of time steps (n)					
Maturity (years)	**Model**	**r(%)**	**20**	**50**	**100**	**300**	**500**	**1000**
5	CIR	5	83.4832	83.4832	83.4832	83.4832	83.4832	83.4832
	BMS		83.6040	83.5409	83.5244	83.5145	83.5115	83.5098
	CNS		84.3837	84.3614	84.3554	84.3538	84.3516	84.3503
5	CIR	11	72.5572	72.5572	72.5572	72.5572	72.5572	72.5572
	BMS		72.6956	72.6166	72.5961	72.5836	72.5802	72.5781
	CNS		73.2609	73.2389	73.2338	73.2319	73.2305	73.2293
15	CIR	5	68.2741	68.2741	68.2741	68.2741	68.2741	68.2741
	BMS		68.4127	68.3836	68.3730	68.3668	68.3657	68.3631
	CNS		69.0981	69.0851	69.0807	69.0802	69.0801	69.0801
15	CIR	11	58.9177	58.9177	58.9177	58.9177	58.9177	58.9177
	BM		59.0348	59.0095	59.0002	58.9947	58.9940	58.9913

Table 3. Call Prices calculated analytically (CIR), using the Box and the Crank Nicolson methods.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 > 0$$

$$\Delta t = 0.05, \Delta r = 0.5\%, r_0 = 8\%, \gamma = 0.5$$

Exercise Price	**Model**	**Bond Price**	Maturity (years) **5**	**4**	**3**	**2**	**1**
	CIR	45.1561[2]	21.8802	19.9509	17.8585	15.5863	13.1552
35	BMS	45.5000[1]	21.9445	20.0181	17.9293	15.6615	13.1957
	CNS	45.8809	22.1132	20.1790	18.0921	15.8450	13.4362
	CIR		18.5163	16.3114	13.9201	11.3233	8.4993
40	BMS		18.5836	16.3605	13.9887	11.3968	8.5789
	CNS		18.7181	16.5076	14.0513	11.5545	8.8015
	CIR		15.1524	12.6719	9.9819	7.0636	3.9137
45	BMS		15.2104	12.7336	10.0482	7.1352	3.9896
	CNS		15.3230	12.8362	10.1531	7.2662	4.1834
	CIR		11.7886	9.0330	6.0560	2.9514	0.4535
50	BMS		11.8433	9.0919	6.1191	3.0126	0.4788
	CNS		11.9820	9.1653	6.1943	3.1020	0.5267
	CIR		8.4257	5.4156	2.3804	0.3118	0.0001
55	BMS		8.4772	5.4705	2.4305	0.3307	0.0001
	CNS		8.5338	5.5143	2.4679	0.3443	0.0000

Table 4. Call Prices calculated analytically (CIR), using the Box Method and the Crank Nicolson method.

$$\alpha_1 = \left(4k\theta - \sigma^2\right)/8 < 0$$

$$\Delta t = 0.05, \Delta r = 0.5\%, r_0 = 8\%, \gamma = 0.5$$

Exercise Price	**Model**	**Bond Price**	Maturity (years) **5**	**4**	**3**	**2**	**1**
	CIR	63.4557	23.9008	22.8564	20.2596	19.8902	16.9798
60	BMS	69.9969	23.9476	22.9006	21.6375	19.9112	16.9769
	CNS	70.8166	32.2997	32.0170	31.4356	30.1946	27.3805
	CIR		20.1770	19.0843	17.7967	16.0922	13.2470
65	BM		20.2200	19.1255	17.8313	16.1109	13.2320
	CNS		28.2373	27.9676	27.4063	26.1936	23.3519
	CIR		16.4887	15.3565	14.0532	12.3971	9.7260
70	BMS		16.5281	15.3950	14.0865	12.4102	9.7061
	CNS		24.1833	23.9313	23.4043	22.2519	19.4636
	CIR		12.8444	11.6829	10.3819	8.8038	6.4487
75	BM		12.8803	11.7194	10.4151	8.8246	6.4317
	CNS		20.1358	19.4299	19.4299	18.3732	15.7371
	CIR		9.2570	8.0789	6.8019	5.3528	3.4558
80	BM		9.2895	8.1135	6.8352	5.3787	3.4527
	CNS		16.0962	15.8990	15.4794	14.5314	12.0601

Table 5. Bond Prices calculated using the Box and the Crank Nicolson methods based on the original CKLS parameters.

Maturity (years)	**r(%)**	**BMS**	**CNS**
5	5	73.9710	74.9387
	8	68.0032	68.8242
	11	62.8360	63.4843
10	5	52.4782	53.2131
	8	47.4292	48.0326
	11	43.2494	43.7180
15	5	36.9671	37.4302
	8	33.3173	33.6941
	11	30.3189	30.6063

Table 6. Call Prices calculated using the Box and the Crank Nicolson methods based on the original CKLS parameters.

$$k = 0.2213, \theta = 0.0786, \sigma = 1.1767, \gamma = 1.4808$$

$$\Delta t = 0.05, \Delta r = 0.5\%$$

			Maturity (years)				
Exercise Price	**Model**	**Bond Price**	**5**	**4**	**3**	**2**	**1**
40	BMS	47.4292	20.2416	18.1490	15.8549	13.3260	10.5193
	CNS	48.0326	20.5324	18.4391	16.1612	13.6756	10.9583
45	BMS		16.8573	14.5175	11.9626	9.1602	6.0511
	CNS		17.1065	14.7647	12.2240	9.4592	6.4189
50	BMS		13.4905	10.9259	8.1616	5.2122	2.1787
	CNS		13.6978	11.1282	8.3707	5.4372	2.3901
55	BMS		10.1617	7.4289	4.6026	1.9276	0.1891
	CNS		10.3262	7.5820	4.7445	2.0392	0.2051
60	BMS		6.9194	4.1675	1.6821	0.1986	0.0001
	CNS		7.0386	4.2640	1.7460	0.2078	0.0003

APPENDIX : Box Method

Our starting position is to integrate equation (10) where $r_a = \frac{r_n + r_{n-1}}{2}$ to $r_b = \frac{r_{n+1} + r_n}{2}$:

$$\int_{r_a}^{r_b} \frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial v}{\partial r}\right)dr - \int_{r_a}^{r_b}\left(\Psi(r)r^{1-2\gamma}cv\right)dr = \int_{r_a}^{r_b}\left(c\Psi(r)r^{-2\gamma}\frac{\partial v}{\partial \tau}\right)dr \qquad \text{(A.1)}$$

For $\frac{\partial u}{\partial t}$ we use the backward Euler approximation as with the Crank Nicolson to obtain the following equation.

$$\frac{\partial v}{\partial t} = \frac{v - v_0}{\Delta t}$$

Such that equation (A.1) becomes:

$$\int_{r_a}^{r_b} \frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial v}{\partial r}\right)dr - \int_{r_a}^{r_b}\left(\Psi(r)r^{1-2\gamma}cv\right)dr = \int_{r_a}^{r_b}\left(c\Psi(r)r^{-2\gamma}\left(\frac{v - v_0}{\Delta t}\right)\right)dr$$

Further rearrangement leads to the following expression

$$-\int_{r_a}^{r_b} \frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial v}{\partial r}\right)dr + \int_{r_a}^{r_b}\left(c\Psi(r)r^{1-2\gamma}v + \frac{\Psi(r)dr^{-2\gamma}}{\Delta t}v\right)dr = \int_{r_a}^{r_b}\left(\frac{\Psi(r)cr^{-2\gamma}}{\Delta t}v_0\right)dr$$

Approximating each of the integrals, we have:

$$-\int_{r_a}^{r_b} \frac{\partial}{\partial r}\left(\Psi(r)\frac{\partial v}{\partial r}\right)dr = -\Psi(r_b)\left(\frac{V_{n+1}^m - V_n^m}{\Delta r}\right) + \Psi(r_a)\left(\frac{V_n^m - V_{n-1}^m}{\Delta r}\right)$$

$$\int_{r_a}^{r_b}\left(c\Psi(r)r^{1-2\gamma}v+\frac{\Psi(r)dr^{-2\gamma}}{\Delta t}v\right)dr$$

$$=\Psi(r_n)\left[\frac{cr_b^{2-2\gamma}}{2-2\gamma}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right)+\frac{cr_b^{1-2\gamma}}{\Delta t(1-2\gamma)}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right)\right]V_n^m \text{ if } \gamma\neq\frac{1}{2} \text{ or } \gamma\neq 1$$

$$=\Psi(r_n)\left[c(r_b-r_a)-\frac{c}{\Delta t}\ln\left(\frac{r_a}{r_b}\right)\right]V_n^m \quad \text{for } \gamma=\frac{1}{2}$$

$$=\Psi(r_n)\left[-c\ln\left(\frac{r_a}{r_b}\right)+\frac{c}{\Delta t}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)\right]V_n^m \quad \text{for } \gamma=1$$

$$\int_{r_a}^{r_b}\left(\frac{\Psi(r)dr^{-2\gamma}}{\Delta t}V_0\right)dr$$

$$=\Psi(r_n)\left[\frac{cr_b^{1-2\gamma}}{\Delta t(1-2\gamma)}\left(1-\left(\frac{r_a}{r_b}\right)^{1-2\gamma}\right)\right]V_n^{m-1} \quad \text{if } \gamma\neq\frac{1}{2} \text{ or } \gamma\neq 1$$

$$=\Psi(r_n)\left[-\frac{c}{\Delta t}\ln\left(\frac{r_a}{r_b}\right)\right]V_n^{m-1} \quad \text{for } \gamma=\frac{1}{2}$$

$$=\Psi(r_n)\left[\frac{c}{\Delta t}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)\right]V_n^{m-1} \quad \text{for } \gamma=1$$

Substituting the above approximations into the original equation yields

$$\alpha_n=\chi_n V_{n-1}^m+\eta_n V_n^m+\beta_n V_{n+1}^m$$

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